Using Set theory in model theory I Helsinki, 2013

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Today's Topics

Overview

1. Overview of a method to use forcing to prove model theoretic results in ZFC
2. Constructing many models in $\aleph_1$
3. Constructing models in the continuum

References

New results are from papers by Baldwin/Larson
and
Baldwin/Larson/Shelah

and from Shelah F1098
and commentaries thereon - Baldwin-Koerwien-Laskowski
that are on my website.

Using Extensions of ZFC in Model Theory

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   - B The conclusion might be absolute

     The elementary equivalence proved in the Ax-Kochen-Ershov theorem

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     The elementary equivalence proved in the Ax-Kochen-Ershov theorem
   - C Consistency may imply truth.

Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

See also the Vaananen article in Model Theoretic Logic volume

Shoenfield Absoluteness Lemma

Theorem (Shoenfield)

If

1. $V \subset V'$ are models of ZF with the same ordinals and
2. ϕ is a lightface Π^1_2 predicate of a set of natural numbers

then for any $A \subset N$, $V \models \phi(A)$ iff $V' \models \phi(A)$.

Note that this trivially gives the same absoluteness results for Σ^1_2-predicates.

Which 'Central Notions'?

Chang's two cardinal theorem (morasses)

'Vaughtian pair is absolute'

saturation is not absolute

Aside: For aec, saturation is absolute below a categoricity cardinal.

'elementarily prime model' is absolute. (countable and atomic)

'algebraically prime model' is open

Study Theories not logics

The most fundamental of Shelah's innovation was to shift the focus from properties of logics (completeness, preservation theorem, compactness)

to

Classifying theories in a model theoretically fruitful way. (stable not decidable)

Classification Theory

Crucial Observation

The stability classification is absolute.

Fundamental Consequence

Crucial properties are provable in ZFC for certain classes of theories.

1. All stable theories have full two cardinal transfer.
2. There are saturated models exactly in the cardinals where the theory is stable.

But this is for FIRST ORDER theories.

Geography

$$L_{\omega,\omega} \subset L_{\omega_1,\omega} \subset \hat{L}_{\omega_1,\omega}(Q) \subset anal.pres.AEC \subset AEC.$$

In a central case explained below

Extensions of ZFC are used for $L_{\omega_1,\omega}$.

$\hat{L}_{\omega_1,\omega}(Q) \subset$ means only negative occurences of Q: e.g. Zilber field, counterexample to absoluteness

Extensions of ZFC are proved necessary for $L_{\omega_1,\omega}(Q)$.

Two notions of 'use'

1. Some model theoretic results 'use' extensions of ZFC
2. Some model theoretic results are provable in ZFC, using models of set theory.

This Talk

1. A quick statement of some results of the first kind
2. Discussion of several examples of the second method.

One Completely General Result

Theorem: ($2^\lambda < 2^{\lambda^+}$) (Shelah)

Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. For any Abstract Elementary class, if amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality λ^+.

Is $2^\lambda < 2^{\lambda^+}$ needed?

Is $2^\lambda < 2^{\lambda^+}$ needed?

1. $\lambda = \aleph_0$:
   a. Definitely not provable in ZFC: There are $L(Q)$-axiomatizable examples
      i. Shelah: many models with CH, $\aleph_1$-categorical under MA
      ii. Koerwien-Todorcevic: many models under MA, $\aleph_1$-categorical from PFA.
   b. Independence Open for $L_{\omega_1,\omega}$

A simple Problem

Let ϕ be a sentence of $L_{\omega_1,\omega}$.

Question

Is the property 'ϕ has an uncountable model' absolute?

False Start

Fact: Easy for complete sentences

If ϕ is a complete sentence in $L_{\omega_1,\omega}$,
ϕ has an uncountable model if and only if there exist countable $M \precneqq_{\omega_1,\omega} N$ which satisfy ϕ.

This property is Σ^1_1 and done by Shoenfield absoluteness.

Note: $L_{\omega_1,\omega}$ satisfies downward Löwenheim-Skolem for sentences but not for theories.

Fly in the ointment

There are uncountable models that have no $L_{\omega_1,\omega}$-elementary submodel.

E.g. any uncountable model of the first order theory of infinitely many independent unary predicates P_i.

So the sentence saying every finite Boolean combination of the P_i is non-empty has an uncountable model and our obvious criteria does not work.

