A MODEL IN $\aleph_2$

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MAIN RESULT

Theorem If a sentence of $L_{\omega_1,\omega}(Q)$ is categorical in $\aleph_1$ then it has a model of cardinality $\aleph_2$.

This result was originally proved by Shelah in Shelah 48; the proof expounded here is from Shelah 88 taking into account some later emendations by Shelah.

ABSTRACT ELEMENTARY CLASSES

Definition 1 *A class of L-structures, $(\mathbf{K}, \preceq_{\mathbf{K}})$, is said to be an* abstract elementary class: AEC *if both $\mathbf{K}$ and the binary relation $\preceq_{\mathbf{K}}$ are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If $M \preceq_{\mathbf{K}} N$ then $M \subseteq N$.*
- **A2**. *$\preceq_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.*
- **A3**. *If $\langle A_i : i < \delta \rangle$ is $\preceq_{\mathbf{K}}$-increasing chain:*
  1. *$\cup_{i<\delta} A_i \in \mathbf{K}$;*
  2. *for each $j < \delta$, $A_j \preceq_{\mathbf{K}} \cup_{i<\delta} A_i$*
  3. *if each $A_i \preceq_{\mathbf{K}} M \in \mathbf{K}$ then $\cup_{i<\delta} A_i \preceq_{\mathbf{K}} M$.*
- **A4**. *If $A, B, C \in \mathbf{K}$, $A \preceq_{\mathbf{K}} C$, $B \preceq_{\mathbf{K}} C$ and $A \subseteq B$ then $A \preceq_{\mathbf{K}} B$.*
- **A5**. *There is a Löwenheim-Skolem number $\kappa(\mathbf{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \mathbf{K}$ with $A \subseteq A' \preceq_{\mathbf{K}} B$ and $|A'| < \kappa(\mathbf{K})$.*

OVERVIEW

The general setting here will be an AEC. We show first that if an AEC is categorical in λ and λ^+ and has no 'maximal triple' in power λ then it has a model in power λ^{++}. Then we show in $L_{\omega_1,\omega}(Q)$ there are no maximal triples in $\aleph_0$.

Model Theoretic Methods

Since we are proving things for all classes of models (in any vocabulary) satisfying certain condition, we are able to repeatedly say WLOG and assume reduce the problem to classes which have a particularly nice presentation.

COMPLETENESS

Definition 2 *A sentence ψ in $L_{\omega_1,\omega}(Q)$ is called* complete *if for every sentence ϕ in $L_{\omega_1,\omega}(Q)$, either $\psi \models \phi$ or $\psi \models \neg\phi$.*

Categoricity implies completeness is no longer trivial.

Fact 3 *ψ is complete implies ψ is small. That is, each model of ψ realizes only countably many $L_{\omega_1,\omega}(Q)$-types.*

SYNTAX AND SEMANTICS I

EASY FACT

Definition 4 *A* $PC(T,\Gamma)$ class *is the class of reducts to* $\tau \subset \tau'$ *of models of a first order theory* τ'*-theory which omit all types from the specified collection* Γ *of types in finitely many variables over the empty set.*

We write $PC\Gamma$ *to denote such a class without specifying either* T *or* Γ*.*

We write $\mathbf{K}$ *is* $PC(\lambda,\mu)$ *if* $\mathbf{K}$ *can be presented as* $PC(T,\Gamma)$ *with* $|T| \leq \lambda$ *and* $|\Gamma| \leq \mu$*.*

(Chang's trick)

Lemma 5 *Every* $L_{\omega_1,\omega}(Q)$*-sentence in a countable language is* ω*-presented. That is, the class of models of* ψ *is a* $PC(\aleph_0,\aleph_0)$*-class.*

SYNTAX AND SEMANTICS II

Theorem 6 *[Presentation Theorem] If* $\mathbf{K}$ *is an AEC with Lowenheim number* $\mathrm{LS}(\mathbf{K}) = \lambda$ *(in a vocabulary* τ *with* $|\tau| \le \mathrm{LS}(\mathbf{K})$*), then* $\mathbf{K}$ *is a* $PC(\lambda, 2^\lambda)$*-class*

Closely related to Chang's trick is:

Lemma 7 *Every* complete *(i.e. small)* $L_{\omega_1,\omega}(Q)$*-sentence in a countable language can be presented as the atomic models of a first order theory.*

