Math 512: Finite Model Theory
Fixed Point logics

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References: E and F: 6., 7; Vaananen chapter 3.

Moschovakis, Elementary Induction on Abstract Structures, North Holland; Barwise and Feferman Model Theoretic Logics, Springer, 1985.

0.1 Definition. Let K be a class of L-structures. An n-ary global relation on K is a function Γ such that for each A, $\Gamma(A)$ is an n-ary relation on A, which is preserved by isomorphism.

Examples: any logical formula; if R is a binary relations symbol, the transitive closure of R, connectivity is a 0-ary global relation on graphs.

Let F be a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$. F is called *inflationary* if $X \subseteq F(X)$. F is called *monotone* if $X \subseteq Y$ implies $F(X) \subseteq F(Y)$. $F^0(X) = \emptyset, F^{i+1}(X) = F(F^i(X))$. For any F, $\hat{F}(X) = X \cup F(X)$ is inflationary.

If F is inflationary or monotone (since we begin with $\emptyset$), $F^i(X) \subseteq F^{i+1}(X)$. Thus, if X is finite, there is an i such that $F_\infty = F^{i+1}(X) = F^i(X)$. $F^i(X)$ is the *least fixed point* of F.

There are several choices for exactly how to form a logic incorporating inductive definitions. We deal first with inflationary and the positive inductive definitions.

For any formula $\phi(\overline{x}, S)$, the operator $F^{\phi(\overline{x},S)\vee S\overline{x}}(A) = \{\overline{a} : A \models \phi(\overline{a}, S) \vee S\overline{a}$ is inflationary. The least fixed point $F_\infty^{\phi(\overline{x},S)\vee S\overline{x}}$ is called the *inflationary fixed point* of ϕ.

0.2 Definition. $F_\infty^{\phi(S)}$ is a global relation. We define a new formula by $[IFP_{\overline{x},X}\phi]\overline{t} = \psi(\overline{t})$ with semantics $A \models \psi(\overline{a})$ iff $\overline{a} \in F_\infty^{X\overline{x}\vee\phi}$. We form *Inflationary Fixed Point logic* IFP by closing under first order operations.

A symbol S occurs positively in a formula ϕ if it is in the scope of an even number of negation symbols (Boolean connectives are $\wedge, \vee, \neg$.). If S occurs positively in ϕ, the operator $F^{\phi(S)}(A) = \{\overline{a} : A \models \phi(\overline{a}, S)$ is monotone. $F_\infty^{\phi(S)}$ is the least fixed point of this operator.

E.g. transitive closure, subgroup generated by (an operator can have more than one fixed point), 'even' or 'odd' elements of a successor chain.

To have a decidable syntax, we add this quantifier only when S occurs positively in ϕ. Gurevich showed it is undecidable to just ask for $F^{\phi(S)}$ monotone.

0.3 Definition. *Least Fixed Point Logic* LPF contains exactly those sentences of IFP in which all instances of the fixed point operator on $(\overline{x}, X)$ have X occuring positively. Thus, $F^{\phi(X)}$ is monotone, so we define a new formula by $[LFP_{\overline{x},X}\phi]\overline{t} = \psi(\overline{t})$ with semantics $A \models \psi(\overline{a})$ iff $\overline{a} \in F_\infty^{X\overline{x}}$.

We don't need $\vee X\overline{x}$ in the superscript because X occurs positively.

0.4 Theorem. *(Gurevich and Shelah) LFP-logic = IFP logic.*

After proving the result we will follow Vaananen and write FP for this logic. But for the proof we need a number of new ideas and lemmas.

0.5 Digression. Comparing logics. What does equality mean in this lemma? For that matter, what is a logic?

0.6 Definition. A *logic* is a pair $\mathcal{L}, \models_{\mathcal{L}}$ such that for each vocabulary (similarity type, language) τ, $\mathcal{L}(\tau)$ is a collection of sentences ϕ such that for τ-structure A either $A \models_{\mathcal{L}} \phi$ or not. This pair should satisfy certain regularity conditions with we don't enumerate here. See E and B article in Barwise et al.

0.7 Definition. $\mathcal{L}_\infty \leq \mathcal{L}_\in$ if for every vocabulary τ and every $\mathcal{L}_\infty(\tau)$-sentence ϕ, there is an $\mathcal{L}_\infty(\tau)$-sentence ψ such that for all A, $A \models_{\mathcal{L}_\infty} \phi$ iff $A \models_{\mathcal{L}_\in} \psi$. Equivalence of logics is derived in the obvious way.

0.8 Example: Simultaneous Induction. Let A, E be a graph. Let

$$\phi_1(x, y, S, T) = Exy \vee (\exists z)(Sxz \wedge Ezy); \phi_2(x, y, S, T) = (\exists z)(\exists w)(Txz \wedge Txw \wedge Ezw).$$

The fixed point of the first formula is the transitive closure of E; of the second is the collection of all elements that are in cycles. (Assignment: explain 7.1.16 of E and F and last paragraph on page 186.)

