Models in ω_1

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

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Preliminary Version

This is an account of Keisler's work showing that if a sentence ϕ of $L_{\omega_1,\omega}$ that has few models in $\aleph_1$ each model of ϕ realizes only countably many types over the emptyset. The proof has two main components. One rephrases the existence of an end extension of a model A that omits a type p as a sentence (in an expanded vocabulary and logic) that is true in A. The other applies the rephrasing. We will state the rephrasing, make the applications of it, and then prove the rephrasing holds. The last of these comprises Theorem 13 (p. 55), 28 (p. 111) and 42 (p. 163) of [**?**]. The first is the rest of chapters 30 and 31 of that book.

Definition 0.1 *A* fragment Δ *of* $L_{\omega_1,\omega}$ *is a subset of* $L_{\omega_1,\omega}$ *closed under subformula, substitutions of terms, finitary logical operations and such that: whenever* $\Theta \subset \Delta$ *is countable and* $\phi, \bigvee\Theta \in \Delta$ *then* $\bigvee\{\exists x\theta : \theta \in \Theta\}$, $\bigvee\{\phi \wedge \theta : \theta \in \Theta\}$, *and* $\bigvee(\{\phi\} \cup \Theta)$ *are all in* Δ. *Further, when dealing with theories with linearly ordered models, we require that if* $\phi, \bigvee\Theta \in \Delta$ *then* $\bigvee(\{\text{for arb large } x)\theta : \theta \in \Theta\}$

Notation 0.2 $L_{\mathcal{A}}$ *is a countable fragment of* $L_{\omega_1,\omega}$. *All models will be equipped with a linear ordering* $<$. $L_{\mathcal{A}}(A)$ *is the language obtained by adding names for the elements of* A. $(B,<)$ *is an* $L_{\mathcal{A}}$ *end extension of* $(A,<)$ *if it is an* $L_{\mathcal{A}}$ *elementary extension and* $(A,<)$ *is an initial segment of* $(B,<)$. A' *denotes* $(A,<,a)_{a\in A}$.

Keisler wants to apply these results to models of set theory and uses a transitive, irreflexive relation rather than requiring a linear order. The added generality is easy to obtain but makes it a bit harder to state results so we ignore it.

Basic idea. M omits $p(\mathbf{x})$ iff $M \models (\forall \mathbf{x}) \bigvee_{\sigma\in p} \neg\sigma(\mathbf{x})$. Our types are over the empty set and in a fixed fragment $L_{\mathcal{A}}$ unless said otherwise.

In the Section 1 we describe the main technical result and give some immediate consequences.

1 Key tool and immediate consequences

Throughout this section we assume that there is a symbol $<$ in the vocabulary and that the theory T makes $<$ a linear order. Note that in any linearly ordered structure we have the quantifier '(for arbitrarily large u)$\chi(u)$': $(\forall x)(\exists u) u > x \wedge \chi(u)$.

The next theorem is the key tool for the main results.

Theorem 1.1 *Fix a countable fragment $L_{\mathcal{A}}$ of $L_{\omega_1,\omega}$, a theory T in $L_{\mathcal{A}}$ such that $<$ is a linear order of each model of T. For each $p(\mathbf{x})$ an $L_{\mathcal{A}}$-type (possibly incomplete) over the empty set, there is a sentence $\theta_p \in \Theta$ in $L_{\omega_1,\omega}$ satisfying the following conditions.*

1. *If p is omitted in an uncountable model $(B,<)$ of T then for any countable $(A,<)$ such that $(B,<)$ is an end $L_{\mathcal{A}}$-elementary extension of $(A,<)$, $(A,<) \models \theta_p$.*
2. *θ_p satisfies:*
   - *(a) If $B \models \theta_p$ then B omits p.*
   - *(b) θ_p is preserved under unions of chains of $L_{\mathcal{A}}$-elementary extensions;*
   - *(c) for any family X of $L_{\mathcal{A}}$-types $\langle p_m : m < \omega\rangle$ over $\emptyset$ and any countable A, if $A \models \theta_{p_m}$ for each M then A has a proper $L_{\mathcal{A}}$-elementary extension that satisfies each θ_{p_m}.*
3. *Let X be a collection of complete $L_{\mathcal{A}'}(\tau')$-types (for some $\mathcal{A}' \subseteq \mathcal{A}$ and $\tau' \subseteq \tau$) over the empty set that are realized in every uncountable model of T. Then, X is countable.*

We defer the proof and make several observations.

