# Math 512: Finite Model Theory
# Monadic $\Sigma^1_1$ and Hanf's theorem

John Baldwin

February 7, 1997

Remark PTIME is closed under complement; it is open whether $NP$ is closed under complement – thus whether $\Sigma^1_1$ is closed under complement. We will show monadic $\Sigma^1_1$ is not.

**0.1 Definition.** Let $K$ be a class of $L$-structures. An $n$-ary global relation on $K$ is a function $\Gamma$ such that for each $A$, $\Gamma(A)$ is an $n$-ary relation on $A$, which is preserved by isomorphism.

Examples: any logical formula; if $R$ is a binary relations symbol, the transitive closure of $R$, connectivity is a 0-ary global relation on graphs.

**0.2 Theorem.** *Connectivity is not monadic-$\Sigma^1_1$.*

We will introduce considerable terminology to prove this result. Refer to Sections 1.3 and 1.4 of E and F.

**0.3 Lemma.** *$A \equiv_m B$ if and only if there exists an $m$-back and forth between them. That is a sequence, $\langle I_j; j \leq m\rangle$, such that*

1. *Every $I_j$ is a nonempty set of partial isomorphisms between $A$ and $B$.*
2. *(forth) If $p \in I_{j+1}$ and $a \in A - \operatorname{dom} p$, there is a $q \in I_j$ with $a \in \operatorname{dom} q$ and $q \supseteq p$.*
3. *(back) If $p \in I_{j+1}$ and $b \in B - \operatorname{rng} p$, there is a $q \in I_j$ with $b \in \operatorname{rng} q$ and $q \supseteq p$.*

In general, if $p \in I_j$, $|\operatorname{dom} p| = m - j$.

Application: In the language with binary relation. $C_r \equiv_m C_s$ if $s, r \geq 2^m$. Let $I_j = \{p \in Part(C_r, C_s) : d_j(a,b) = d_j(p(a), p(b))\}$ where $d_j(a,b)$ is the distance from $a$ to $b$ if it is less than $2^{j+1}$ or else infinity.

We can extend this idea to an arbitrary relational structure by $d(a,b) = 1$ if for some $\overline{a}$, $a, b \in \overline{a}$ and for some $R$, $A \models R(\overline{a})$ and take lengths of paths.

In this sense, $S(r,a) = \{b : d(a,b) < r\}$. An $r$-sphere type of an element $a$ is the isomorphism type of the $r$-sphere about $a$. $S(r,\overline{a}) = \cup_i S(r,a_i)$.

Trivial case: only unary predicates. The $r$-sphere type of $a$ is just which unary predicates hold and don't hold of it. More generally, unary predicates don't affect distance.

**0.4 Theorem.** *(Hanf) Let $A$ and $B$ be structures such that the $3^m$-spheres in $A, B$ have less than $e$ elements. Suppose also that for each $n < 3^m$ and $n$-sphere type $\mathbf{t}$:*

1. *$A$ and $B$ have the same number ($< e \cdot 3^m$) of elements of $n$-sphere type $\mathbf{t}$.*

2. *Both $A$ and $B$ have more than ($e \cdot 3^m$) elements of $n$-sphere type $\mathbf{t}$.*

Proof. Let

$$I_j = \{p \in Part(C_r, C_s) : S(3^j, \overline{a}) \approx S(3^j, p(\overline{a}))\}.$$

We must verify the back and forth; say forth. So suppose, $\langle \overline{a}, p(\overline{a})\rangle = p \in I_{j+1}$ and $a \in A - \operatorname{dom} p$. Let $\pi : S(3^{j+1}, \overline{a}) \approx S(3^{j+1}, p(\overline{a})$.

If $a \in S(2 \cdot 3^j, \overline{a})$, $S(3^j, \overline{a}a) \subseteq S(3^{j+1}, \overline{a})$ and we let $q(a) = \pi(a)$.

If $a \notin S(2 \cdot 3^j, \overline{a})$ then $S(3^j, \overline{a}) \cap S(3^j, a) = \emptyset$. Let $\mathbf{t}$ be the $3^j$-type of $a$. Clearly (since isomorphic under $\pi$), $S(2 \cdot 3^j, \overline{a})$ and $S(2 \cdot 3^j, p(\overline{a}))$ contain the same number of elements with $3^j$-type $\mathbf{t}$ and this number must be less than length of $\overline{a}$ times $e$. So there must be a $b \notin S(2 \cdot 3^j, p(\overline{a}))$ with $3^j$-type $\mathbf{t}$. Let $q(a) = b$.

Proof of 0.2. Suppose for contradiction that the $\Sigma^1_1$-formula $\phi$ with (first order) quantifier degee $m$ axiomatized connectivity. Say $\phi$ contains unary predicates $P_1 \dots P_r$. Note that if $s > 2 \cdot 3^m + 1$, then each $3^m$-sphere in $C_s$ has exactly $2 \cdot 3^m + 1$ elements. Let $i$ be number of isomorphism types of $3^m$ spheres in expansions of long cycles by $r$ unary predicates. Let $\ell_0 = (i+1)(2 \cdot 3^m + 1)$. If $\ell \geq \ell_0$, $(C_\ell, P_1 \dots P_r)$ contains disjoint and isomorphic $3^m$-spheres.

Then any such $(C_\ell, P_1 \dots P_r) \models \phi$. We will construct $(D_\ell, P_1 \dots P_r)$ which also satisfies $\phi$ but is not connected. Namely take $a, b \in C_\ell$, such that $S(3^m, a) \cap S(3^m, b) = \emptyset$. Suppose we have $E(a^-, a$ and $E(b^-, b)$ (with $a^-$ and $b^-$ in different intervals between $a$ and $b$. Form $D_\ell$ by replacing the edge from $b^-$ to $b$ with one from $b$ to $a^-$ and the edge from $a^-$ to $a$ with one from $a$ to $b^-$. Since $S(3^m, a) \approx S(3^m, b)$, $(C_\ell, P_1 \dots P_r)$ and $(D_\ell, P_1 \dots P_r)$ satisfy the hypotheses of Hanf's theorem and are $m$-equivalent in $(R, P_1 \dots P_n)$.

Easily, the class of non-connected graphs is monadic $\Sigma^1_1$. So,

**0.5 Theorem.** *Monadic $\Sigma^1_1$ is not closed under complement.*