WHAT IS AN EXTENSION AXIOM?

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GENERIC STRUCTURES

standard name: Homogeneous Universal

Dense Linear Order, The random graph (Hausdorff, Rado)

$\leq$ is a relation between models which is at least as strong as substructure.

Definition. $\mathbf{K_0}$ is a class of finite relational structures. The countable model M is $(\mathbf{K_0}, \leq)$-*generic* if

i) If $A \leq M, A \leq B \in \mathbf{K_0}$, then there exists $B' \leq M$ such that $B \cong_A B'$,

ii) For finite A_i with $A_i \leq A_{i+1}$, $M = \cup_{i<\omega} A_i$.

EXTENSION AXIOMS

If $A \leq B$, every instance of A extends to an instance of B.

Dense Linear Order:

$$(\forall v_0)(\exists z)\, v_0 < z$$
$$(\forall v_0)(\exists z)\, z < v_0$$
$$(\forall v_0, v_1)(\exists z)\, v_0 < z < v_1$$

The random graph:

Axioms ϕ_k :

$(\forall v_0 \dots v_{k-1} w_0 \dots w_{k-1})(\exists z) \bigwedge_{i<k}(Rzv_i \wedge \neg Rzw_i)$

ALMOST SURE THEORIES

Fix a finite relational language L. Let $\mathbf{K_n}$ be a collection of L-structures with universe n. Let P_n be a probability measure on $\mathbf{K_n}$.

For any formula ϕ, let

$$P_n(\phi) = \Sigma\{P_n(B) : B \models \phi, |B| = n\}.$$

E.g. $\mathbf{K_n}$ is all graphs of size n; P_n is the uniform distribution (edge probability 1/2).

T is an *almost sure theory* if for some $(\mathbf{K_n}, \mathbf{P_n})$, $\phi \in T$ iff $\lim_{n\to\infty} P_n(\phi) = 1$.

0-1 law for finite graphs (Glebski et al, Fagin):

The (theory of the)random graph is almost sure with respect to the uniform distribution as each extension axiom has limit probability 1.

SIMILARITIES and DIFFERENCES

These theories are:

i) $\aleph_0$-categorical

ii) $\forall\exists$-axiomatizable

iii) quantifier eliminable

iv) unstable

Dense linear order is finitely axiomatizable; the random graph is not.

The random graph is an 'almost-sure' theory; dense linear order is not.

The random graph is simple.

Dense linear order has the strict order property.

QUESTIONS

Is there a stable but not superstable, $\aleph_0$-categorical theory?
(Lachlan 1974)

Yes (Hrushovski, 1988)

Is there a finitely axiomatizable example?

WEIGHT

Let $\mathbf{K_0}$ be a universal class of finite graphs and $\mathbf{K}$ a class of structures such that if $M \in \mathbf{K}$ then every finite substructure of $M \in \mathbf{K_0}$. Define a preweight by:

Let δ map members of $\mathbf{K_0}$ into the real numbers by

$$\delta(\langle A, R\rangle) = |A| - \alpha e(R)$$

for some α where $e(R)$ is the number of edges of the graph.

STRONG SUBSTRUCTURES

For each $N \in \mathbf{K}$ and finite $X \subseteq N$, the *weight* of X in N is

$$d_N(X) = \min\{\delta(X') : X \subseteq X' \subseteq_\omega N\}.$$

For $N \subseteq M$, we say that N is *strong* in M, and write $N \leq M$, if for all finite $X \subseteq N$,

$$d_N(X) = d_M(X).$$

THE AMALGAMATION PROPERTY

$A \longrightarrow B$ means there is a strong embedding f from A to B.

Definition. The class $(\mathbf{K_0}, \leq)$ satisfies the *amalgamation property* $(\mathbf{AP})$ if for any situation:

$$\begin{tikzcd}[row sep=tiny] & B \\ A \arrow[ur] \arrow[dr] & \\ & C \end{tikzcd}$$ there exist a D and strong embeddings,

such that $$\begin{tikzcd}[row sep=tiny] & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D. \\ & C \arrow[ur] & \end{tikzcd}$$

A, B, C, D are in $\mathbf{K_0}$; A may be empty,

Theorem. If $(\mathbf{K_0}, \leq)$ is a collection of finite relational structures that satisfies natural properties on $\leq$ and has the amalgamation property then there is a countable $\mathbf{K_0}$-generic model M.

PARAMETERS OF THE CONSTRUCTION

We consider classes $(\mathbf{K_0}, \leq)$.
$\leq$ is usually derived from a function δ_α, by

$$A \leq B \text{ if for each } B' \subseteq B, \delta_\alpha(B'/A) \geq 0.$$

By varying $\mathbf{K_0}$ and $\leq$ (that is, α) independently many important examples arise.

MODEL COMPLETENESS

A theory T is model complete if every formula $\phi(\overline{x})$ is equivalent in T to an existential formula. (Equivalently, every submodel is an elementary submodel.)