Corollary 1.2 *Let $(A,<)$ be countable and suppose for each $m < \omega$ $A \models \theta_{p_m}$. Then there is an uncountable end extension B of A omitting all the p_m.*

Proof. By condition Theorem 1.1 2c) there is a proper elementary end extension of A_1 of A satisfying all the θ_{p_m}. Iterate this construction through ω_1 using Theorem 1.1 2b) at limit stages. By Theorem 1.1 2a) the limit model omits all the p_m. $\square_{1.2}$

The following example (Marker correcting an example of Baldwin) shows the significance of *end* extension in the statement above.

Remark 1.3 Our fragment is first order logic. In our base model M we have points of two colors–say red and blue and the red points and blue points each form a copy of (Z, s). Let p be the type which says there are no new red points and q the type which says no new blue points. Of course it is true that M

has elementary extensions of cardinality $\aleph_1$ omitting p and extensions of size $\aleph_1$ omitting q but none omitting both.

But–this observation does not take into account the *end* extension. We also have some linear order $>$ of A. One of the following holds: a) there exists x all $y > x$ are the same color b) for all x there are $y, z > x$ of different colors.

if a) holds then only one of p or q can be omitted in an elementary end extension.

if b) holds then neither p nor q can be omitted in an elementary end extension.

2 Main Consequences

In this section, we do not assume there is a linear ordering in the language, but we will add one in order to obtain the situation of the previous section.

Definition 2.1 *$\boldsymbol{K}$ is a PC_δ class in $L_{\omega_1,\omega}$ if $\boldsymbol{K}$ is the class of reducts to $\tau(\boldsymbol{K})$ of the class of models of a sentence of $L_{\omega_1,\omega}$ in some expansion τ' of τ.*

This is the same as what Shelah calls $PC(\aleph_0, \aleph_0)$ and I call $PC\Gamma(\aleph_0, \aleph_0)$.

Theorem 2.2 *If a PC_δ over $L_{\omega_1,\omega}$ class $\boldsymbol{K}$ has an uncountable model then for any countable fragment $L_{\mathcal{A}}$, there are only countably many $L_{\mathcal{A}}$-types over $\emptyset$ realized in* every *uncountable member of $\boldsymbol{K}$.*

Proof. Let ϕ be a τ'-sentence of $L_{\omega_1,\omega}$ such that $\boldsymbol{K}$ is the class of τ reducts of models of ϕ. Let $L_{\mathcal{A}}(\tau')$ be the smallest fragment that contains ϕ. Let X be the collection of $L_{\mathcal{A}}(\tau)$-types over $\emptyset$ realized in *every* uncountable model of ϕ.

Take an uncountable model of ϕ and well-order it in order type ω_1 to get $(B, <)$. Let T' be the $L_{\mathcal{A}}(\tau')$-theory of $(B, <)$. We can construct $(A, <)$ such that $(B, <)$ is an uncountable $L_{\mathcal{A}}$-end extension of $(A, <)$. If p is realized in every uncountable model in $\boldsymbol{K}$ then p is realized in every uncountable model of T'. So applying Theorem 1.1.3 to T' and X we have the result. $\square_{2.2}$

Theorem 2.3 *If a PC_δ over $L_{\omega_1,\omega}$ class $\boldsymbol{K}$ has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then for any countable fragment $L_{\mathcal{A}}$, then* every *member of $\boldsymbol{K}$ realizes only countably many $L_{\mathcal{A}}$-types over $\emptyset$.*

Proof. Let $T' = \{\phi\}$ be a τ'-sentence of $L_{\omega_1,\omega}$ such that $\boldsymbol{K}$ is the class of τ reducts of models of ϕ. Let $L_{\mathcal{A}}(\tau')$ be the smallest fragment that contains ϕ.

If the conclusion fails for some natural number p there are uncountably many $L_{\mathcal{A}} - p$-types over the empty set realized in some model $B \in \boldsymbol{K}$; wolog $|B| = \aleph_1$. First note that we can expand the language with a unary predicate and functions so that there is a set U of p-tuples that realize distinct p-types and

U has the same cardinality as the universe. This can be expressed by a sentence of $L_{\omega_1,\omega}$, so we have a PC_δ over $L_{\omega_1,\omega}$-class $\boldsymbol{K}''$ such that *every* uncountable model realizes uncountably many types. We will show $\boldsymbol{K}''$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$. Indeed they are pairwise not mutually embeddible.