Note that if we add the requirement that each type is realized at most once, then every model has cardinality $\leq 2^{\aleph_0}$.

Remember History

The really basic proof

Karp (1964) had proved completeness theorems for $L_{\omega_1,\omega}$, and Keisler (late 60's/ early 70's) for $L_{\omega_1,\omega}(Q)$, Barwise-Kaufmann-Makkai for L(aa) $L_{\omega_1,\omega}(\mathrm{aa})$.

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The rest of the talk illustrates a different argument with many applications.

A Correct Characterization

Larson's characterization

Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})(\tau)$,
the existence of a τ-structure of size $\aleph_1$ satisfying ϕ
is equivalent to
the existence of a countable model of ZFC° containing $\{\phi\} \cup \omega$ which thinks there is a τ-structure of size $\aleph_1$ satisfying ϕ.

This property is Σ^1_1 and done by Shoenfield absoluteness.

Method: 'Consistency implies Truth'

Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let $\mathcal{A}$ be the countable ω-model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct $\mathcal{B}$, an uncountable model of set theory, which is an elementary extension of $\mathcal{A}$
such that $\mathcal{B}$ is correct about uncountability. Then the model of ϕ in $\mathcal{B}$ is actually an uncountable model of ϕ.

How to build $\mathcal{B}$

- **MT** Iterate a theorem of Keisler and Morley (refined by Hutchinson).
- **ST** Iterations of ‘special’ ultrapowers.

ZFC° denotes a sufficient subtheory of ZFC for our purposes.

How to build $\mathcal{B}$

The main technical tool is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by NS_{ω_1}.

The ultrafilter

Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U).

The Ultrapower

Given such M and U, we can form the generic ultrapower $\mathrm{Ult}(M, U)$, which consists of all functions in M with domain ω_1^M,

where for any two such functions f, g, and any relation R in $\{=, \in\}$, fRg in $\mathrm{Ult}(M, U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$.

Nota Bene

If M is countable, $\mathrm{Ult}(M, U)$ is countable.

By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M, U)$ with its Mostowski collapse.

One step in the construction

We use three crucial properties of ZFC° .

1. The theory ZFC° holds in every structure of the form $H(\kappa)$ or V_κ, where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$
2. If P is a c.c.c. notion of forcing, $2^{|P|^{\aleph_1}|} < \theta$, satisfying natural technical conditions, and X is an elementary submodel of $H(\theta)$ with $P \in X$, then any forcing extension by P of the transitive collapse of X satisfies ZFC° .
3. Whenever M is a model of ZFC° and U is an M-ultrafilter on ω_1^M, M is elementarily embedded in $\mathrm{Ult}(M, U)$.
4. If U constructed as above, $\mathrm{Ult}(M, U)$ increases exactly the sets that M thinks are uncountable.

Iterations

Definition

Let M be a model of ZFC° and let γ be an ordinal less than or equal to ω_1.

An iteration of M of length γ consists of models

$$M_\alpha : \quad (\alpha \leq \gamma),$$

sets

$$G_\alpha : \quad (\alpha < \gamma),$$

and a commuting family of elementary embeddings

$$j_{\alpha\beta} : M_\alpha \to M_\beta : \quad (\alpha \leq \beta \leq \gamma)$$

such that the successor stages are the ultrapowers just discussed.

What is this good for?

Fact

Suppose that M is a model of ZFC°, and that M_{ω_1} is the final model of an iteration of M of length ω_1.
Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable.

So consistent sentences of $L_{\omega_1,\omega}(Q)$ are provable.

One can also make M_{ω_1} correct about stationarity, extending the absoluteness results to $L_{\omega_1,\omega}(aa)$.

Many Iterations

Remark

We emphasize that for any countable model *M* of ZFC° there are $2^{\aleph_0}$ many *M*-generic ultrafilters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$.

It follows that there are $2^{\aleph_1}$ many iterations of *M* of length ω_1.

Really distinct interations

Theorem (Larson)

If M is a countable model of $\mathrm{ZFC}^\circ + \mathrm{MA}_{\aleph_1}$ and

$$\langle M_\alpha, G_\alpha, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

and

$$\langle M'_\alpha, G'_\alpha, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\alpha,$$

where α is least such that $G_\alpha \neq G'_\alpha$.

G_α not defined for $\alpha = \omega_1$.

The Model Theory

Theorem: (Keisler, new proof Larson)

Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.

