Math 502: SUBSTITUTION

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This account comes almost word for word from Shoenfield.

An occurence of the variable x in the formula ϕ is bound if it occurs in a part of ϕ of the form: $(\exists x)\psi$; otherwise it the occurence is free.

We say x is free (bound) in ϕ if some occurence is.

Note that if x and y are different the free occurences of x in $\neg\phi$, $\phi \wedge \psi$ and $(\exists y)\psi$ are the same as the free occurences of x in ϕ and ψ. But x is not free in $(\exists x)\psi$.

If t and a are terms $t_x[a]$ designates the expression obtained by replacing each occurence of x in t by a; for a formula ϕ, $\phi_x[a]$ designates the expression obtained by replacing each *free* occurence of x in ϕ by a.

Usually, $\phi_x(a)$ says the same thing about the element designated by a as ϕ says about the element designated by x. But consider, $\phi : (\exists y)x = 2y)$ which asserts that x is even. $\phi_x[2y+1]$ is false. This motivates:

Definition The term t is substitutable for x in ϕ if no variable which occurs in t, occurs in the scope of a quantifier in ϕ. Henceforth, we will use the notation $\phi_x(t)$ only when t is substitutable for x in ϕ.

These notions extend naturally to the *simultaneous substitution* of $t_1, \ldots t_n$ for $x_1, \ldots x_n$.

0.1 Lemma. *Let A be an L-structure. Suppose t is a variable free term in $L(A)$ and a is the name of t^A. If s is a term of $L(A)$ whose only variable is x, $s_x^A(t) = s_x^A(a)$; if ϕ is a formula whose only free variable is x, $A \models \phi_x(t)$ if and only if $A \models \phi_x(a)$.*

Proof. Induction on terms and then formulas. If s is a constant or a variable the result is evident. If $s = ft_1, \ldots t_n$, $s_x^A(a) = f^A t_1^A[a], \ldots t_n^A[a]$ which by induction is $f^A t_1^A[t], \ldots t_n^A[t]$ which equals $s_x^A(t)$.

For formulas the induction is straightforward; the only interesting case is: $(\exists y)\psi$. Note both t and a are variable-free terms. Now $A \models \phi_x(t)$ iff $A \models (\exists y)\psi_x(t)$ iff $A \models \psi_{x,y}(t, c)$ where c names an element of A. By induction this holds iff $A \models \psi_{x,y}(s, c)$ where c names an element of A. But this is equivalent to $A \models (\exists y)\psi_x(s)$ and we finish.