Suppose that $\boldsymbol{K}''$ is axiomatized in the fragment $L_{\mathcal{A}''}$ and let B'' be an uncountable model. Now fix $A'' = A_\emptyset$ as a countable submodel so that B'' is an $L_{\mathcal{A}''}$-end extension of A'' and $p_\emptyset$ as any $L_{\mathcal{A}}$-p-type over $\emptyset$ realized in A''. We construct a family of countable τ''-models A_s for $s \in 2^{<\omega_1}$ and $L_{\mathcal{A}}$-types p_s over the empty set to satisfy the following conditions:

1. if $s < t$ then A_t is an $L_{\mathcal{A}''}$-end extension of A_s;
2. if $s \leq t$ then A_t realizes p_s;
3. if $s < t$ and $s\frown i \not\leq t$ (for $i \in \{0,1\}$) then $A_t \models \theta_{p_{s\frown i}}$ and so omits $p_{s\frown i}$.

Then if $\sigma \in 2^{\omega_1}$ and $s \in 2^{<\omega_1}$, $M_\sigma = \bigcup_{s \subset \sigma} M_s$ realizes p_s iff $s < \sigma$. This clearly suffices as $\sigma \neq \tau \in 2^{\omega_1}$ implies M_σ cannot be embedded in M_τ.

Now for the construction. For the limit stage we need to know that if we have an increasing chain M_i such that for $i_0 < j < \alpha$, $M_j \models \theta_{p_{i_0}}$ then so does M_α. This is immediate from Theorem 1.1.2b.

Now for the successor stage. We have an A_s satisfying the conditions. That is, A_s realizes p_t if $t \leq s$ and $A_s \models \theta_{p_{t\frown i}}$ if $t < s$ and $t\frown i \not\leq s$. Let $\boldsymbol{K}^3$ be the class of all τ-reducts of $L_{\mathcal{A}''}$-end extensions of A_s that omit $p_{t\frown i}$ if $t < s$ and $t\frown i \not\leq s$. Corollary 1.2 gives us an uncountable $L_{\mathcal{A}''}$-end extension B_s of A_s in $\boldsymbol{K}^3$. By Theorem 2.2 only countably many $L_{\mathcal{A}}$ types over $\emptyset$ are realized in all models in $\boldsymbol{K}'''$. So we can choose $p_{s\frown 0}$ that is realized in B_s but omitted in some uncountable model $L_{\mathcal{A}''}$-end extension of A_s, $B_1 \in \boldsymbol{K}^3$. Choose $A_{s\frown 0}$ as a countable $L_{\mathcal{A}''}$-end extension of A_s that realizes $p_{s\frown 0}$ and with

$$A_s \prec_{\mathcal{A}''} A_{s\frown 0} \prec_{\mathcal{A}''} B_s.$$

By Theorem 1.1.1, $A_{s\frown 0} \models \theta_{p_t}$ for $t < s$ and $t\frown i \not\leq s$.

To choose $p_{s\frown 1}$ and $A_{s\frown 1}$, we now apply Theorem 2.2 to $\boldsymbol{K}^4$ obtained by requiring in addition to $\boldsymbol{K}^3$ that $p_{s\frown 0}$ is omitted. We know B_1 is one $L_{\mathcal{A}''}$-end extension of A_s that is in $\boldsymbol{K}^4$. Since B_1 realizes $\aleph_1$-types there must be a type $p_{s\frown 1}$ realized in B_1 and omitted in some uncountable $L_{\mathcal{A}''}$-end extension of $A_{s\frown 0}$; thus $A_{s\frown 0} \models \theta_{p_{s\frown 1}}$. Let $A_{s\frown 1}$ be a countable $L_{\mathcal{A}''}$-end extension of A_s with B_1 an $L_{\mathcal{A}''}$-end extension of $A_{s\frown 1}$ so that $A_{s\frown 1}$ realizes $p_{s\frown 1}$; by Theorem 1.1.1 $A_{s\frown 1}$ satisfies condition 3). This completes the construction.