Corollary (Shelah using ch)

If a sentence in $L_{\omega_1,\omega}$ has less that $2^{\aleph_1}$ models in $\aleph_1$ then it is (syntactically) ω-stable.

CH used twice.

Sketching New Proof:

Let F be the smallest fragment of $L_{\omega_1,\omega}(Q)$ containing ϕ.

1. Choose θ so that $H(\theta)$ contains the model and θ is large enough for preservation of ZFC° .
2. Choose X a countable elementary submodel of $H(\theta)$.
3. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse.
4. Let M' be a c.c.c. forcing extension of M satisfying Martin's Axiom. to get really distinct ultrapowers.
5. Build a tree of generic ultrapower iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M', each of length ω_1.
6. Since F-types can be coded by reals using an enumeration of F in M, the images of N_0 under these iterations will pairwise realize just countably many F-types in common.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an abstract elementary class: AEC if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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3. Downward Löwenheim-Skolem.

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First order and $L_{\omega_1,\omega}$-classes
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Analytically Presented AEC: descriptive set theory version

Definition

An abstract elementary class **K** with Löwenheim number $\aleph_0$ is analytically presented if the set of countable models in **K**, and the corresponding strong submodel relation $\prec_{\mathbf{K}}$, are both analytic.

Definition

An AEC $\boldsymbol{K}$ is $PC\Gamma(\aleph_0, \aleph_0)$-presented:

if the models are reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types
and the submodel relation is given in the same way.

AKA:

1. Keisler: PC_δ over $L_{\omega_1,\omega}$
2. Shelah: $PC(\aleph_0, \aleph_0)$, $\aleph_0$-presented

More Precisely

Theorem

If $\boldsymbol{K}$ is AEC then $\boldsymbol{K}$ can be analytically presented iff and only if its restriction to $\aleph_0$ is the restriction to $\aleph_0$ of a $PC\Gamma(\aleph_0, \aleph_0)$-AEC.

The following is basically folklore.

Countable case

The countable τ-models of an analytically presented class can be represented as reducts to τ of a sentence in $L_{\omega_1,\omega}(\tau')$ for appropriate $\tau' \supseteq \tau$.
Moreover the class of countable pairs (M, N) such that $M \prec_{\boldsymbol{K}} N$ is also a $PC\Gamma(\aleph_0, \aleph_0)$-class.

Uncountable Case

Theorem

All τ-models of an analytically presented AEC $\boldsymbol{K}$ can be represented as reducts to τ of a sentence θ^* in $L_{\omega_1,\omega}(\tau^*)$ for appropriate $\tau^* \supseteq \tau$.
Moreover the class of pairs (M, N) such that $M \prec_{\boldsymbol{K}} N$ is the class of reducts to τ' of models of θ^*.

Proof combines the countable case with the idea of the proof of the presentation theorem.

Example

Groupable partial orders (Jarden varying Shelah)

Let $(\boldsymbol{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order (equivalently admits a group structure)
with $M \prec N$ if $M \subseteq N$ and no component is extended.

This AEC is analytically presented.
Add a binary function and say it is a group on each component.
But it has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.

Recall: this 'is' the pseudo-elementary counterexample to Vaught' s conjecture.

Galois Types

Let $M \prec_{\mathbf{K}} N_0$, $M \prec_{\mathbf{K}} N_1$, $a_0 \in N_0$ and $a_1 \in N_1$ realize the same Galois Type over M iff
there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 : N_0 \rightarrow N$ and $f_1 : N_1 \rightarrow N$ such that $f_0|M = f_1|M$ and $f_0(a_0) = f_1(a_1)$.

Realizing the same Galois type (over countable models) is an equivalence relation

$$E_M$$

if $\boldsymbol{K}_{\aleph_0}$ satisfies the amalgamation property.

The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property for models of cardinality at most $\aleph_0$

and has at most $\aleph_1$-Galois types over models of cardinality $\leq \aleph_0$

then there is an $\aleph_1$-monster model $\mathbb{M}$ for $\boldsymbol{K}$ and the Galois type of a over a countable M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

So E_M is an equivalence relation on $\mathbb{M}$.

Some stability notions

Definition

1. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be Galois ω-stable if for each countable $M \in \mathbf{K}$, E_M has countably many equivalence classes.
2. The abstract elementary class $(\mathbf{K}, \prec)$ is almost Galois ω-stable if for each countable $M \in \mathbf{K}$, no E_M has a perfect set of equivalence classes.

