MODEL THEORY: FINITE, COUNTABLE, UNCOUNTABLE

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THEMES

I. Descriptive Complexity and Logic on Finite Models

II. Approximating the finite by the countable.

III. Stability theory as analog and actor.

What are model theoretically significant invariants of a first order formula?

DESCRIPTIVE COMPLEXITY

Fact: If $\mathbf{K}$ is a class of finite structures that is defined by a first order sentence then $\mathbf{K}$ is decided in polynomial time.

Definition: $\mathbf{FP}$ denotes the logic obtained by adding to first order logic formulas expressing the fixed point of inductive definition over a formula $\phi(\overline{x}, X)$ where X occurs positively.

Theorem. (Immerman/Vardi)
On *ordered structures* $\mathbf{FP} = \mathbf{PTIME}$.

MAJOR PROBLEM: Is there a logic which captures polynomial time on arbitrary finite structures?

INFINITARY LOGIC ON FINITE MODELS ???

FO^k denotes first order logic with only k-variables *free or bound.*

$L^k_{\omega_1,\omega}$ denotes $L_{\omega_1,\omega}$ with only k-variables *free or bound.*

$L^\omega_{\omega_1,\omega} = \cup_k L^k_{\omega_1,\omega}$.

Theorem: (Barwise) **FP** is contained in $L^\omega_{\omega_1,\omega}$.

For finite A, B,
if $A \equiv_{FO^k} B$ then $A \equiv_{L^k_{\omega_1,\omega}} B$

It is more convenient to analyze $L^\omega_{\omega_1,\omega}$ by means of 'pebble games' than to work directly with fixed-point logics.

THE SEARCH FOR A LOGIC FOR P

Considerable work has attempted to find a logic which captures polynomial time, e.g. by adding Lindstrom quantifiers.

Theorem: (Cai, Furer, Immerman) No extension of $L^{\omega}_{\omega_1,\omega}$ by a finite number of Lindstrom Quantifiers captures polynomial time.

This is reminiscent of the situation in model theory in the late 60's - e.g. dependence on set theory for two-cardinal theorems. The solution then was to classify theories.

FINITE VARIABLE LOGIC

Definition.

i) For $m \leq k$ and A finite, $\overline{a} \in A^m$, an m-type in FO^k, p, is the collection of FO^k formulas satisfied by $\overline{a} \in A$.

ii) $S^k_k(A)$ is the collection of k-ary FO^k-types realized in A.

iii) The k-size of A is $|S^k_k(A)|$.

We have Scott sentences for finite models.

Lemma. For any finite structure A and any finite k there is a sentence of FO^k which characterizes A up to FO^k-equivalence. Moreover, the quantifier rank of the Scott sentence is less than $|S^k_k(A)|$.

LOWENHEIM SKOLEM FOR FINITE VARIABLE LOGIC

Lemma. (Dawar) For finite A there are functions

i) (Lowenheim) $L_k(n)$ such that $|S_k^k(A)| \leq n$ implies there is a model B with $B \equiv_k A$ and $|B| < L_k(n)$.

ii) (Hanf)and $H_k(n)$ such that $|S_k^k(A)| \leq n$ and there is a model B with $B \equiv_k A$ and $|B| > H_k(n)$ implies there are arbitrarily large finite models k-equivalent to A.

Proof sketch: E.g.

$$L_k(n) = \max\{|A| : k - size(A) \leq n$$

$$\text{and A is smallest in its } \equiv_k \text{ class}\}.$$

Problem: Grohe has shown L_k can not be recursive; what about H_k?

STABLE FORMULAS

Lemma. (Baldwin-Dawar) If a complete theory T in FO^k has arbitrarily large finite models then no FO^k-formula has the strict order property.

Lemma. (Shelah) If the k-variable formula ϕ does not have the have the independence property then there is a k'-formula ψ with the strict order property.

Questions: Must k' be greater than k? Suppose a formula does not have the strict order property but is unstable; can you derive a formula with the independence property; how complicated? Need to consider (a la Grossberg/Ensley) the n-order property etc.

Is there a reasonable definition of a k-variable type *with parameters*?

APPROXIMATING THE FINITE BY THE COUNTABLE

General Program: Find a countable structure (or elementary equivalence class of structures) which represents a class of finite structures.

Specific Programs:

i) homogeneous structures

ii) smoothly approximable structures

iii) model theoretic forcing

iv) almost sure theories

The first two are the most highly developed but require that the countable structure be $\aleph_0$-categorical. We talk today about iv).