3 The Omitting Types and End Extension Theorems

Theorem 3.1 (Omitting types theorem) *Let $L_{\mathcal{A}}$ be a countable fragment of $L_{\omega_1,\omega}$ and T a set of $L_{\mathcal{A}}$-sentences. Further, for each m, let p_m be a set of $L_{\mathcal{A}}$-formulas $\phi(x_1, \ldots x_{k_m})$. Suppose*

1. *T has a model*
2. *and for each $m < \omega$ and any $\phi(x_1, \ldots x_{k_n}) \in L_{\mathcal{A}}$, if $T \cup (\exists \mathbf{x})\phi(\mathbf{x})$ has a model then so does $T \cup (\exists \mathbf{x})(\phi(\mathbf{x}) \wedge \neg\sigma_m)$ for some $\sigma_m \in p_m$.*

Then there is a model of T omitting all the p_m.

Proof sketch: Add a new set of constants C and obtain $M_{\mathcal{A}}$ by including any substitution of a finite sequence $\mathbf{c}$ for a finite sequence of variables $\mathbf{x}$ in an $L_{\mathcal{A}}$-formula.

Let S be the set of sets s of the form:

$$s = s_0 \cup T \cup \{ \bigvee_{\sigma \in p_m} \neg\sigma(\mathbf{c}) : \mathbf{c} \in C, m < \omega\}$$

where s_0 is a finite set of $M_{\mathcal{A}}$-sentences such that $T \cup s_0$ has a model and only finitely many of the new constants occur in s_0. We claim S is a consistency property.

The crux is $C4$ in Marker's compilation:

Let X be a countable set of $L_{\mathcal{A}}$ formulas. $\bigvee X \in s \in S$ implies for some $\phi \in X$, $s \cup \{\phi\} \in S$.

So consider a specific s of the form: $\bigvee X \in s \in S$.

If $\bigvee X \in s_0 \cup T$, this is immediate by the definition of truth of a disjunction.

So, suppose for some $\mathbf{c}$, $\bigvee X = \bigvee\{\neg\sigma(\mathbf{c}) : \sigma \in p_m\}$ for some m. Let $\mathbf{d}$ be the new constants which occur in s_0 but not in $\mathbf{c}$ – we write $s_0 = s_0(\mathbf{c}, \mathbf{d})$. Since $s \in S$.

$$T \cup \{(\exists \mathbf{x})(\exists \mathbf{y}) \bigwedge s_0(\mathbf{x}, \mathbf{y})\}$$

has a model. By hypothesis 2) for some $\sigma \in p_m$,

$$T \cup \{(\exists \mathbf{x})(\exists \mathbf{y}) \bigwedge s_0(\mathbf{x}, \mathbf{y}) \wedge \neg\sigma(\mathbf{x})\} \tag{1}$$

has a model. So

$$s \cup \{\neg\sigma(\mathbf{c})\} \in S$$

because $\neg\sigma(\mathbf{c}) \in X$ and $T \cup s_0 \cup \{\neg\sigma(\mathbf{c})\}$ has a model. By Marker's Exercise 3.7, the set of formulas 1 has a model with all its elements named by members of C. This completes the proof. $\square_{3.1}$

Definition 3.2 *Let $(A, < \ldots)$ be a countable linearly ordered structure and $L_{\mathcal{A}}$ be a countable fragment of $L_{\omega_1,\omega}$. $(A, < \ldots)$ is $L_{\mathcal{A}}$-*extendible *if*

1. *$<$ has no last element;*
2. *(exists arbitrarily large x) $\bigvee_n \phi_n \rightarrow \bigvee_n$ (exists arbitrarily large x) ϕ_n;*
3. *(exists arbitrarily large x) $(\exists y)\phi(x, y) \rightarrow (\exists y)$ (exists arbitrarily large x) $\phi(x, y) \vee$ (exists arbitrarily large y)$(\exists x)\phi(x, y)$;*

where $\bigvee_n \phi_n$ and $(\exists y)\phi$ are in $L_{\mathcal{A}}$.

Note that there is an $L_{\mathcal{A}}$-sentence over the empty set, θ^{ext}, such that for any countable $(B, <)$, $B \models \theta^{\text{ext}}$ if and only if B is extendible.

Exercise 3.3 *Verify that the conditions for extendibility are true in a model of power ω_1 that is ordered by ω_1.*

We quote the next result in the proof of Theorem 1.1.