Galois equivalence is Σ^1_1

On an analytically presented AEC, having the same Galois type over M is an analytic equivalence relation, E_M. So by Burgess's theorem we have the following trichotomy.

Theorem

An analytically presented abstract elementary class $(\mathbf{K}, \prec)$ is

1. Galois ω-stable or
2. almost Galois ω-stable or
3. has a perfect set of Galois types over some countable model

Again basically folklore.

Keisler for Galois types

Theorem: (B/Larson)

Suppose that

1. **K** is an analytically presented abstract elementary class;
2. N is a **K**-structure of cardinality $\aleph_1$, and N_0 is a countable structure with $N_0 \prec_{\mathbf{K}} N$;
3. P is a perfect set of E_{N_0}-inequivalent members of ω^ω;
4. N realizes the Galois types of uncountably many members of P over N_0.

Then there exists a family of $2^{\aleph_1}$ many **K**-structures of cardinality $\aleph_1$, each containing N_0 and pairwise realizing just countably many P-classes in common.

$L_{\omega_1,\omega}$-case

Fact: Hyttinen-Kesala, Kueker

If a sentence in $L_{\omega_1,\omega}$, satisfying amalgamation and joint embedding, is almost Galois ω-stable
then it is Galois ω-stable.

What about analytically presented?

Analytically presented Strictly Almost Galois ω-stable example

The 'groupable partial order' is almost Galois stable

Let $(\boldsymbol{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended.

Since there are only $\aleph_1$-isomorphism types of components this class is almost Galois ω-stable.

This AEC is analytically presented.

Complexity of (almost) ω-stability

1. first order syntactic: Π^1_1
2. $L_{\omega_1,\omega}$-syntactic: Π^1_1
3. analytically presented AEC: Galois ω-stable: perhaps boldface Π^1_4
4. analytically presented AEC: almost Galois ω-stable: boldface Π^1_2

Absoluteness of $\aleph_1$-categoricity

1. $\aleph_1$-categoricity of a class $\boldsymbol{K}$ defined in $L_{\omega_1,\omega}$ is absolute between models of set theory that satisfy any one of the following conditions.
   1. $\boldsymbol{K}$ is ω-stable;
   2. $\boldsymbol{K}$ has arbitrarily large members and $\boldsymbol{K}$ has amalgamation in $\aleph_0$;
   3. $2^{\aleph_0} < 2^{\aleph_1}$.

   `http://homepages.math.uic.edu/~jbaldwin/pub/singsep2010.pdf`
2. $\aleph_1$-categoricity of an analytically presented AEC $\boldsymbol{K}$ is absolute between models of set theory in which $\boldsymbol{K}$ is almost Galois ω-stable, satisfies amalgamation in $\aleph_0$, and has an uncountable model.

Why is this absoluteness of $\aleph_1$-categoricity true for AEC?

Fact

Suppose that **K** is an analytically presented AEC. Then the following statements are equivalent.

1. There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that:
   - $M \prec_{\mathbf{K}} N$;
   - the set of Galois types over M realized in N is countable;
   - some Galois type over M is not realized in N.
2. There is a countable model of ZFC° whose ω_1 is well-founded and which contains trees on ω giving rise to $\mathbf{K}$, $\prec_{\mathbf{K}}$ and the associated relation $\sim_0$, and satisfies statement 1.

Smallness

Definition

1. A τ-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many $L^*(\tau)$-types (i.e. only countably many $L^*(\tau)$-n-types for each $n < \omega$).
2. A τ-structure M is called small or $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.

Why Smallness matters

Fact

Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.

{

The importance of small models

1. Downward Lowenheim Skolem
2. The entire Shelah investigation of categoricity in $L_{\omega_1,\omega}$ is built on a reduction to atomic models of first order theories using small models.
3. So crucial if absoluteness of $\aleph_1$-categoricity is to be proved.
4. Perhaps a tool for Vaught's conjecture.

Vaught's Conjecture

Vaught's Conjecture

A sentence of $L_{\omega_1,\omega}$ has either countably many or a perfect set of countable models.

Morley's theorem

A sentence of $L_{\omega_1,\omega}$ has either $\leq \aleph_1$ or a perfect set of countable models.