And there is a harder result (using Lopez-Escobar for $L_{\omega_1,\omega}(Q)$):

Theorem 8 *If* ψ *is an* $\aleph_1$*-categorical sentence in* $L_{\omega_1,\omega}(Q)$ *then* ψ *is implied by a complete sentence that has a model of cardinality* $\aleph_1$*.*

From $L_{\omega_1,\omega}(Q)$ to AEC

Definition 9 *Let ψ be a small $L_{\omega_1,\omega}(Q)$-complete sentence with vocabulary τ in the countable fragment L^* of $L_{\omega_1,\omega}(Q)$. Form τ' by adding predicates for infinitary formulas and also add for each formula $(Qx)\phi(x,\overline{y})$ a predicate $R_{(Qx)\phi(x,\overline{y})}$ and add the axiom*

$$(\forall x)[(Qx)\phi(x,\overline{y}) \leftrightarrow R_{(Qx)\phi(x,\overline{y})}].$$

Let ψ' be the conjunction of the $L_{\omega_1,\omega}(Q)$-τ'-axioms encoding this expansion. Let K_1 be the class of atomic models of $T(\psi)$, the first order τ' theory containing all first order consequences of ψ'.

TWO APPROXIMATIONS

Notation 10 *1. Let $\leq^*$ be the relation on $\mathbf{K}_1$: $M \leq^* N$ if $M \prec_{\tau'} N$ and for each formula $\phi(x, \overline{y})$ and $\overline{m} \in M$, if $M \models \neg R_{(Qx)\phi(x,\overline{m})}$ then $R_{\phi(x,\overline{m})}$ has the same solutions in M and N.*

2. Let $\leq^{**}$ be the relation on $\mathbf{K}_1$: $M \leq^* N$ if $M \prec_{L'} N$ and for each formula $\phi(x, \overline{y})$ and $\overline{m} \in M$, if $M \models \neg R_{(Qx)\phi(x,\overline{m})}$ if and only if $R_{\phi(x,\overline{m})}$ has the same solutions in M and N.

ALTERNATIVE NOTION

It is easy to check that $(\mathbf{K}_1, \leq^*)$ is an AEC, but $(\mathbf{K}_1, \leq^{**})$ is not an AEC. It can easily happen that each of a family of models $M_i \leq^{**} M$ but $\bigcup_i M_i \not\leq^{**} M$.

An AEC* class is one which satisfies the axioms of an AEC, but A3.3 is replaced by:
if each $A_i \preceq_{\mathbf{K}} M_0 \preceq_{\mathbf{K}} M \in \mathbf{K}$ and M_0 is strictly contained in M then $\bigcup_{i<\delta} A_i \preceq_{\mathbf{K}} M$.

This condition is satisfied by $(\mathbf{K}_1, \leq^{**})$ and suffices for the argument in this talk.

GOING UP

Definition 11 *We say* (M, N) *is a* proper pair *in* λ*, witnessed by* a*, if we mean* $M \preceq_{\mathbf{K}} N$ *and* $a \in N - M$ *and* $|M| = |N| = \lambda$*.*

The fixed a is not used in the next Lemma but plays a central role in the proof of Lemma 14.

Lemma 12 *If an AEC* $\mathbf{K}$ *is categorical in* λ *and has a proper pair* (M, N) *in* λ *then there is a model in* $\mathbf{K}$ *with cardinality* λ^+*.*

Proof. Let $M_0 = M$. For any α, given M_α, choose $M_{\alpha+1}$ so that $(M, N) \approx (M_\alpha, M_{\alpha+1})$ and take unions at limits. The union of M_α for $\alpha < \lambda^+$ is as required. $\square_{S12}$

MAXIMAL TRIPLES

Definition 13 *A* maximal triple *is a triple* (M, a, N) *such that* a *witnesses that* (M, N) *is a proper pair and if* (M', N') *satisfies* $M \preceq_{\mathbf{K}} M'$*,* $M \neq M'$*,* $N \preceq_{\mathbf{K}} N'$ *and* $M' \preceq_{\mathbf{K}} N'$ *then* $a \in M'$*.*

