Abstract Elementary Classes Various Directions Abelian Groups

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Topics

Two Goals

General

Can we extend the methods of first order stability theory to generalized logics – e.g. $L_{\omega_1,\omega}$?

Special

Can the model theory of infinitary logic solve 'mathematical problems' (as the model theory of first order logic has)?

A background principle

Slogan

To study a structure A, study $\mathrm{Th}(A)$.

e.g.
The theory of algebraically closed fields to investigate $(\mathcal{C}, +, \cdot)$.

The theory of real closed fields to investigate $(\mathcal{R}, +, \cdot)$.

DICTA

But there is no real necessity for the ‘theory’ to be complete.

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Strong Slogan

Classes of structures and the relations between them are more interesting than singleton structures.

ABSTRACT ELEMENTARY CLASSES defined

Definition

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an abstract elementary class: AEC if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism and satisfy the following conditions.

- **A1**. If $M \prec_{\mathbf{K}} N$ then $M \subseteq N$.
- **A2**. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
- **A3**. If $\langle A_i : i < \delta \rangle$ is $\prec_{\mathbf{K}}$-increasing chain:
  1. $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
  2. for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
  3. if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.

- **A4**. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$.
- **A5**. There is a Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and
  $|A'| \leq \mathrm{LS}(\mathbf{K}) + |\mathrm{A}|$.

Examples

1. First order complete theories with $\prec_{\mathbf{K}}$ as elementary submodel.
2. Models of $\forall\exists$-first order theories with $\prec_{\mathbf{K}}$ as substructure.
3. L^n-sentences with L^n-elementary submodel.
4. Varieties and Universal Horn Classes with $\prec_{\mathbf{K}}$ as substructure.
5. Models of sentences of $L_{\kappa,\omega}$ with $\prec_{\mathbf{K}}$ as: elementary in an appropriate fragment.
6. Models of sentences of $L_{\kappa,\omega}(Q)$ with $\prec_{\mathbf{K}}$ carefully chosen.
7. Robinson Theories with Δ-submodel
8. 'The Hrushovski Construction' with strong submodel

Eventual Categoricity

Modern history begins with the decision to assume Amalgamation. (also JEP and arbitrarily large models)

Eventual Categoricity

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- Shelah: There is a κ such that if **K** is categorical in λ^+ greater than κ then **K** is categorical on $[\kappa, \lambda^+]$.
- Grossberg-Vandieren: If **K** is λ^+-categorical with $\lambda > \mathrm{LS}(\mathbf{K})$ and $(< \lambda, \infty)$-tame then **K** is categorical in all $\theta \geq \lambda^+$.

Second yields new proof even for first-order upward categoricity.

‘concrete’ excellence in $L_{\omega_1,\omega}$

- Shelah: ($2^{\aleph_n} < 2^{\aleph_{n+1}}$): Categoricity up to $\aleph_\omega$ implies excellence.
- Shelah: If **K** is excellent, categoricity in one uncountable power implies categoricity in all uncountable powers.
- Zilber: Quasimimal excellent classes are categorical in all uncountable powers. Covers of $(\mathcal{C}, *)$ and ‘psuedoexponentiation’ are quasiminimal excellent.
- Zilber: Certain ‘arithmetical’ properties of semi-abelian varieties are equivalent to excellence.

‘abstract excellence/superstability’ in AEC

- Shelah: good frames (at least three long papers)
- Grossberg-Kolesnikov: superior classes

Classification of AEC via stronger hypotheses

This direction originated with Grossberg-VanDieren.

- Tameness as a working hypothesis:
  - Stability spectrum: Grossberg-Vandieren and Baldwin-Kueker-VanDieren
  - categoricity: Grossberg-Vandieren and Lessmann
- Finitary Abstract Elementary Classes: Hyttinen-Kesala

Stability Spectrum

- Suppose **K** is $(\mathrm{LS}(\mathbf{K}), \infty)$-tame.
  - Grossberg-VanDieren: If **K** is stable in μ, it is stable in every κ with $\kappa^\mu = \kappa$.
  - Baldwin-Kueker-VanDieren: If **K** is stable in κ, it is stable in κ^{+n} for each n.
- Baldwin-Kueker-VanDieren:
  If **K** is (∞, ∞)-local then ω-stable implies stable in all cardinalities.