0.9 Definition. (Simultaneous Fixed Points)

1. if $S_0 = T_0 = \emptyset$, $S_{i+1} = \{\overline{a} : \phi(\overline{a}, S^i, T^i), T_{i+1} = \{\overline{b} : \phi(\overline{a}, S^i, T^i)$, there is a j with $S_{j+1} = S_j$ and $T_{j+1} = T_j$, (S_j, T_j) is the *simultaneous fixed point.*
2. A (rather awkward) syntax for a simultaneous fixed point logic is described at E and F pg. 173.

Remark. If S_∞ and T_∞ are fixed points of $\phi(\overline{x}, S)$, $\psi(\overline{x}, T)$ then $S_\infty \cap T_\infty$ is a simultaneous fixed point of the system $[LFP_{\overline{x}S}\phi(\overline{x}, S)], [LFP_{\overline{x}T}\phi(\overline{x}, T)], [LFP_{\overline{x}R}S(\overline{x} \wedge T(\overline{x})]$. (Note that R does not occur in the third formula.) This is a step toward showing that every formula in FP is equivalent to one involving only one application of the fixed point operator.

As shorthand Vaananen writes $\psi(\overline{x}, S, R, T) \leftrightarrow [LFP_{\overline{x}R}S(\overline{x} \wedge T(\overline{x})]$.

0.10 Lemma. *Let $\phi_1(\overline{x}, S, T)$ and $\phi_2(\overline{y}, S, T)$ both be positive in the k-ary predicates S and T. Then there is a formula $\theta(t, \overline{x}, \overline{y}, U)$, positive in the $2k+1$-ary predicate U, such that the simultaneous fixed points of S and T are sections of the fixed point of U.*

Proof. Fix $c_0, c_1, \overline{a}, \overline{b}$ (of length k) in A. ($\overline{a}, \overline{b}$ could be sequences of c_i). Let $\theta(t, \overline{x}, \overline{y}, U)$ be

$$[t = c_0 \wedge \phi_1(\overline{x}, \{\overline{y}' : U(c_0, \overline{y}', \overline{a})\}, \{\overline{x}' : U(c_1, \overline{b}, \overline{x})\}] \vee [t = c_1 \wedge \phi_2(\overline{y}, \{\overline{y}' : U(c_0, \overline{y}', \overline{a}\}, \{\overline{x}' : U(c_1, \overline{b}, \overline{x})\}].$$

Now,

$$[LFP_{\overline{x},S}\phi_1]\langle v_1, \ldots v_k\rangle = [LFP_{w_\circ \overline{xy},U}\theta]c_0\overline{va}$$

$$[LFP_{\overline{y},T}\phi_2]\langle v_{k+1}, \ldots v_{2k}\rangle = [LFP_{w_\circ \overline{xy},U}\theta]c_1\overline{b}\langle v_{k+1}, \ldots v_{2k}\rangle.$$

0.11 Definition. Suppose $\phi(\overline{x}, S)$ is positive in S. On a structure A we define:

1. $|\phi| =$ least i such that $S^{i+1} = S^i$.
2. $$|a| = \begin{cases} \text{least } i \text{ such that } \overline{a} \in S^i, & \text{if } \overline{a} \in S^\infty \\ |\phi| + 1 & \text{otherwise} \end{cases}$$

The result is due to Moschovakis; this proof is by Leivant. In fact, the argument is part of the more general argument of Gurevich-Shelah for IFP=LFP.

Lemma 0.12 (Stage Comparison Theorem). *Suppose $\phi(\overline{x}, S)$ is positive in S. The global relations:*

$$\overline{x} \prec_\phi \overline{y} \leftrightarrow |\overline{x}| < |\overline{y}|$$

$$\overline{x} \leq_\phi \overline{y} \leftrightarrow |\overline{x}| \leq |\overline{y}| \wedge \overline{x} \in S^\infty$$

and their complements $\not\prec_\phi$ and $\not\leq_\phi$ are in FP.

Proof. Define also

$$\overline{x} \prec_1^{\phi} \overline{y} \leftrightarrow |\overline{x}| + 1 = |\overline{y}|$$

To save clutter we drop the bars from $\overline{x}, \overline{y}$; we write $\cdot \prec y$ for $\{u : u \prec y\}$. We simultaneously induct on the following 5 formulas (see pg 25 of Vaananen for interpretation):

1. $[LFP_{xy,\prec}(\exists z)(x \leq z \prec_1 y)]$
2. $[LFP_{xy,\leq}\phi(x, \cdot \prec y)]$
3. $[LFP_{xy,\prec_1}$
   $\{\phi(x, \cdot \prec x) \wedge \neg\phi(y, \neg\cdot \not\prec x) \wedge [(\phi(y, \cdot \leq x) \vee (\forall z)(\phi(z, \neg\cdot \not\leq x) \rightarrow \phi(z, \cdot \prec x))]\}]$
4. $[LFP_{xy,\not\prec}(\exists z)(x \not\leq z \prec_1 y \vee \phi(y, \emptyset) \vee (\forall z)\phi(z, \emptyset)]$
5. $[LFP_{xy,\not\leq}\neg\phi(x, \neg\cdot \not\prec y)]$

We need to make two observations: the above formulas are positive in the new predicates so a fixed point exists; the simultaneous fixed point $(\prec, \leq, \prec_1, \not\prec, \not\leq)$ of this induction is $(\prec_\phi, \leq_\phi, \prec_1^\phi, \not\prec_\phi, \not\leq_\phi)$.