Lemma 14 *If there are no maximal triples of cardinality* λ *and there is a proper pair of cardinality* λ *then there is a proper pair of cardinality* λ^+*.*

Proof. Let a witness that (M_0, N_0) is a proper pair in λ. Since there are no maximal triples, we can construct proper pairs (M_i, N_i) such that M_{i+1} is a proper $\preceq_{\mathbf{K}}$ extension of M_i and N_{i+1} is a $\preceq_{\mathbf{K}}$ extension of N_i but no M_i contains a; that is, the properness of each (M_i, N_i) is witnessed by the same a. So $(\bigcup_{i<\lambda^+} M_i, \bigcup_{i<\lambda^+} N_i)$ is the required proper pair. $\Box_{14}$

PROOF SKETCH

We have shown that if there are no maximal triples in λ and $\mathbf{K}$ is λ^+-categorical then there is a model in λ^{++}. We will show there are no maximal triples in $\aleph_0$ if $\mathbf{K}$ is $\aleph_1$-categorical and has few models in $\aleph_1$. For this, we need another definition.

Definition 15 *$M \preceq_{\mathbf{K}} N$ is a* cut-pair *if there exist models N_i for $i < \omega$ such that $M \preceq_{\mathbf{K}} N_{i+1} \preceq_{\mathbf{K}} N_i \preceq_{\mathbf{K}} N$ and $\cap_{i<\omega} N_i = M$.*

Let $(\mathbf{K}, \preceq_{\mathbf{K}})$ be the collection of dense linear orders with elementary submodel and let $(Q, <)$ be the rational order. Then $((-\infty, \sqrt{2}), (\sqrt{2}, \infty))$ is a cut-pair.

Need an example of a maximal triple

CUT PAIR + MAXIMAL TRIPLE IMPLIES MANY MODELS

Lemma 16 *Suppose* $\mathbf{K}$ *is* λ*-categorical. If* $\mathbf{K}$ *has a cut-pair in cardinality* λ *and it has a maximal triple in* λ*, then* $I(\lambda^+, \mathbf{K}) = 2^{\lambda^+}$*. Moreover, each of these models is a union of a* $\preceq_{\mathbf{K}}$*-increasing chain of length* λ^+*.*

Proof. Let (M, N) be a cut-pair. For S a stationary subset of λ^+, define M_i^S for $i < \lambda^+$ so that

(M_i, M_{i+1}) is isomorphic to (M, N) if i is 0 or a successor ordinal.

But if i is a limit ordinal.

Let (M_i, M_{i+1}) be a cut-pair if $i \notin S$;

and for some a, let (M_i, a, M_{i+1}) be a maximal triple if $i \in S$.

Then, let $M^S = \cup_{i<\lambda^+} M_i^S$.

Now, if $S_1 - S_2$ is stationary, $M^{S_1} \not\approx M^{S_2}$.

If f is an isomorphism between them, we find a contradiction by intersecting $S_1 - S_2$ with the cub E consisting of those $\delta < \lambda^+$ such that $M_\delta^{S_1}$ and $M_\delta^{S_2}$ both have domain δ and $i < \delta$ implies $f(i) < \delta$.

If δ is in the intersection, as $\delta \in S_1$, $a_\delta^{S_1} \in M_{\delta+1}^{S_1} - M_\delta^{S_1}$; $f(a_\delta^{S_1}) \in M^{S_2} - M_\delta^{S_2}$.

But, $M_\delta^{S_2} = \cap_{n<\omega} M_\delta^{S_2,n}$ for appropriate $M_\delta^{S_2,n}$, since $(M_\delta^{S_2}, M_{\delta+1}^{S_2})$ is a cut pair. So $f(a_\delta^{S_1}) \notin M_\delta^{S_2,n}$ for some n. Let N denote $f^{-1}(M_\delta^{S_2,n})$. Then for some $\gamma \in E \cap (S_1 - S_2)$, $N \preceq_{\mathbf{K}} M_\gamma^{S_1}$. But then $(N, a_\delta^{S_1}, M_\gamma^{S_1})$ properly extends $(M_\delta^{S_1}, a_\delta^{S_1}, M_{\delta+1}^{S_1})$ and this contradiction yields the theorem. $\square_{16}$

