Morley's Proof
Canadian Mathematical Society MITACS Winnipeg
June 3, 2007

John T. Baldwin

June 3, 2007

Modern Model Theory Begins

Theorem (Morley 1965)

If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.

Outline

SELF-CONSCIOUS MATHEMATICS

- A vocabulary (or signature) L is a collection of relation and function symbols.
- A structure for that vocabulary (L-structure) is a set with an interpretation for each of those symbols.

Languages

- The first order language ($L_{\omega,\omega}$) associated with L is the least set of formulas containing the atomic L-formulas and closed under **finite** Boolean operations and quantification over finitely many individuals.
- The $L_{\omega_1,\omega}$ language associated with L is the least set of formulas containing the atomic L-formulas and closed under **countable** Boolean operations and quantification over finitely many individuals.

Model Theory and Mathematics

Prior to 1960: Use of basic model theoretic notions

compactness, quantifier elimination: Erdos-Rado
applications
Ax-Kochen-Ershov

Later: Increasing use of sophisticated first order model theory

stability theory;
Shelah's orthogonality calculus;
o-minimality:
applications
Wilkie's proof of o-minimality of $(\Re, +, \cdot, \exp)$.
Hrushovski's proof of geometric Mordell-Lang.

The Transition

'...what makes his paper seminal are its new techniques, which involve a systematic study of Stone spaces of Boolean algebras of definable sets, called type spaces. For the theories under consideration, these type spaces admit a Cantor Bendixson analysis, yielding the key notions of Morley rank and ω-stability.'

Citation awarding Michael Morley the 2003 Steele prize for seminal paper.

Categoricity

A class **K** is categorical in κ if all members of **K** with cardinality κ are isomorphic.

ALGEBRAICALLY CLOSED FIELDS

Fundamental structure of Algebraic Geometry
Axioms for fields of fixed characteristic and for each n

$$(\forall a_1, \ldots a_n)(\exists y)\Sigma_{i=1}^{n} a_i y^i = 0$$

FUNDAMENTAL EXAMPLE

The theory T_p of algebraically closed fields of fixed characteristic has exactly one model in each uncountable cardinality. (Steinitz)

That is, T_p is *categorical* in each uncountable cardinality

First Order Categorical Structures

I. $(\mathcal{C}, =)$

II. $(\mathcal{C}, +, =)$ vector spaces over Q.

III. $(\mathcal{C}^*, \times, =)$

IV. $(\mathcal{C}, +, \times, =)$ Algebraically closed fields - Steinitz

Zilber's Precept

Fundamental canonical mathematical structures like I-IV should admit logical descriptions that are categorical in power.

Another Canonical Structure

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

The integers are defined as $\{a : e^{2a\pi i} = 1\}$.
This makes the first order theory unstable, provides a two cardinal model .... The theory is clearly not categorical.

Thus first order axiomatization can not determine categoricity.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty.
Fix Z by adding the axiom:

$$(\forall x) e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi i.$$

Two Themes

1. Contexts where Morley's Theorem is generalized
2. Differing proofs and how they generalize

Contexts

Syntactic Description

$L_{\omega,\omega}$, $L_{\kappa,\omega}$, $L(Q)$, $L_{\infty,\omega}$, $L_{\kappa,\mu}$, continuous logics

Semantic Description

1. Homogeneous Model Theory
2. Abstract Elementary Classes
3. CATS

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónssson:

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónssson:

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;
2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónssson:

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;
2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

Fundamental Ideas

Two Definitions of (first order) Type

1. Let $A \subset M$. A syntactic 1-type is an ultrafilter in the Boolean algebra of 1-ary formulas with parameters from A.
2. The *first order type* of b over A (in M) is the collection formulas $\phi(x, \mathbf{a})$ with parameters from A that are satisfied by b.

These are equivalent but note the dependence on M.
$S(A)$ is the set of types over A.

Definition: first order Saturation

M is *κ-saturated* if
every 1-type over a subset A of M with $|A| < \kappa$ is realized.

We say saturated if $\kappa = |M|$.

Prime Models

M is prime over A if every elementary embedding of A in to $N \models T$ extends to an elementary embedding of M into N.