Regimenting $L_{\omega_1,\omega}$: Scheme 0: For a scattered class

Morley:

For sentence ϕ, $\boldsymbol{K} = \mathrm{mod}(\phi)$ is scattered if for every countable fragment L^* only countably many L^*-types are realized in any model in $\boldsymbol{K}$.
If ϕ has $< 2^{\aleph_0}$ countable models then $\boldsymbol{K}$ is scattered.

For a scattered ϕ:

Define a continuous increasing chain of countable fragments L_α such that each type in L_α realized in some model in $\boldsymbol{K}$ is a formula in $L_{\alpha+1}$.

A ubiquitous notion

Definition

K is pseudo-elementary in $L_{\omega_1,\omega}$ if ***K*** $= \{M \restriction \tau : M \models \phi\}$ (for ϕ an $L_{\omega_1,\omega}(\tau^+)$ sentence and $\tau^+ \supseteq \tau$).

This notion is also called *PC*Γ AEC, a $PC(\aleph_0, \aleph_0)$-AEC, an analytically presented AEC, a countably presented AEC ...

Locally Small

Recall M is L^*-small if it realizes only countably many L^*-types over the emptyset.

Definition

A model M of cardinality $\aleph_1$ is locally τ-small if it is L^*-small for every countable fragment L^* of $L_{\omega_1,\omega}(\tau)$.

Theorem (Keisler)

If a (pseudo) elementary class ***K*** in $L_{\omega_1,\omega}$ has less than $2^{\aleph_1}$ models in $\aleph_1$, every model in ***K*** is locally τ-small.

Regimenting $L_{\omega_1,\omega}$: Scheme I: for locally small models

Suppose the uncountable model M is locally τ-small (Remember: this means L^*-small for every countable fragment L^* of $L_{\omega_1,\omega}(\tau)$.)

In particular, if M is a member of a scattered class

Define a continuous increasing chain of countable fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$.

(A little slower than the Morley analysis)

Getting small models I

Theorem: Shelah

Suppose $\boldsymbol{K}$ is a (pseudo)-elementary class in $L_{\omega_1,\omega}$ and some $M \in \boldsymbol{K}$ is locally τ-small.

Then SOME model of cardinality $\aleph_1$ in $\boldsymbol{K}$ is $L_{\omega_1,\omega}(\tau)$-small.

Pseudo: $\boldsymbol{K}$ is $\{M \restriction \tau : M \models \phi\}$ (for ϕ a τ^+ sentence and $\tau^+ \supseteq \tau$).

Makkai proved a slightly weaker version of this theorem using the more complicated machinery of saturation in admissible model theory.

Coding L_α-equivalence

Extend the similarity type to τ' by adding:

1. a binary relation $<$, interpreted as a linear order of M with order type ω_1;
2. new $2n+1$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $n+1$-ary functions f_n.

Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type
where the L_α are defined via M locally small.

Let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ implies

$$f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \mathbf{b}).$$

What the coding means

Properties of the E_n: (same L_α-type)

1. $E_n(\beta, \mathbf{y}, \mathbf{z})$ refines $E_n(\alpha, \mathbf{y}, \mathbf{z})$ if $\beta > \alpha$;
2. $E_n(0, \boldsymbol{a}, \mathbf{b})$ implies $\boldsymbol{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \boldsymbol{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\boldsymbol{a}, c_2\mathbf{b})$ and
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \mathbf{y}, \mathbf{z})$ has only countably many classes.

The hammer: Lopez-Escobar

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau')$ sentence χ. Let Δ^* be the smallest τ'-fragment containing $\phi \wedge \chi$.

By Lopez-Escobar there is a structure N of cardinality $\aleph_1$ satisfying $\phi \wedge \chi$ such that $<$ is not well-founded on N.

The beauties of descending chains

Fix an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N.
For each n, define $E_n^+(\mathbf{x}, \mathbf{y})$ if for some i, $E_n(d_i, \mathbf{x}, \mathbf{y})$.

Now using 1), 2) and 3) prove by induction on the quantifier rank of ϕ for every $L_{\omega_1,\omega}(\tau)$-formula ϕ that

$N \models E_n^+(\boldsymbol{a}, \mathbf{b})$ implies
$N \models \phi(\boldsymbol{a})$ if and only if $N \models \phi(\mathbf{b})$.