Theorem 3.4 (Theorem 28) *Let $(A, < \ldots)$ be a countable linearly ordered structure and $L_{\mathcal{A}}$ be a countable fragment of $L_{\omega_1,\omega}$. The following are equivalent:*

1. *$(A, <)$ has an $L_{\mathcal{A}}$-end-elementary extension;*
2. *$(A, <)$ has an $L_{\mathcal{A}}$-end-elementary extension with cardinality ω_1;*
3. *$(A, <)$ is $L_{\mathcal{A}}$-extendible.*

Proof. We sketch the easy parts first. Clearly ii) implies i). Assuming i and iii are equivalent, i) implies ii) by an easy elementary chain argument. I check the second case of i) implies iii).

Suppose B is an end extension of A and A (*a fortiori* B) satisfy (exists arbitrarily large x) $\bigvee_n \phi_n(\boldsymbol{a}, x)$ with $\boldsymbol{a} \in A$. So for some $b \in B - A$ and some $c > b$, $B \models \bigvee_n \phi_n)(\boldsymbol{a}, c)$. But then for some n, $B \models \phi_n(\boldsymbol{a}, c)$. Now B and thus A satisfies (exists arbitrarily large x) $\phi_n(\boldsymbol{a}, x)$.

Now we prove iii) implies i). Add to τ constant symbols a for each $a \in A$ and a further d to obtain a vocabulary τ'. We will write A' for the structure $(A, a)_{a \in A}$. Let T^A be the collection of all $L_{\mathcal{A}}(A)$-sentences true in A' along with sentences $\theta(d)$ where $\theta(x) \in L_{\mathcal{A}}$ and

$$A \models (\exists y)(\forall x)[x > y \rightarrow \theta(x)].$$

Let ϕ be the $L_{\mathcal{A}}(A)$ sentence expressing that each type of a new element that is not above all members of A is omitted. Formally,

$$\phi := \bigwedge_{a \in A} (\forall y) \bigvee_{b \in A} (y = b \vee a < y).$$

Thus,

$$B \models T^A \cup \{\phi\}$$

implies B is a proper $L_{\mathcal{A}}$-elementary (by T^A), end (by ϕ) extension of A.

We will use the omitting types theorem to show $T \cup \{\phi\}$ is consistent. We need to show:

1. T^A is consistent
2. If $(\exists y)\psi(y) \in L_{\mathcal{A}}(A)$ is consistent with T^A then for each $a \in A$, there is a $b \in A$ such that: $(\exists y)\psi(y) \wedge (y = b \vee y > a)$ is consistent with T^A.

If 2) holds, by the omitting types theorem we can omit each type: $p_a = \{y < a, y \neq b : b < a\}$.

To prove 1) we add to τ' a further countable set C of constants. Let S be the set of all finite subsets s (with constants $c_1, \ldots c_n, d$) of $L_{\mathcal{A}}(ACd)$-sentences such that:

$$A' \models (\text{for arb large } x)(\exists \mathbf{u}) \bigwedge s(\mathbf{u}, x).$$

We now show S is a consistency property; the crucial step verifying $C4$ is given by clause 2) in the definition of extendible. Suppose $\bigvee_{\theta \in X} \in s \in S$. Then, trivially,

$$A' \models (\text{for arb large } x)(\exists \mathbf{u}) \bigvee_{\theta \in X} (\bigwedge s \wedge \theta))$$

and so since existential quantification distributes over disjunction,

$$A' \models (\text{for arb large } x) \bigvee_{\theta \in X} (\exists \mathbf{u})(\bigwedge s \wedge \theta))$$

But then by extendibility for some $\theta \in X$,

$$A' \models (\text{for arb large } x)(\exists \mathbf{u})(\bigwedge s \wedge \theta)).$$

So, $s \cup \{\theta\} \in S$.

Since A has arbitrarily large elements, $A' \models (\text{for arb large } x) x = x$. Thus $\{c = c\} \in S$ and S is non-empty.

Now to show T has a model it suffices to show that

$$S' = \{s \cup T^A : s \in S\}$$

is a consistency property. For this, suppose $s \in S$ and $\psi \in T^A$; we deduce $s \cup \{\psi\} \in S$. Recall from the definition of T^A that $\theta(d) \in T^A$ where $\theta(x) \in L_{\mathcal{A}}$ implies

$$A \models (\exists y)(\forall x)[x > y \rightarrow \theta(x)].$$

Since A has no last element,

$$A' \models (\text{for arb large } x)(\exists \mathbf{u})(\bigwedge s).$$

Together these two statements imply

$$A' \models (\text{for arb large } x)(\psi(x) \wedge (\exists \mathbf{u})(\bigwedge s))$$

so $s \cup \{\psi\} \in S$. So T^A is consistent.