κ-stability

K is κ-stable if for every $M \in \mathbf{K}$ with $|M| = \kappa$, $|S(M)| = \kappa$.

Categoricity implies stability

arbitrarily large models: : Ehrenfeucht-Mostowski models give stability below the categoricity cardinal.

κ-stability

K is κ-stable if for every $M \in \mathbf{K}$ with $|M| = \kappa$, $|S(M)| = \kappa$.

Categoricity implies stability

arbitrarily large models: : Ehrenfeucht-Mostowski models give stability below the categoricity cardinal.

$2^{\aleph_0} < 2^{\aleph_1}$: For $L_{\omega_1,\omega}$, categoricity in $\aleph_1$ implies $\aleph_0$-stability.

Variants on Morley's Theorem

A rough classification

1. First order logic (4)
2. Assume Arbitrarily Large Models
   - Cofinal Categoricity (1)
   - Eventual Categoricity (9)
3. Build Large Models (3)

Morley's Proof (1965)

Saturation means first order saturated.

Theorem

If **K***, the class of models of a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.*

1. Saturated models of the same cardinality are isomorphic.
2. κ-categoricity implies $< \kappa$-stable. ω stable implies stability in all powers.
3. For any κ, κ-stable implies there is an $\aleph_1$-saturated model of cardinality κ.

Morley's Proof continued

4. ω-stable implies
   - If there is a nonsaturated model, there is a countable model M with a countable subset X such that:
     a) M contains an infinite set of indiscernibles over X;
     b) Some $p \in S(X)$ is omitted in M.
5. Taking prime models over sequences of indiscernibles, Item 4) implies:
   If there is a nonsaturated model, then there is a model in every cardinal that is not $\aleph_1$-saturated.

Key Ideas - Morley

1. ω-stability
2. Ehrenfeucht-Mostowski Models
3. saturation
4. omitting types
5. indiscernibles
6. prime models

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
If points are closed the structure is called a geometry.

STRONGLY MINIMAL

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

The complex field is strongly minimal.

Baldwin-Lachlan proof (1971)

Theorem

If **K***, the class of models of a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.*

1. κ-categoricity implies $< \kappa$-stable. ω stable implies stability in all powers.
2. upwards: $\aleph_1$-cat implies κ-cat
   - There exists a strongly minimal set
   - Every model is prime and minimal over the strongly minimal set
   - strongly minimal sets have dimension.
3. downwards: κ-cat implies $\aleph_1$-cat:
   If an ω-stable theory has a two-cardinal model in $\aleph_1$ then it has one in every cardinal.

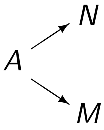
Key Ideas - B-L

1. ω-stability
2. Ehrenfeucht-Mostowski Models
3. strongly minimal sets
4. two-cardinal models
5. prime models

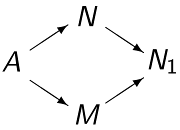
Prime models are essential for both the upwards and downwards arguments. Saturation is not used.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



Assumptions

A.P. etc

An AEC **K** has 'a.p. etc' means **K** has:

1. arbitrarily large models
2. amalgamation over models
3. joint embedding

The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property then we can work inside a monster model (universal domain) $\mathcal{M}$ that is $|\mathcal{M}|$-model homegeneous. That is,

If $N \prec_{\mathbf{K}} \mathcal{M}$ and $N \prec_{\mathbf{K}} M \in \mathbf{K}$ and $|M| < |\mathcal{M}|$ there is embedding of M into $\mathcal{M}$ over N.

Galois Types

Fix a monster model $\mathbb{M}$ for $\mathbf{K}$.

Definition

Let $A \subseteq M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The *Galois type* of a over M $(\in \mathbb{M})$ is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

The set of Galois types over A is denoted $\mathcal{S}(A)$.

Galois Saturation

Definition

The model M is μ-*Galois saturated* if for every $N \prec_{\mathbf{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.

Theorem

For $\lambda > \mathrm{LS}(\mathbf{K})$, If M, N are λ-Galois saturated with cardinality λ then $M \approx N$.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Distinct Galois types differ on a small submodel.