CUT PAIRS EXIST

Lemma 17 *Suppose* $(\mathbf{K}, \preceq_{\mathbf{K}})$ *is an AEC* that is* $\aleph_0$*-categorical and is a* $PC\Gamma(\aleph_0, \aleph_0)$ *class. If* $\mathbf{K}$ *has a model of power* $\aleph_1$*, then there is a cut pair in* $\aleph_0$*.*

Proof. Recall that $\mathbf{K}$ is the class of τ-reducts of models of a first order theory T, which omit a countable set Γ of types. Let $M \in \mathbf{K}$ be a model with universe $\aleph_1$; write M as $\cup_{i<\aleph_1} M_i$. For simplicity, assume the universe of M_0 is $\aleph_0$. Expand M to a τ^*-structure M^* by adding the order $<$ on $\aleph_1$ and a binary function g such that $g(i, x)$ is a τ-isomorphism from M_0 to M_i. Note that a unary predicate P naming M_0 and a binary relation $R(x, y)$ such that $R(a, i)$ if and only $a \in M_i$ are easily definable from g. Moreover, for each i, $\{x : R(x, i)\}$ is closed under the functions of τ^*.

Let ψ be a sentence in $L_{\omega_1,\omega}(\tau^*)$ describing this situation; the existence of ψ follows since $\mathbf{K}$ is a $PC\Gamma(\aleph_0, \aleph_0)$ class. By Lopez-Escobar, there is a model N^* of ψ with cardinality $\aleph_0$ in which $<$ is not well-founded. For any $b \in N^*$, let

$$N_b = \{x \in N^* : R(x, b)\}.$$

Let a_i for $i < \omega$ be a properly descending chain. Then if $N_i = N_{a_i}$, which has universe $\{x \in N^* : R(x, a_i)\}$,

$$N_i \restriction \tau \preceq_{\mathbf{K}} N^* \restriction \tau$$

and because of g, each N_i is τ-isomorphic to $P(N^*)$. Let I be the set of $b \in N^*$ such that for every i, $b < a_i$.

For any $b \in I$ and any $i < \omega$ we have $N_b \preceq_{\mathbf{K}} N_{i+1} \preceq_{\mathbf{K}} N_i$ so by the revised axiom A3.3, $N_I \preceq_{\mathbf{K}} N_i$, which is exactly what we need.

Our required cut-pair is (N_I, N_0). $\square_{17}$

CONCLUSION FOR AEC

Theorem 18 *If* $\mathbf{K}$ *is a* $\aleph_0$*-categorical* $PC\Gamma(\aleph_0, \aleph_0)$ *class that is also an AEC* and has a unique model of power* $\aleph_1$*, then there is a model of power* $\aleph_2$*.*

Proof. By Lemma 17, there is a cut-pair in $\aleph_0$. Since ψ is $\aleph_1$-categorical, Lemma 16 implies there is no maximal triple in $\aleph_0$. So by Lemma 14 there is a proper pair in $\aleph_1$ and then by Lemma 12, there is a model of power $\aleph_2$. $\square_{18}$

Corollary 19 *An* $\aleph_1$*-categorical sentence* ψ *in* $L_{\omega_1,\omega}(Q)$ *has a model of power* $\aleph_2$*.*

$L_{\omega_1,\omega}(Q)$

What are the difficulties for $L_{\omega_1,\omega}(Q)$?

We don't have A3.3?. The weaker version suffices to find a cut pair, Lemma 17.

But, the application of Lemma 16, no maximal triples, is subtle. It could very well happen that a sentence in $L_{\omega_1,\omega}(Q)$ has few models in $\aleph_1$, but there are many models of the associated class $\mathbf{K}_1$ of 'weak models'. But Lemma 16, yielded $2^{\aleph_1}$ models which were unions of $\aleph_1$ chains. Therefore, they are standard models of ψ and we finish.

Finally, how do we translate to a $PC(\aleph_0, \aleph_0)$ class?

CONTEXT

Shelah's original proof:

1. Assumed $\diamond$,
2. Used much more stability theoretic machinery.

Morley's theorem for $L_{\omega_1,\omega}$:

1. Assume weak gch - exp is increasing.
2. Assume categoricity up to $\aleph_\omega$ (essential)
3. Use much stability theoretic machinery.

Morley's theorem for $L_{\omega_1,\omega}(Q)$:
Still open