Abelian Groups

At CRM-Barcelona last week, I asked.

1. Does the notion of AEC provide a general framework to describe some work in Abelian group theory?
2. Certain AEC of abelian groups provide interesting previously unknown examples for the general study of AEC. Can this work be extended?

The group group

AIM meeting July 2006

J. Baldwin, W. Calvert, J. Goodrick, A. Villaveces, & A. Walczak-Typke, & Jouko Väänänen

I described some very preliminary results of this group to emphasize the exploratory nature of this program.

Strong Submodel

Notation

Consider various subclasses $\mathbf{K}^{foo}$ of the class $\mathbf{K}^{ab}$ of all abelian groups (e.g. $foo = div, red(p), \ldots$).

1. "$\leq$" *denotes subgroup.*
2. $G \prec_{pure} H$ *means* G *is a pure subgroup of* H*:*
3. "$G \prec_{sum} H$" *means that* G *is a direct summand of* H*;*
4. "$G \prec_{foo} H$" *means that* G *is a pure subgroup of* H *and* $H/G \in \mathbf{K}^{foo}$.

Properties of $(\mathbf{K}^{cyc}, \prec_{sum})$

Fact

Abbreviating $(\mathbf{K}^{cyc}, \prec_{sum})$ as $\mathbf{K}^{cyc}$, we ought to be able to prove the following:

- $\mathbf{K}^{cyc}$ *is not an elementary class.*
- $\mathbf{K}^{cyc}$ *is a tame AEC with amalgamation and Löwenheim-Skolem number $\aleph_0$.*
- $\mathbf{K}^{cyc}$ *is not categorical.*
- $\mathbf{K}^{cyc}$ *has a universal model at every infinite cardinal λ.*
- $\mathbf{K}^{cyc}$ *is (galois-)stable at every cardinal.*
- $I(\mathbf{K}^{cyc}, \aleph_d) = |d + \omega|^{\omega}$.

But it fails **A3.3**

Reflection

Why do we want A.3.3?

THE PRESENTATION THEOREM

Every AEC is a PCΓ

More precisely,

Theorem

If K is an AEC with Lowenheim number $\mathrm{LS}(\mathbf{K})$ (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbf{K})$), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then
$M' \restriction L \prec_{\mathbf{K}} N' \restriction L$.

What's so great about PCΓ

We care because PCΓ gives

- Ehrehfeucht Mostowski models;
- omitting types (for Galois types);
- a handle on making non-splitting extensions.

Still a PCΓ

$(\mathbf{K}^{cyc}, \prec_{sum})$ is a PCΓ class by adding a predicate for a basis and using omitting types to translate $L_{\omega_1,\omega}$-axioms.

Andrew Coppola introduces the notion of a Q-AEC which generalizes the notion and still allows the presentation theorem to hold. This notion might be relevant here although the motivation was very different - equicardinality quantifiers.

Synergy

Baldwin, Eklof, Trlifaj (last week):

Theorem

1. *For an abelian group N, the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class if and only if N is a cotorsion module.*
2. *For any R-module N, over an hereditary ring R, if N is a pure-injective module then the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class.*

What do the words mean?

Definition

1. ${}^{\perp}N = \{A : Ext(A, N) = 0\}$
2. For $A \subseteq B$ both in ${}^{\perp}N$, $A \prec_N B$ if $B/A \in {}^{\perp}N$.

Generalizes the class of Whitehead groups: $Ext(G, \mathcal{Z}) = 0$.