Definition

We say $\mathbf{K}$ is (χ, μ)-*tame* if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \upharpoonright N_0 = q \upharpoonright N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut_N}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut_M}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose the AEC **K** has a.p. etc.

Theorem (Grossberg-Vandieren: 2006)

*If $\lambda > \mathrm{LS}(\mathbf{K})$, **K** is λ^+-categorical and $(\lambda, < \infty)$-tame then **K** is categorical in all $\theta \geq \lambda^+$.*

Theorem (Lessmann)

*If **K** with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then **K** is categorical in all uncountable cardinals*

AEC categoricity

Suppose the AEC **K** has a.p. etc.
Saturation is Galois saturation. There is no use of prime models.

1. saturated models of same cardinality are isomorphic.
2. Categoricity in any power κ implies stability below κ.
3. κ-stable implies there is a saturated model of cardinality κ for every (regular)-κ.

Grossberg-VanDieren 2006: tame AEC upward-categoricity

Proof Sketch

4. Categoricity in κ and κ^+ implies there is no Vaughtian pair with respect to a minimal type over a model of cardinality κ.
5. Condition 4) implies every saturated model of cardinality κ^{++} is saturated (tameness crucial)
6. Now induct on cardinality.

Shelah 1999: AEC downward-categoricity

Theorem

K *has a.p. etc. If* **K** *is categorical in some* λ^+ *above* H_2, **K** *is categorical on* $[H_2, \lambda^+]$.

4. Using Morley omitting types theorem/ two cardinal theorem,
   1. the model in H_2 is Galois saturated.
   2. **K** is $(< H_1, \lambda^+)$-tame.
   3. the model in H_2 does not admit a Vaughtian pair.
5. Argue as in Grossberg-VanDieren to move from H_2-categoricity to categoricity on $[H_2, \lambda^+]$.

Another Approach (1971)

Keisler, Chudnovsky and Shelah: $L_{\omega_1,\omega}$

Replace the omitting types argument (4-5 of Morley's proof) by Morley's omitting types theorem and two cardinal theorem for cardinals far apart.

Problem

But if one restricts to types that are realized in models of **K**, (weak types $S^*(A)$) the uniqueness of 'saturated' models fails.

Finitary AEC (2006)

Assume A.P. etc.
Hytinnen and Kesala resurrect this general idea in the context of finitary AEC.

1. Develop stability theory over arbitrary sets;
2. transfer 'weak categoricity'.
3. But with tame get the strongest known categoricity transfer conclusion:
   a.p etc, tame, finitary, simple implies categoricity in one uncountable cardinal implies categoricity in all
4. Kueker shows close connection to $L_{\omega_1,\omega}$

Categoricity Transfer in $L_{\omega_1,\omega}$

1. Zilber (Baldwin-Lachlan style): Upwards only
2. Shelah (Morley style)

MODEL THEORETIC CONTEXT

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case.

QUASIMINIMAL EXCELLENCE

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,

QUASIMINIMAL EXCELLENCE

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.

QUASIMINIMAL EXCELLENCE

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the 'excellence condition' which follows.

Excellence

Definition

K is excellent if there is a prime model over any countable independent n-system.

Extending isomorphism

Quasiminimal excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

Zilber: 2005

Quasiminimal Excellence implies Categoricity

Theorem. Suppose the quasiminimal excellent class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Shelah's Approach

K is the class of atomic models (realize only principal types) of a first order theory.
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

ω-stability I

Definition

The atomic class **K** is λ-stable if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{\mathrm{at}}(M)| = \lambda$.

Theorem (Keisler-Shelah)

If **K** is $\aleph_1$-categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then **K** is ω-stable.

Categoricity Transfer in $L_{\omega_1,\omega}$

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

WGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

Sketchy Sketch

1. Develop a notion of independence for ω-stable atomic classes
2. Define excellence in that context.
3. Prove categoricity transfer in excellent classes.
4. Prove few models implies excellence.

Reprise

1. First order
2. Homogeneous Models (not covered)
3. ap etc.
4. $L_{\omega_1,\omega}$

Thank you, Canada.
Merci beaucoup, Canada