We now prove 2). First we note:

(*) there is a model of $T^A \cup \{\psi(d)\}$ iff $A' \models (\text{for arb large } x)\psi(x)$.

If $A' \models (\text{for arb large } x)\psi(x)$, then $\{\psi(d)\} \in S$ so $T^A \cup \{\psi(d)\} \in S'$ and has a model. The converse is just translating formulas.

Now suppose $(\exists y)\psi(d, y)$ is consistent with T^A and choose $a \in A$. Then

$$A' \models (\text{for arb large } x)[(\exists y)(\psi(x, y) \wedge a < y) \vee (\exists y)(\psi(x, y) \wedge y \leq a)].$$

By e) in the definition of extendible, we can distribute (for arb large x).

Case I (for arb large x)$(\exists y)(\psi(x, y) \wedge a < y)$. Trivially, for any $b \in A$, (for arb large x)$(\exists y)(\psi(x, y) \wedge (y = b \vee a < y))$. By (*), $(\exists y)(\psi(d, y) \wedge (y = b \vee a < y))$ is consistent with T^A and 2) holds.

Case II (for arb large x)$(\exists y)(\psi(x, y) \wedge y \leq a)$. Clearly $A' \models \neg(\text{for arb large } y)(\exists x)(\psi(x, y) \wedge y \leq a)$. So by 3) in definition of extendible

$$A' \models (\exists y)(\text{for arb large } x)(\psi(x, y) \wedge y \leq a).$$

In particular, for some $b \in A$, $A' \models (\text{for arb large } x)(\psi(x, b)$. Thus, $(\exists y)\psi(d, y) \wedge y = b \vee a < y)$ is consistent with T^A and again we have 2).

This completes the proof. $\square_{3.4}$

4 Proof of the Key Tool

In this section we complete the proof of Theorem 1.1. We must first define the sentence θ_p. Let Γ be the set of all pairs $\langle \gamma, S \rangle$ where γ is a formula in $L_{\mathcal{A}}$ with all free variables displayed as $\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y})$ and $S = (S\mathbf{u})$ is a finite sequence of $(\exists u_i)$ and 'for arbitrarily large u_j'. We fix the length of the $\mathbf{x}$-sequence but $\mathbf{u}$ and $\mathbf{y}$ can have any finite length).

Now for any type p in the variables $\mathbf{x}$, let θ_p be the conjunction of the extendability sentence θ^{ext} and the $L_{\omega_1,\omega}(\emptyset)$-sentence:

$$\bigwedge_{\langle\gamma,S\rangle\in\Gamma}(\forall\mathbf{y})\left[(S\mathbf{u})(\exists\mathbf{x})\gamma(\mathbf{x},\mathbf{u},\mathbf{y})\rightarrow\bigvee_{\sigma\in p}(S\mathbf{u})(\exists\mathbf{x})(\gamma(\mathbf{x},\mathbf{u},\mathbf{y})\wedge\neg\sigma)\right].$$

Part 1). Given an uncountable model $(B,<)$ with order type ω_1 omitting p, we can choose a countable $(A,<)$ so that $(B,<)$ is an $L_{\mathcal{A}}$-elementary end extension of $(A,<)$. Then $(A,<)$ satisfies the extendibility sentences by Theorem 3.4. Applying 2) in Definition 3.2, it is straightforward to prove the result by induction on the length of the string S.

Before proving part 2, we need some further notation.

Definition 4.1 **($C_{p,A}$)** *We say $C_{p,A}$ holds if for every $\langle\gamma(x,\mathbf{u},ybar),S\rangle\in\Gamma$, partition of $\mathbf{y}$ into $\mathbf{y}_1,\mathbf{y}_2$ and every substitution of an $\boldsymbol{a}\in A$ for $\mathbf{y}_2$ to yield $\gamma(\mathbf{x},\mathbf{u},\mathbf{y}_1,\boldsymbol{a})$ in $L_{\mathcal{A}}(A)$,*

$$A\models\left[(S\mathbf{u})(\exists\mathbf{x})\gamma(\mathbf{x},\mathbf{u},\mathbf{y}_1,\boldsymbol{a})\rightarrow\bigvee_{\sigma\in p}(S\mathbf{u})(\exists\mathbf{x})(\gamma(\mathbf{x},\mathbf{u},\mathbf{y}_1,\boldsymbol{a})\wedge\neg\sigma(\mathbf{x}))\right].$$

The following essential claim is easy to check using the observation that θ_p contains universal quantifiers while $C_{p,B}$ contains their instantiations over B.