Whitehead Groups

Definition

We say A is a Whitehead group if $Ext(A, \mathcal{Z}) = 0$. That is, every short exact sequence

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow A \rightarrow 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathcal{Z}$.

Under V=L, Whitehead groups are free; hence PCΓ. What about in ZFC?

Easy Part

A1 and **A2** are immediate for any R and N.
And in this context **A.3.1** easily implies **A.3.2**
And **A4** is equally immediate if

Definition

R is hereditary if and only if for every pair $A \subset B$ of R-modules and any N, $Ext(B, N) = 0$ implies $Ext(A, N) = 0$.

Interesting part

A3.1 is immediate from Eklof's Lemma:

Lemma

Let C be a module. Suppose that $A = \bigcup_{\alpha<\mu} A_\alpha$ *with* $A_0 \in {}^{\perp}N$ *and for all* $\alpha < \mu$, $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$ *then* $A \in {}^{\perp}N$.

Very Interesting part

Fact

For abelian groups, the following are equivalent:

1. *N is cotorsion;*
2. *$^\perp N$ is closed under direct limits;*
3. *$Ext(Q, N) = 0$.*

Now it is easy to show that $^\perp N$ is closed under direct limits implies $^\perp N$ satisfies **A.3.3**.

$^\perp N$ satisfies **A.3.3** implies $Ext(Q, N) = 0$.

Very Very Interesting part

For arbitrary rings the conditions for closure of ${}^{\perp}N$ under direct limits don't seem to be well-understood.

Refinements

Definition

For any right R-module A and any cardinal κ, a
(κ, N)-refinement of length σ of A
is a continuous chain $\langle A_\alpha : \alpha < \sigma \rangle$ of submodules such that:

- $A_0 = 0$,
- $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$, and
- $|A_{\alpha+1}/A_\alpha| \leq \kappa$ for all $\alpha < \sigma$.

Lowenheim-Skolem Number

Finally, (using the generalized Hill's Lemma) it is straightforward to show that $({}^{\perp}N, \prec_N)$ has $\mathrm{LS}({}^{\perp}N) = \kappa$ if there is a $({}^{\perp}N, \kappa)$ refinement of each $M \in {}^{\perp}N$. But the existence of refinements is an active research area.

$({}^{\perp}N, \prec_N)$ as an AEC

Lemma

$({}^{\perp}N, \prec_N)$ is an AEC under any of the following conditions.

1. *N is cotorsion and R is a Dedekind domain.*
2. *N is pure-injective and R is hereditary.*
3. *(V=L) N is arbitrary and R is hereditary and ${}^{\perp}N$ is closed under direct limits.*

Some Examples

Lemma

The class $\mathbf{K}^{ab}$ of all abelian groups forms an AEC with amalgamation and joint embedding under either $\leq$ or $\prec_{pure}$, with Löwenheim-Skoelm number $\aleph_0$. Moreover, under $\leq$ it is stable in all cardinals.

But what does stable mean?

Model Homogeneity

Definition

M is μ-model homogenous if for every $N \prec_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \prec_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.

To emphasize, this differs from the homogenous context because the N must be in $\mathbf{K}$. It is easy to show:

Monster Model

Lemma

(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then
$M_1 \approx M_2$.

Theorem

If $\mathbf{K}$ has the amalgamation property and $\mu*^{<\mu*} = \mu*$ and $\mu* \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu*$ which is $\mu*$-model homogeneous.

GALOIS TYPES: General Form

Define:

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

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$$f(a) = f'(a').$$

'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N''
if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.

GALOIS TYPES: Algebraic Form

Suppose $\mathbf{K}$ has the amalgamation property.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathcal{M}$ and $a \in \mathcal{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathcal{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathcal{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- **locality** Any increasing chain of types has at most one upper bound;
- **tameness** two distinct types differ on a finite set;
- **compactness** an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Galois and Syntactic Types

Work in $(\mathbf{K}^{ab}, \leq)$.