ALMOST SURE THEORIES

Fix a finite relational language L. Let $\mathbf{K_n}$ be a collection of L-structures with universe n. Let P_n be a probability measure on $\mathbf{K_n}$.

For any formula ϕ, let

$$P_n(\phi) = \Sigma\{P_n(B) : B \models \phi, |B| = n\}.$$

E.g. $\mathbf{K_n}$ is all graphs of size n; P_n is the uniform distribution (edge probability 1/2).

T is an *almost sure theory* if for some $(\mathbf{K_n}, \mathbf{P_n})$, $\phi \in T$ iff $P(\phi) = \lim_{n\to\infty} P_n(\phi) = 1$.

0-1 law for finite graphs (Glebski et al, Fagin):

The (theory of the)random graph is almost sure with respect to the uniform distribution as each extension axiom has limit probability 1.

ALMOST EVERYWHERE EQUIVALENCE

Definition. The logics $\mathcal{L}$ and $\mathcal{L}'$ are *almost everywhere equivalent* with respect to the probability measure P if there exists a collection C of finite models such that $P(C) = 1$ and for every sentence θ of $\mathcal{L}$ there is a sentence θ' of $\mathcal{L}'$ such that θ and θ' are equivalent on C (and conversely).

Theorem. (Hella, Kolaitis, Luosto) FO and $L^{\omega}_{\infty,\omega}$ are almost everywhere equivalent with respect to the uniform distribution.

RYLL-NARDJEWSKI FOR FINITE VARIABLE LOGIC

Let T_{ran} be the almost sure first order theory of class $\mathbf{K}$ with respect to a probability $P = \langle P_n : n < \omega \rangle$.

An FO^k-type p is a maximal set of FO^k formulas consistent with T_{ran}, p is k-principal if it is generated by a formula in FO^k

Mcarthur using results of Dawar, Lindell, and Weinstein showed.

Theorem. For each k, if there are only finitely many FO^k-types over T_{ran} in at most k elements, then $L^k_{\infty,\omega}$ is almost everywhere reducible to first order on $\mathbf{K}$.

We will see applications below where T_{ran} is not $\aleph_0$-categorical.

MODEL THEORETIC INTERLUDE: WEIGHT

Let $\mathbf{K_0}$ be a universal class of finite graphs and $\mathbf{K}$ a class of structures such that if $M \in \mathbf{K}$ then every finite substructure of $M \in \mathbf{K_0}$. Define a preweight by:

Let δ map members of $\mathbf{K_0}$ into the real numbers by

$$\delta(\langle A, R\rangle) = |A| - \alpha e(R)$$

for some α where $e(R)$ is the number of edges of the graph.

STRONG SUBSTRUCTURES

For each $N \in \mathbf{K}$ and finite $X \subseteq N$, the *weight* of X in N is

$$d_N(X) = \min\{\delta(X') : X \subseteq X' \subseteq_\omega N\}.$$

For $N \subseteq M$, we say that N is *strong* in M, and write $N \leq M$, if for all finite $X \subseteq N$,

$$d_N(X) = d_M(X).$$

THE AMALGAMATION PROPERTY

$A \longrightarrow B$ means there is a strong embedding f from A to B.

Definition. The class $(\mathbf{K_0}, \leq)$ satisfies the *amalgamation property* (**AP**) if for any situation:

$$\begin{tikzcd}[row sep=small, column sep=small] & B \\ A \arrow[ur] \arrow[dr] & \\ & C \end{tikzcd}$$ there exist a D and strong embeddings,

such that $$\begin{tikzcd}[row sep=small, column sep=small] & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D. \\ & C \arrow[ur] & \end{tikzcd}$$

A, B, C, D are in $\mathbf{K_0}$; A may be empty,

Theorem. If $(\mathbf{K_0}, \leq)$ is a collection of finite relational structures that satisfies natural properties on $\leq$ and has the amalgamation property then there is a countable $\mathbf{K_0}$-generic model M.

PARAMETERS OF THE CONSTRUCTION

We consider classes $(\mathbf{K_0}, \leq)$.
$\leq$ is usually derived from a function δ_α, by

$$A \leq B \text{ if for each } B' \subseteq B, \delta_\alpha(B'/A) \geq 0.$$

By varying $\mathbf{K_0}$ and $\leq$ (that is, α) independently many important examples arise. These include Hrushovski's examples to the Lachlan and Zil'ber conjectures, an almost strongly minimal projective plane. Other variants include fusions. We concentrate today on the following variation.

RANDOM GRAPHS

Let B be a graph with $|B| = n$. Let

$$P_n(B) = n^{-\alpha|e(B)|} \cdot (1 - n^{-\alpha})^{\binom{n}{2} - e(B)}.$$

Let α be irrational $0 < \alpha < 1$.