For each n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E_n^+(\mathbf{x}, \mathbf{y})$ and by 4) $E_n(d_0, \mathbf{x}, \mathbf{y})$ has only countably many classes; so N is small.

Getting small models: II

Theorem: Baldwin/Shelah/Larson

Suppose $\boldsymbol{K}$ is a (pseudo)-elementary class in $L_{\omega_1,\omega}$,
$\boldsymbol{K} = \{M \restriction \tau : M^+ \models \phi\}$ with $\phi \in L_{\omega_1,\omega}(\tau)$
If some $M \in \boldsymbol{K}$ is locally τ-small

and $\boldsymbol{K}$ has only countably many models in $\aleph_1$ (or $\aleph_0$).

Then ALL models of cardinality $\aleph_1$ in $\boldsymbol{K}$ are $L_{\omega_1,\omega}(\tau)$-small.

Regimenting $L_{\omega_1,\omega}$: Scheme II

Suppose the uncountable model $M \in \boldsymbol{K}$ is locally τ-small but not $L_{\omega_1,\omega}(\tau)$-small.
We construct a sequence of τ^+-structures $\{N_\alpha^+ : \alpha < \omega_1\}$ each with cardinality $\aleph_1$ and an increasing continuous family of countable fragments $\{L'_\alpha : \alpha < \omega_1\}$ of $L_{\omega_1,\omega}(\tau)$ and sentences χ_α such that:

1. $L'_0(\tau)$ is first order logic on τ;
2. All the $N_\alpha^+ \models \phi$;
3. All $N_\alpha^+ \restriction \tau$ are $L_{\omega_1,\omega}(\tau)$-small;
4. χ_α is the $L_{\omega_1,\omega}(\tau)$-Scott sentence of N_α;
5. $L'_{\alpha+1}(\tau)$ is the smallest fragment of $L_{\omega_1,\omega}(\tau)$ containing $L'_\alpha(\tau) \cup \{\neg\chi_\alpha\}$;
6. For limit δ, $L'_\delta(\tau) = \bigcup_{\alpha<\delta} L'_\alpha(\tau)$;
7. For each α, $N_\alpha \equiv_{L'_\alpha(\tau)} M$.

What the construction yields

By the first theorem we construct at each stage α a small model M_α which is not L_α-equivalent to M.

M_α is a countable $L_{\omega_1,\omega}(\tau)$-elementary submodel of N_α.

The M_α are $\aleph_1$ uncountable models of ϕ.
The N_α are $\aleph_1$ extendible countable models of ϕ.

Connections to Vaught's conjecture

A countable model is extendible if it has an $L_{\omega_1,\omega}$-elementary extension.
Since every counterexample $\boldsymbol{K}$ to Vaught's conjecture is locally small, we have shown every such $\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.

Corollary: Baldwin/Shelah/Larson

Vaught's conjecture is equivalent to Vaught's conjecture for extendible models.

This also follows from earlier work of Gao and Becker.

Counterexamples Large Models

ϕ is a minimal counterexample to V.C. if for every ψ either

$$\phi \wedge \psi \text{ or } \phi \wedge \neg\psi$$

has only countably many countable models.

Harnik Makkai

Every counterexample to VC extends to a minimal counterexample.
Every minimal counterexample has a large model.

B-Larson-Shelah

All large models of a minimal counterexample to VC satisfy the same sentences of $L_{\omega_1,\omega}$.

Finding models of large Scott Rank

Theorem: Baldwin/Shelah/Larson

Suppose $\boldsymbol{K}$ is a (pseudo)-elementary class in $L_{\omega_1,\omega}$, some uncountable $M \in \boldsymbol{K}$ is locally τ-small but not $L_{\omega_1,\omega}(\tau)$-small.

and $\boldsymbol{K}$ has only countably many models in $\aleph_1$ (or $\aleph_0$).

Then $\boldsymbol{K}$ has small models in $\aleph_1$ of unbounded Scott rank in $L_{\omega_1,\omega}$.

Proof Set up

Suppose $|M| = \aleph_1$ is locally τ-small but not $L_{\omega_1,\omega}(\tau)$-small.

Expand M to M' by naming equivalence relations as in step I.

Expand $H(\theta)$ (for a sufficiently large θ) with the same predicates.