0.13 Proof of Theorem 0.4. Clearly $LFP \leq IFP$. For the converse, it suffices to show inductively that for any LFP-formula $\phi(\overline{x}, X)$ (where X may occur negatively, $[IFP_{\overline{x},X}\phi]\overline{t}$ is equivalent to a formula in LFP.

First rewrite $\phi(\overline{x}, X)$ as $\psi(\overline{x}, Y, X)$ which is positive in both X and Y, by replacing each occurence of X with $\neg Y$. So $\phi(x, X) \leftrightarrow \psi(\overline{x}, X^c, X)$ defines the inductive step. Note that we can define $\leq_\phi$ just as for a monotone formula. Moreover:

$$\models_{fin} [IFP_{\overline{x},X}\phi] \leftrightarrow \overline{x} \leq_\phi \overline{x}$$

and

$$\{u : u \leq_\phi x\}^c = \{u : x \prec_\phi u\}.$$

So it suffices to show (by a variant of the stage comparison theorem) that $\leq^\phi$ is definable as a LFP. We simultaneously induct on the following 2 formulas.

$$[LFP_{xy,\leq}(\phi(x, \emptyset) \wedge \phi(y, \emptyset)) \vee (\exists z)z \prec y \wedge \psi(x, \cdot \leq z, z \prec \cdot) \wedge \psi(y, \cdot \leq z, z \prec \cdot)]$$

$$[LFP_{xy,\prec}((\phi(x, \emptyset) \wedge \neg(\phi(y, \emptyset)) \vee (\exists z)(z \leq z \wedge \psi(x, \cdot \leq z, z \prec \cdot) \wedge \neg\psi(y, \neg z \prec \cdot, \neg\cdot \leq z)].$$

These formulas are positive in $\leq, \prec$ so there is a fixed point. Let $\leq^n_\phi$, $\prec^n_\phi$ denote the nth iterate towards this fixed point. We prove by simultaneous induction on n:

$$a \leq^n_b \text{ iff } a, b \in F^n_\phi \text{ and } a \leq^n_\phi b$$

$$a \prec^n_b \text{ iff } a, b \in F^n_\phi \text{ and } a \prec^n_\phi b$$

Here is the induction step for the first equivalence:
$a \leq^{n+1} b$ iff

$$(\phi(a, \emptyset) \wedge \phi(b, \emptyset)) \vee (\exists z)(z \prec^n b \wedge \psi(a, \cdot \leq^n z, z \prec^n \cdot) \wedge \psi(b, \cdot \leq^n z, z \prec^n \cdot)).$$

By induction this is equivalent to

$$(\phi(a, \emptyset) \wedge \phi(b, \emptyset)) \vee (\exists z)(z \in F^n_\phi \wedge \psi(a, \cdot \leq^n_\phi z, z \prec^n_\phi \cdot) \wedge \psi(b, \cdot \leq^n_\phi z, z \prec^n_\phi \cdot)).$$

By the choice of ϕ, ψ this is equivalent to:

$$(\phi(a, \emptyset) \wedge \phi(b, \emptyset)) \vee (\exists z)(z \prec^n_\phi y \wedge a, b \in F^\phi(F^\phi_n)).$$

Which is then equivalent to: $a \leq^n_\phi b \wedge a, b \in F^\phi_{n+1}$.

This shows that the stage comparison relations of H can be given by a positive inductive definition (i.e. are in LPF).

0.14 Corollary. *(Immerman) Every formula of LFP can be given by a single positive induction.*

Proof. (idea) We checked 'and' earlier. For the existential quantifier and iterating LPF see 7.2.4 of E and F. This is the transitivity lemma. We need to show that $\neg[LFP_{\overline{x},S}\phi]\overline{t}$. The key is that $\neg[LFP_{\overline{x},S}\phi]\overline{t}$ is equivalent to

$$(\forall \overline{x})\neg\phi(\overline{x}, \emptyset) \vee ((\exists \overline{x})\phi(\overline{x}, \emptyset) \wedge ((\exists \overline{z})\phi_{|\phi|}(\overline{z}) \wedge \overline{z} \prec_\phi \overline{t}))$$

where $\phi_{|\phi|}(\overline{z})$ means $(\forall \overline{v})(\overline{v} \leq_\phi \overline{z} \vee (\neg\psi(\overline{v}, \neg\overline{z} \prec \cdot, \neg\cdot \leq \overline{z}))$. (No element comes in at the stage after $\overline{z}$.) Note that the number $|\phi|$ is depends on the structure A.

Coming attraction:

0.15 Lemma. $FO \subseteq FP \subseteq PTIME$.