Lemma

Suppose that G_1 is a subgroup of both G_2 and G_3, $a \in G_2 - G_1$, and $b \in G_3 - G_1$. the following are equivalent:

1. $\text{ga-tp}(a, G_1, G_2) = \text{ga-tp}(b, G_1, G_3)$;
2. *There is a group isomorphism from $\langle G_1, a\rangle_{G_3}$ onto $\langle G_1, b\rangle_{G_3}$ that fixes G_1 pointwise;*
3. $\mathrm{tp}_{qf}(a/G_1) = \mathrm{tp}_{qf}(b/G_1)$.

But this equivalence is far from true of all AEC's of Abelian groups.

Stability

Theorem

1. *The AEC of Abelian groups under subgroup is stable in all cardinals.*
2. *The AEC of Abelian groups under pure subgroup is stable in all cardinals λ with $\lambda^\omega = \lambda$.*
3. *For N an Abelian group, $({}^\perp N, \prec_N)$ is stable in all cardinals λ with $\lambda^\omega = \lambda$.*

But, the actual stability class of various $({}^\perp N, \prec_N)$ is open.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\mathbf{K}$ is (χ, μ)-tame if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

*If $\lambda > \mathrm{LS}(\mathbf{K})$, **K** is λ^+-categorical and $(\lambda, < \infty)$-tame then **K** is categorical in all $\theta \geq \lambda^+$.*

Theorem (Lessmann)

*If **K** with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then **K** is categorical in all uncountable cardinals*

Fact

In studying categoricity of short exact sequences, Zilber has proved equivalences between categoricity in uncountable cardinals and 'arithmetic properties' of algebraic groups. These are not proved in ZFC but an independent proof of tameness would put them in ZFC.

Two Examples that are not tame

1. 'Hiding the zero'
   For each $k < \omega$ a class which is $(\aleph_k, \infty)$-tame but not $(\aleph_{k+1}, \aleph_{k+2})$-tame. Baldwin-Kolesnikov (building on Hart-Shelah)
2. Coding EXT
   A class that is not $(\aleph_0, \aleph_1)$-tame.
   A class that is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame. (Baldwin-Shelah)

Categoricity does not imply tameness

Theorem For each $k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not $\aleph_{k-2}$-Galois stable;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the disjoint amalgamation property;
5. ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
6. ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

Locality and Tameness

Definition

K has (κ, λ)-local galois types if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of **K** with $|M| = \lambda$ and for any $p, q \in \mathcal{S}(M)$: if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.

Lemma

If $\lambda \geq \kappa$ and $\mathrm{cf}(\kappa) > \chi$, then (χ, λ)-tame implies (κ, λ)-local. If particular, $(\aleph_0, \aleph_1)$-tame implies $(\aleph_1, \aleph_1)$-local.

Key Example

Shelah constructed (page 228 of Eklof-Mekler, first edition) of a group with the following properties.

Fact

There is an $\aleph_1$-free group G of cardinality $\aleph_1$ which is not Whitehead.
Moreover, there is a countable subgroup R of G such that G/R is p-divisible for each prime p.

THE AEC EXAMPLE

Let $\mathbf{K}$ be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\mathbf{G}, \mathbf{Z}, \mathbf{I}, \mathbf{H})$.

G is a torsion-free Abelian Group. Z is a copy of $(Z, +)$. I is an index set and H is a family of infinite groups.

Each model in **K** consists of

1. a torsion free group G,
2. a copy of $\mathcal{Z}$
3. and a family of extensions of Z by G.

Each of those extensions is coded by single element of the model so the Galois type of a point of this kind represents a specific extension. The projection and embedding maps from the short exact sequence are also there.

$M_0 \prec_{\mathbf{K}} M_1$ if

M_0 is a substructure of M,

but $\mathbf{Z}^{M_0} = \mathbf{Z}^{M}$

and $\mathbf{G}^{M_0}$ is a pure subgroup of $\mathbf{G}^{M_1}$.