Claim 4.2 *For any B, $B\models\theta_p$ is equivalent to $C_{p,B}$ holds.*

For part 2a) note that by Claim 4.2, we may assume $C_{p,B}$ holds. But, if $C_{p,B}$ holds then B omits p since for any $\mathbf{b}\in B$ we can take $\theta\in L_{\mathcal{A}}(B)$ from $C_{p,B}$ to be $\mathbf{x}=\mathbf{b}$.

Part 2b) is immediate noting that if the hypothesis of one conjunct of θ_p is satisfied in some A_α, then a particular one of the disjunctions in the conclusion of the implication is true in A_α and so in every $L_{\mathcal{A}}$-elementary extension of A_α.

For part 2c), let A be a countable model of T and X a countable set of types over the empty set. We will concentrate on a single p, just noting at the key point that the omitting types theorem will allow us to lift all the θ_p for $p\in X$ to a single model.

Note that θ_p is actually in the form asserting a family of types is omitted (if we write $F\rightarrow\bigvee_i G_i$ as $\neg F\vee\bigvee_i G_i$). More precisely, define for each $\gamma(\mathbf{x},\mathbf{u},\mathbf{y})\in L_{\mathcal{A}}(A)$ and each string $(S\mathbf{u})$ the $L_{\mathcal{A}}(A)$-type:

$$\lambda_{p,\gamma,S}(\mathbf{y})=\{(S\mathbf{u})(\exists\mathbf{x})\gamma(\mathbf{x},\mathbf{u},\mathbf{y})\}\cup\{\neg(S\mathbf{u})(\exists\mathbf{x})\gamma(\mathbf{x},\mathbf{u},\mathbf{y})\wedge\neg\sigma:\sigma\in p\}.$$

Now, just checking the definition,

Lemma 4.3 *For any B, B omits $\lambda_{p,\gamma,S}(\mathbf{y})$ for each γ,S with $\gamma\in L_{\mathcal{A}}(\emptyset)$ if and only if $B\models\theta_p$.*

Let T^A be the theory in $L_{\mathcal{A}}((A) \cup \{d\})$ introduced in proving Theorem 3.4, let ϕ be the end extension sentence from that proof. We want to show $T^A \cup \{\phi\}$ has a model (thus a proper $L_{\mathcal{A}}$-elementary end extension $(B, <)$ of $(A, <)$) omitting each $\lambda_{p,\gamma}$.

Lemma 4.4 *Let A be countable and suppose $A \models \theta_p$. For any $L_{\mathcal{A}}(A)$-formula, $\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y})$, any type $p(\mathbf{x})$ and any formula $\pi(d, \mathbf{y}) \in L_{\mathcal{A}}(Ad)$, if $(\exists \mathbf{y})\pi(d, \mathbf{y})$ is consistent with T^A then*

- $(\exists \mathbf{y})(\pi(d, \mathbf{y}) \wedge \neg(S\mathbf{u})(\exists \mathbf{x})\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}))$ *is consistent with T^A, or*
- *for some $\sigma \in p$,*

$$(\exists \mathbf{y})(\pi(d, \mathbf{y}) \wedge (S\mathbf{u})(\exists \mathbf{x})\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \neg\sigma)$$

  is consistent with T^A.

Proof. Since $A \models \theta_p$, by Claim 4.2, $C_{p,A}$ holds. Because of this observation we have suppressed additional parameters $\boldsymbol{a}$ which may occur in the formulas π and γ. Suppose $(\exists \mathbf{y})\pi(d, \mathbf{y})$ is consistent with T^A but

$$T^A \models \neg(\exists \mathbf{y})(\pi(d, \mathbf{y}) \wedge \neg(S\mathbf{u})(\exists \mathbf{x})\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y})).$$

Then

$$T^A \models (\forall \mathbf{y})(\pi(d, \mathbf{y}) \rightarrow (S\mathbf{u})(\exists \mathbf{x})\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y})).$$

