Metamathematics Fall 2001
Problem Set 1:

The first problem will be discussed on Wednesday Aug. 22. The other problems will be reviewed in class on Monday Aug. 27. Tentatively, 1-4 are due September 3.

A. How many different answers are there to the question $720 \div 120 \div 20 \div 6$? What are they? More generally suppose a structure has a single binary operation $\#$? How many different values could the expression $a_1 \# a_2 \# \ldots \# a_n$? take?

1. A truth function is a map from the set of truth values $\{T, F\}$ to itself. The truth function H is definable from $H_1, \ldots H_n$ if $H(a_1, \ldots a_n) = \ldots$ where the r.h.s is built up from $H_1, \ldots H_n$, $a_1, \ldots a_n$, commas and parentheses in the natural way.

a) Let $H_{D,n}$ be the truth function defined by $H_{D,n}(a_1, \ldots a_n) = T$ iff some $a_i = T$ and let $H_{C,n}(a_1, \ldots a_n) = T$ iff every $a_i = T$. Let $H_\neg(a_1) = T$ if $a_1 = F$ and vice versa. Show that every truth function is definable in terms of $H_\neg$ and certain of the $H_{C,n}, H_{D,n}$.

More precisely,

b) Show every truth function is definable from $H_\neg$ and $H_\wedge = H_{C,2}$.

c) What are $H_\rightarrow$ and $H_\leftrightarrow$?

d) Show not every truth function is defined from $H_\wedge, H_\vee, H_\leftrightarrow$.

e) The Scheffer stroke is the truth function H_S defined by $H_S(a, b) = T$ iff $a = b = T$. Show every truth function is definable from the Scheffer stroke.

2. a) Suppose $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas and ψ is a Boolean combination of $\phi_1, \ldots, \phi_n$. Then there is $S \subseteq \mathcal{P}(\{1, \ldots, n\})$ such that

$$\models \psi \Leftrightarrow \bigvee_{X \in S} (\bigwedge_{i \in X} \phi_i \wedge \bigwedge_{i \notin X} \neg \phi_i).$$

b) Show that every formula is equivalent to one of the form $Q_1 v_1 \ldots Q_m v_m \, \psi$ where ψ is quantifier free and each Q_i is either $\forall$ or $\exists$.

3. a) Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. Show that there is a sentence ϕ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

b) Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence ϕ such that $\mathcal{N} \models \phi$ if and only if $\mathcal{N} \cong \mathcal{M}$.