NOT LOCAL

Lemma

$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_1, \aleph_1)$-local. That is, there is an $M^0 \in \mathbf{K}$ of cardinality $\aleph_1$ and a continuous increasing chain of models M_i^0 for $i < \aleph_1$ and two distinct types $p, q \in \mathcal{S}(M^0)$ with $p \restriction M_i^0 = q \restriction M_i$ for each i.

Let G be an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. There is an H such that,

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow G \rightarrow 0$$

is exact but does not split.

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z\rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\}\rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\}\rangle$

Let $p = \mathrm{tp}(t_1/M^0, M^1)$ and $q = \mathrm{tp}(t_2/M^0, M^2)$.
Since the exact sequence for $\mathbf{H}^{M^2}$ splits and that for $\mathbf{H}^{M^1}$ does not, $p \neq q$.

NOT $\aleph_1$-LOCAL

But for any countable $M_0' \prec_{\mathbf{K}} M_0$, $p \restriction M_0' = q \restriction M_0'$, as

$$0 \rightarrow Z \rightarrow H_i' \rightarrow G' \rightarrow 0. \tag{1}$$

splits.
$G' = \mathbf{G}(M_0')$, $H' = \pi^{-1}(t_i, G')$.

NOT $\aleph_0$-TAME

It is easy to see that if $(\mathbf{K}, \prec_{\mathbf{K}})$ is $(\aleph_0, \aleph_0)$-tame then it is $(\aleph_1, \aleph_1)$-local, so
$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_0, \aleph_0)$-tame.
So in fact, $(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\chi, \aleph_0)$-tame for any χ.

Question

Could this example be formulated more naturally as
$\{Ext(G, Z) : G \text{is torsion-free}\}$
(with projection and injection maps?

Incompactness

Theorem

Assume $2^{\aleph_0} = \aleph_1$, *and* $\diamondsuit_{\aleph_1}, \diamondsuit_{S_1^2}$ *where*

$$S_1^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}.$$

Then, the last example fails either $(\aleph_1, \aleph_1)$ *or* $(\aleph_2, \aleph_2)$*-compactness.*

BECOMING TAME ??

Grossberg and Van Dieren asked for $(\mathbf{K}, \prec_{\mathbf{K}})$, and $\mu_1 < \mu_2$ so that $(\mathbf{K}, \prec_{\mathbf{K}})$ is not (μ_1, ∞)-tame but is (μ_2, ∞)-tame.

Tameness gained

Theorem

There is an AEC with the closure property in a countable language with Lowenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Proof Sketch: Repeat the previous example but instead of letting the quotient be any torsion free group

1. insist that the quotient is an $\aleph_1$-free group;
2. add a predicate R for the group R G/R is divisible by every prime p where G is Shelah's example of a non-Whitehead group.

This forces $|G| \leq 2^{\aleph_0}$ and then we get $(2^{\aleph_0}, \infty)$-tame. But $\aleph_1$-free groups fail amalgamation ??

Lemma

For any AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ *which admits closures there is an associated AEC* $(\mathbf{K}', \prec_{\mathbf{K}})$ *with the same (non) locality properties that has the amalgamation property.*

Theorem

There is an AEC with the amalgamation property in a countable language with Lowenheim-Skolem number $\aleph_0$ *which is not* $(\aleph_0, \aleph_1)$*-tame but is* $(2^{\aleph_0}, \infty)$*-tame.*

Summary

The true significance of first order stability theory became clear when one found a wide variety of mathematically interesting theories at various places in the stability hierarchy.

Zilber's work and $({}^{\perp}N, \prec_N)$ suggest we may find a similar future for AEC.

References

Much is on the web at www.math.uic.edu/jbaldwin including:

1. Categoricity: a 200 page monograph introducing AEC,
2. Some examples of Non-locality (with Shelah)
3. Categoricity, amalgamation and Tameness (with Kolesnikov)
4. And see Grossberg, VanDieren, Shelah

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