Recall from the definition of T^A that the consistency of $\pi(d, \mathbf{y})$ means

$$A' \models (\text{arb large } x)(\exists \mathbf{y})\pi(x, \mathbf{y}).$$

Combining the last two,

$$A' \models (\text{arb large } x)(\exists \mathbf{y})(S\mathbf{u})(\exists \mathbf{x})(\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \pi(x, \mathbf{y})).$$

Let $S'x\mathbf{y}\mathbf{u}$ denote $(\text{arb large } x)(\exists \mathbf{y})(S\mathbf{u})$. With this notation, we have

$$A' \models (S'x\mathbf{y}\mathbf{u})(\exists \mathbf{x})(\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \pi(x, \mathbf{y})).$$

Now, since $C_{p,A}$ holds, for some $\sigma \in p$,

$$A' \models (S'x\mathbf{y}\mathbf{u})(\exists \mathbf{x})[\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \pi(x, \mathbf{y}) \wedge \neg\sigma(\mathbf{x})].$$

Again using the definition of T^A, we conclude

$$(\exists \mathbf{y})(S\mathbf{u})(\exists \mathbf{x})[\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \pi(d, \mathbf{y}) \wedge \neg\sigma(\mathbf{x})]$$

is consistent with T^A, whence

$$(\exists \mathbf{y})[\pi(d, \mathbf{y}) \wedge (S\mathbf{u})(\exists \mathbf{x})\gamma(\mathbf{x}, \mathbf{u}, \mathbf{y}) \wedge \neg\sigma(\mathbf{x})]$$

is consistent with T^A as required. $\square_{4.4}$

We conclude with the argument for Part 2c).

Lemma 4.5 *For any countable model* $(A,<)$*, for each* $p \in X$ *where* X *is countable,if* $A \models \theta_p$ *then* $(A,<)$ *has a proper* $L_{\mathcal{A}}$*-elementary end extension* $(B,<)$ *so that* $B \models \theta_p$*.*

Proof. Since $(A,<)$ is extendible (implied by θ_p), each of the countably many types whose omission is encoded in ϕ is 'non-principal'. By Lemma 4.4 the same holds for each $\lambda_{p,\gamma,S}(\mathbf{y})$ with $\gamma \in L_{\mathcal{A}}(A)$. By the omitting types theorem, $T^A \cup \{\phi\}$ has a model (thus a proper $L_{\mathcal{A}}$-elementary end extension $(B,<)$ of $(A,<)$) omitting $\lambda_{p,\gamma,S}$ for each $\gamma \in L_{\mathcal{A}}(A)$ and in particular for each $\gamma \in L_{\mathcal{A}}(\emptyset)$. Thus $B \models \theta_p$. $\square$4.5

Finally, we prove Part 3) of Theorem 1. Let X be a collection of complete $L_{\mathcal{A}'}(\tau')$-types (for some $\mathcal{A}' \subseteq \mathcal{A}$ and $\tau' \subseteq \tau$) over the empty set that are realized in every uncountable model of T. If $p \in X$, then for any such $(B,<)$ that is an $L_{\mathcal{A}}$-elementary end extension of a countable $(A,<)$,

$$(A,<) \models \neg\theta_p.$$

That is, there is a formula ψ_p such that:

$$(A,<) \models (S\mathbf{u})(\exists\mathbf{x})\psi_p,$$

but also for some $\sigma \in p$:

$$(A,<) \models \neg(S\mathbf{u})(\exists\mathbf{x})(\psi_p \wedge \neg\sigma).$$

Note that while there are potentially continuum many formulas θ_p (infinite disjunction over p), there are only countably many possible formulas ψ_p. So to conclude that there only countably many possible p, we need only show that if $p \neq q$ then $\psi_p \neq \psi_q$. Since X is a collection of *complete* $L_{\mathcal{A}'}(\tau')$-types, there is some $L_{\mathcal{A}}(\tau')$-formula σ with $\sigma \in p$ and $\neg\sigma \in q$.

$$(A,<) \models \neg(S\mathbf{u})(\exists\mathbf{x})\psi_p \wedge \neg\sigma.$$

$$(A,<) \models \neg(S\mathbf{u})(\exists\mathbf{x})\psi_q \wedge \neg\neg\sigma.$$

But of $\psi_p = \psi_q$, this contradicts that "arb large" and "there exists" distribute over disjunction. So we finish.

References

[Kei71] H.J Keisler. *Model theory for Infinitary Logic.* North-Holland, 1971.