Let $\langle X_\alpha : \alpha < \omega_1 \rangle$ be a $\subseteq$-increasing continuous chain of countable elementary submodels of $H(\theta)$ such that

- ϕ, M' and the sentence χ coding the representation of the Scott analysis are in X_0;
- for each $\alpha < \omega_1$, $(\omega_1 \cap X_{\alpha+1}) \setminus X_\alpha$ is nonempty.

Collapse

For each $\alpha < \omega_1$, let P_α be the transitive collapse of X_α, and let $\rho_\alpha : X_\alpha \to P_\alpha$ be the corresponding collapsing mapping.

Then $\rho_\alpha(\omega_1) = \omega_1^{P_\alpha}$ is the ordinal $X_\alpha \cap \omega_1$, and the ordinals $\omega_1^{P_\alpha}$ $(\alpha < \omega_1)$ constitute an increasing unbounded sequence in ω_1.

Expand

For each countable transitive P_α there is an elementary extension P'_α with corresponding elementary embedding $\pi_\alpha : P_\alpha \to P'_\alpha$ such that

$\omega_1^{P'_\alpha}$ is ill-founded and uncountable, and the critical point of π_α is $\omega_1^{P_\alpha}$.

It follows that the ordinals of each P'_α are well-founded at least up to $\omega_1^{P_\alpha}$.

Calculating Scott rank

Let $N_\alpha = \pi_\alpha(\rho_\alpha(M))$. It has cardinality $\aleph_1$ in V.
For each α, the model P'_α thinks that N_α is locally small but not $L_{\omega_1,\omega}(\tau)$-small.

Since the universe of $\pi_\alpha(\rho_\alpha(M))$ is the ill-founded $\omega_1^{P'_\alpha}$, the argument for N in Step I shows each model N_α is $L_{\omega_1,\omega}(\tau)$-small.

The ordinals of each P'_α are well-founded at least up to $\omega_1^{P_\alpha}$.

Therefore (in V) the Scott rank of N_α is at least $\omega_1^{P_\alpha}$.

Real Goal

By a more elaborate proof combining features of the arguments described in detail here:

Theorem (B-Larson-Shelah)

If a (pseudo) elementary class in $L_{\omega,\omega}$ has countably many models in $\aleph_1$ and there are only $\aleph_1$-Galois types over each countable model then there are only $\aleph_0$-Galois types over each countable model.

Further Connections to Vaught's conjecture

We have shown any failure of Vaught's conjecture has at least $\aleph_1$ models in $\aleph_1$.

But Harrington had long ago shown:
Any failure of Vaught's conjecture has at least $\aleph_2$ models in $\aleph_1$.
Marker's notes: `http://homepages.math.uic.edu/~marker/harrington-vaught.pdf`

I had observed: Any first order failure of Vaught's conjecture has $2^{\aleph_1}$ models in $\aleph_1$.

Question

Does any failure of Vaught's conjecture have $2^{\aleph_1}$ models in $\aleph_1$?

Method: 'Consistency implies Truth'

Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let $\mathcal{A}$ be the countable model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct $\mathcal{B}$, an uncountable model of set theory, which is an elementary extension of $\mathcal{A}$
such that $\mathcal{B}$ is correct about uncountability. Then the model of ϕ in $\mathcal{B}$ is actually an uncountable model of ϕ.

Results using this Set Theoretic Method

1. The set theoretic method provides a uniform proof for for various infinitary logics for Keisler's: few models in $\aleph_1$ implies locally small.
2. Other uses of the method:
   a. B-Larson introduce analytically presented AEC and showed:
      i. The Keisler-Shelah 'few models implies ω-stability' theorem for this setting (for Galois types).
      ii. $\aleph_1$-categoricity is absolute for Almost Galois ω-stable AEC with amalgamation.
   b. B-Larson Shelah Assuming countably many models in $\aleph_1$: Almost Galois ω-stable implies Galois ω-stable
   c. (elucidating Shelah) Failure of exchange implies many models in $\aleph_1$.

Summary

1. The set theoretic method provides a uniform method for studying models of various infinitary logics
2. We introduced analytically presented AEC and showed:
   - i analytically presented = $PC\Gamma(\aleph_0, \aleph_0)$
   - ii Extended Keisler's few models implies ω-stability theorem to this class
   - iii Assuming countably many models in $\aleph_1$: Almost Galois ω-stable implies Galois ω-stable
   - iv $\aleph_1$-categoricity absolute for Almost Galois ω-stable with amalgamation.
3. (Shelah) New notions of independence and psuedominimality