Math 503: 6th problem set: Ultrafilters

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All you ever want to know about ultrafilters is in Chang-Keisler.

1. Define the ultrafilter D on I to be *principal* if for some $X \in I$, $D = \{Y : X \subseteq Y\}$. Prove D is principal iff $D = \{Y : \{i\} \in\subseteq Y\}$ for some $i \in I$. (Note the definition here makes sense for principal filters.)

2. Show that if D is principal, for any structure A, $A^I/D \approx A$.

3. Show that if A is finite, for any ultrafilter D, $A^I/D \approx A$.

4. An ultrafilter D is countably incomplete if there is countable sequence of members of D whose intersection is not in D. Show that if D is a countably complete ultrafilter on a countable I, then D is principal. Show a principal filter is countably complete.

5. An ultrafilter is α-regular if there exists a set $E \subset D$ with $|E| = \alpha$ such that each $i \in I$ belongs to only finitely many $e \in E$. Show that if D is an $|I|$-regular ultrafilter on I and A is infinite $|A^I/D| = |A^I|$. Show that for every infinite I, there is a $|I|$-regular ultrafilter on I. (Hint: Actually build the ultrafilter on the set of finite subsets of I.)