A $\forall\exists$-axiomatizable categorical theory is model complete.

The $(\mathbf{K_0}, \subseteq)$ generic for a universal class $\mathbf{K_0}$ is quantifier eliminable.

EXAMPLES

Theorem. [Ehrenfeucht, Henson, Glassmire, Ash] Let $L = \{E, P_n\}$ with E binary and P_n n-ary. Consider graphs. Let $A \leq B$ if P_n picks out a maximal complete n-graph in A which remains maximal in B. The E-reduct of the generic is model complete.

The almost sure theory of a triangle-free graph (with respect to the uniform distribution) is the complete bipartite graph, which is model complete. [Kolaitis, Rothchild, ...]

δ-INVARIANT AMALGAMATION CLASSES

Aref'ev, Baldwin, Mazzucco:

$\alpha = 1$. $\mathbf{K_0}$ chosen carefully.

Let $\mathbf{K_0}$ be a class of finite graphs;

$$\delta(A) = |A| - e(A).$$

$\mathbf{K_0}$ is δ-*invariant* if for any finite A, $A \in \mathbf{K_0}$, if and only if for every subgraph B of A, there is a $B' \in \mathbf{K_0}$ such that $v(B) = v(B')$

Theorem There are only countably many δ-invariant amalgamation classes. All are ω-stable.

i) Some are categorical; some are not.

ii) Some are model complete; some are not.

iii) Sometimes the generic is saturated; sometimes not.

$\aleph_1$-CATEGORICAL EXAMPLES

α varies. $\mathbf{K_0}$ chosen carefully; violates δ-invariance.

Theorem. For appropriate $\mathbf{K_0}$,

i) If $\alpha = 1$ the generic model is strongly minimal. (Hrushovski)

ii) If $\alpha = .5$ the generic model is an $\aleph_1$-categorical non-Desarguesian projective plane. (Baldwin)

Holland showed the first theory is model complete; Baldwin extended this result to the second.

STRONGLY MINIMAL EXTENSION AXIOMS

Holland provides the following $\forall\exists$ axioms for the new strongly minimal set D.

(Q, F) is a *primitive pair* if $\delta(Q/F) = 0$ and $\delta(Q'/F) > 0$ for all $\emptyset \neq Q' \subseteq Q$.

i) No structure of negative weight can be embedded in D.

ii) For each (minimal) primitive pair (Q, F) there is an integer $m = m_{Q,F}$ such that: there are at most m disjoint copies of Q over F in D.

iii) For all primitive pairs (Q, F), (P, G) with $QF \subseteq PG$, there is an $m' \leq m$, depending on (P, G) and a $\forall\exists$ axiom which asserts the existence of m' copies of Q over F.

FUSIONS

Holland has devised a uniform (syntactic) framework for these constructions and Hrushovski's method of fusing two strongly minimal sets to obtain a proper strongly minimal expansion.

Theorem The fusion of two model complete strongly minimal sets is model complete.

STRICTLY STABLE EXAMPLES

α irrational $0 < \alpha < 1$.

i) $\mathbf{K_0}$ chosen carefully; the theory T_α of the generic model is a strictly stable $\aleph_0$-categorical first order theory. (Hrushovski)

ii) $\mathbf{K_0}$ all graphs with hereditarily positive dimension. The theory T_α of the generic model is a strictly stable first order theory. (Baldwin-Shi)

In the first case the generic is ω-saturated. In the second, the theory has no countable saturated model. Neither theory is model complete.

NEAR MODEL COMPLETENESS

A theory T is *nearly model complete* if every formula $\phi(\overline{x})$ is equivalent in T to a Boolean combination of existential formulas.

Nearly model complete implies 1-model complete (Every 1-submodel is an elementary submodel.) A $\forall\exists\forall$-axiomatizable categorical theory is 1-model complete.

RANDOM GRAPHS

Let B be a graph with $|B| = n$. Let

$$P_n(B) = n^{-\alpha|e(B)|} \cdot (1 - n^{-\alpha})^{\binom{n}{2} - e(B)}.$$

Let α be irrational $0 < \alpha < 1$.

Theorem. [Spencer-Shelah] For each first order sentence ϕ, $\lim_{n\to\infty} P_n(\phi)$ is 0 or 1.

Theorem.[Baldwin-Shelah] The almost sure theory is stable and nearly model complete. (It does not have the finite cover property.)

Baldwin and Shelah have extended this result to random expansions of successor.

CONTEXT

For any finite graph,

$$\delta_\alpha(A) = |A| - \alpha \cdot e(A).$$

$\mathbf{K}_\alpha$ is the collection of finite graphs A with $\delta_\alpha(A)$ *heredetarily* positive.

$\mathbf{K}$ is collection of graphs whose every finite substructure is in $\mathbf{K}_\alpha$.