Key Point: The expected value of embedding B over A into a random graph of size n is $n^{\delta(B/A)}$.

Theorem. [Spencer-Shelah] For each first order sentence ϕ, $\lim_{n\to\infty} P_n(\phi)$ is 0 or 1.

Theorem.[Baldwin-Shelah] The almost sure theory is stable and nearly model complete.

TWO ALMOST SURE THEORIES

THE RANDOM GRAPH –uniform distribution

i) simple, not stable

ii) $\aleph_0$-categorical

iii) has the finite cover property

iv) $L^{\omega}_{\infty,\omega}$ almost equivalent to first order.

THE RANDOM GRAPH –edge probability $n^{-\alpha}$, α irrational.

i) stable

ii) not $\aleph_0$-categorical; not small

iii) does not have the finite cover property

iv) $L^{\omega}_{\infty,\omega}$ is not almost equivalent to first order.

THE RAMSEY QUANTIFIER

Consider the quantifier $(Q_{ram,f})$ defined by $Q^n_f \overline{x}\phi(\overline{x},\overline{y})$ which holds in a finite model $|A|$ if there is a homogeneous subset for ϕ of cardinality at least $f(|A|)$.

Theorem. If f is unbounded, the logic $L_{\omega,\omega}(Q_{ram,f})$ is almost everywhere equivalent to first order logic on graphs with respect to either the uniform distribution or edge probability $n^{-\alpha}$.

Proof Sketch.

1. Baldwin-Kueker: The Ramsey quantifier is eliminable from T in the $\aleph_0$ interpretation if T is either $\aleph_0$-categorical or does not have the finite cover property.

2. Baldwin-Shelah: The almost sure theory T^α does not have the finite cover property.

EXPLORING AND EXPLOITING T^α

Theorem. (McArthur and Spencer) Neither $L^\omega_{\infty,\omega}$ nor Fixed Point logic is almost equivalent to first order on T^α.

Conjecture. (Lynch) Transitive Closure Logic is almost equivalent to first order on T^α.

Rosen, Shelah, and Weinstein have exploited this theory to investigate the class of k-universal graphs - those which satisfy all existential sentences of $L^k_{\infty,\omega}$.

Clearly T^α has the finite model property.

QUESTION: For which k, does the the collection of first order sentences in k-variables which are true in T^α have a finite model?

DEFINABILITY TRIVIA FOR SECOND ORDER LOGIC

Let $L = L_0 \cup \{R\}$. Let T be a complete L_0-theory. Fix an L-sentence $\theta(R)$.

$\theta(R)$ is *totally generic* for T if for every model $\langle M, S\rangle$ of $T \cup \{\theta(R)\}$, and for every permutation α of M,$\langle M, \alpha S\rangle \models \theta(R)$,

A Definability Theorem. If θ of R is invariant for T, there is a sentence $\chi(R)$ involving only the symbol R) such that for every $M \models T$, $\{S : M \models \theta(S)\} = \{S : M \models \chi(S)\}$.

Proof. Easy, by compactness and recursive saturation.

We say the query $\theta(R)$ *collapses* to equality.

DEFINABILITY THEOREMS FOR WEAK SECOND ORDER LOGIC

$\theta(R)$ is *generic* for T if for every model $\langle M, S\rangle$ of $T \cup \{\theta(R)\}$, (with R *finite*) and for every permutation α of M,$\langle M, \alpha S\rangle \models \theta(R)$,

We say order generic if the permutation must preserve an underlying order.

Benedikt et al showed: Order generic queries over o-minimal structures collapse to order.

Belegradek et al showed: Order generic queries over quasi o-minimal structures collapse to order.

Benedikt and I are working on:

Generic queries over stable structures collapse to equality.

STABILITY CONJECTURES

The proofs lead to the following conjectures.
If M is a saturated model of a stable theory and I is a set of indiscernibles in M then

1) (probably proved) Every permutation of I extends to an automorphism of the universe.
2) (M, I) is a stable structure.

INVARIANTS OF FORMULAS

i) quantifier complexity (π_2)

ii) quantifier rank

iii) number of variables

iv) multiplicity of solutions

REFERENCES

k-variable logic: Poizat JSL-82, Thomas JSL 86,

Dawar,Weinstein, Grohe -see home page via FMT homepage

http://www.informatik.rwth-aachen.de/WWW-math/FMT.html.

I have notes, links and papers at

http://www.math.uic.edu/ jbaldwin