$A \leq B$ if for every B' with $A \subset B' \subseteq B$, $\delta_\alpha(B'/A) > 0$.

Fact. $(\mathbf{K}_\alpha, \leq)$ has the amalgamation property (by no edges amalgamation).

INTRINSIC CLOSURE

Definition. For $A, B \in S(\mathbf{K}_\alpha)$, $A \leq_i B$ if $A \subseteq B$ but there is no A' with $A \subseteq A' < B$. If $A \leq_i B$, we say B is an *intrinsic* extension of A.

Definition.

i) For any $M \in \mathbf{K}$, any $m \in \omega$, and any $A \subseteq M$,

$$\mathrm{cl}_M^m(A) = \cup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$

ii) $\mathrm{cl}_M(A) = \cup_m \mathrm{cl}_M^m(A)$.

No edges amalgamation: If $B \cup C \subseteq M$ and $B \cap C = A$ then B and C are freely joined over A if M imposes no relations on $B \cup C$ other than those on B or C.

SEMIGENERICITY

The countable model M is $\mathbf{K}_\alpha$-semigeneric, if

i) $M \in \mathbf{K}$

ii) If $A \leq B \in \mathbf{K}_\alpha$ and $g : A \mapsto M$, then for each finite m there exists an embedding $\hat{g}$ of B into M which extends g such that

(a) $\mathrm{cl}^m_M(\hat{g}B) = \hat{g}B \cup \mathrm{cl}^m(A)$;

(b) $cl^m_M(gA)$ and $\hat{g}B$ are freely joined over gA.

GENERICITY VS SEMIGENERICITY

Theorem. The theory of the $\mathbf{K}_\alpha$-semigeneric models is nearly model complete. More precisely, If T_1 is an extension of the theory of the $\mathbf{K}_\alpha$-semigenerics such that for any models M, N of T_1, $\mathrm{cl}_M(\emptyset) \equiv \mathrm{cl}_N(\emptyset)$, then T_1 is complete.

Theorem. The $\mathbf{K}_\alpha$-generic model is $\mathbf{K}_\alpha$-semigeneric and stable.

(The second result is not true when working with built-in successor.)

Π_3-EXTENSION AXIOMS

There exist $\forall\exists\forall$ formulas $\phi^m_{A,B,C}$ such that the structure $N \in \mathbf{K}$ is semigeneric if and only if for each $A \leq B$ and $C \in \mathcal{D}_A$ and each $m < \omega$, $N \models \phi^m_{A,B,C}$

Theorem. If $A \leq B$ and $A \leq_i C$ with $|C - A| < m$ then

$$\lim_{n \to \infty} P_n(\phi^m_{A,B,C}) = 1.$$

With probability 1, the closure of the empty set is empty.

With built-in successor, the closure of the empty set is not empty.

GENERIC STRUCTURES: PAST AND FUTURE

This method has been applied to:

i) exotic stable theories

ii) random models

iii) applications to finite model theory

iv) specialized algebraic examples

v) infinitary classification theory

Further directions:

i) Study finite approximations of generics.

ii) Explore the consequences of 'generic'.

iii) Construct further algebraic examples.

iv) Find new simple theories.

SOME SPECIFIC QUESTIONS

i) Is there a strictly superstable generic?

ii) Can one construct a rank two field?

iii) Can the Hrushovski strictly stable $\aleph_0$-categorical theory be realized as an almost sure theory?

iv) Is there an $\aleph_0$-categorical strictly stable group?

RANDOM EXPANSIONS OF GEOMETRIES

Let $\mathbf{K_0} = \cup_{\mathbf{n}<\omega}\mathbf{K_n}$ where K_n is

the set of expansions of (n, S) by a symmetric binary relation;

or the set of expansions of an n-dimensional vector space over a finite field by a symmetric binary relation.

In either case, define a probability P_n with edge probability $n^{-\alpha}$.

Theorem.[Shelah, Baldwin] In either case, when α is irrational and $0 < \alpha < 1$, there is a 0-1 law for first order sentences on $\mathbf{K_0}$. Moreover, the almost sure theory is stable.

OBSTRUCTIONS

i) $\delta(B/A)$ may be 0.

ii) Amalgamations may identify.

iii) $\mathrm{cl}(\emptyset)$ may not be empty.

iv) K_n is not closed under substructure.

SOLUTION: 0-1 LAW

Essentially due to Shelah.

Define away Obstruction i);

deal with obstructions ii) and iii) by varying the argument;

ignore obstruction iv).

Define away means $A \leq B$ if $\delta_\alpha(B/A) > 0$.

Variances: Show $\mathrm{cl}_M(\emptyset) \equiv \mathrm{cl}_N(\emptyset)$ for M, N semigeneric.

Count only 'appropriate' substructures.

Don't construct generic; a 'q.e. argument'.

SOLUTION: STABILITY

The major difficulty is obstruction iv).