Foundations
A Model Theoretic Perspective

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January 2, 2009

What are foundations?

Steve Simpson

If X is any field of study, "foundations of X" refers to a more-or-less systematic analysis of the most basic or fundamental concepts of field X.

I add

If X is a mathematical subject, the foundations of X also include an investigation of the basic methodologies and proof techniques of the subject.

Traditional Foundation

1. A single foundation for all mathematics;
2. Goal is guaranteeing truth via derivability.

Weaknesses

1. Coding into a single system (e.g. SO arithmetic or ZFC) obscures mathematical meaning.
2. The focus on derivability obscures the nature of proof.

Derivability asks, 'Is there a sequence of correct deductions from A to B. It does not ask about clarity, irredundance, or ideas.'

Foundations of Algebraic Geometry

Fundamental Notions

curve, surface, variety, genus, Zariski topology
abstract variety, manifold, point, generic point,
scheme, cohomology

Methods

algebraic vrs transcendental
induction on dimension

We will return to model theoretic explanations of some of these phenomena.

foundationS of mathematics

Thesis I:

Studying the model of different(complete first order) theories provides a framework for the understanding of the foundations of specific areas of mathematics.

This study cannot be carried out by interpreting the theory into an über theory such as ZFC; too much information is lost.

The choice of fundamental concepts

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We have isolated the fundamental concepts for studying a tame structure M if we can choose a recursive language L such that:

1. M admits quantifier elimination as an L-structure.
2. M admits elimination of imaginaries as an L-structure.

E.g. Study orderable fields with the order.
Arithmetic is not tame.

Mid-Atlantic Model Theory 2008

1. Model theory and non-archimedean geometry
2. The valuation inequality for complex analytic structure
3. Cherlin's Conjecture and Generix's Adventures in Groupland
4. ω-stable semi-Abelian varieties
5. O-minimal triangulation respecting a standard part map
6. Some modest attempts at defining the notions of groups and fields of dimension one, and establishing their algebraic properties
7. Dependent theories: limit model existence and recounting the number of types
8. The non-elementary model theory of analytic Zariski structures
9. Difference fields, model theory and applications
10. Model Theory of the Adeles

Three types of model theoretic analysis:

1. Properties of first order logic (1930-1965)
2. Properties of complete theories (1950-present)
3. Properties of classes of theories (1970-present)

Properties of first order logic (1930-1965):

1. Completeness and Compactness
2. Lowenheim-Skolem
3. syntactic characterization of preservation theorems
4. Interpolation

Complete Theory

A theory T is complete if for every sentence ϕ,

$$T \vdash \phi$$

or

$$T \vdash \neg\phi$$

Note that for any structure M,

$$\mathrm{Th}(M) = \{\phi : M \models \phi\}$$

is a complete theory.

Properties of complete theories (1950's):

1. Complete Theories (A. Robinson)
2. Elementary Extension (Tarski-Vaught)
3. Models Generated by Indiscernibles (Ehrenfeucht-Mostoski)
4. quantifier elimination and model completeness (Robinson and Tarski)

Algebraic examples: complete theories

Algebraic Geometry

Algebraic geometry is the study of definable subsets of algebraically closed fields
Not quite: definable by positive formulas

Chevalley-Tarski Theorem

Chevalley: The projection of a constructible set is constructible.
Tarski: Acf is admits elimination of quantifiers.

Algebraic consequences for complete theories

1. Artin-Schreier theorem (A. Robinson)
2. Decidability and qe of the real field (Tarski)
3. Decidability and qe of the complex field (Tarski)
4. Decidability and model completness of valued fields (Ax-Kochen-Ershov)
5. quantifier elimination for p-adic fields (Macintyre)

On Mathematical methodology

Thesis II:

Studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics.

Properties of classes of theories (1970-present)

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

The stability hierarchy: examples

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields, complex compact manifolds

strictly superstable

$(\mathcal{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$,

strictly stable

$(\mathcal{Z}, +)^\omega$, separably closed fields,

unstable

Arithmetic, Real closed fields, complex exponentiation, random graph

Two Mathematical Methodologies

1. chain conditions: topology, ring theory, group theory, commutative algebra
2. dimension: vector spaces, fields, fields, algebraic geometry, analysis, ...

Methodology: The descending chain condition

General form: There is no proper infinite descending sequence of X.
e.g. X might be ideals in rings or closed sets in some topology.
This looks like a second order condition.

Methodology: Definable descending chain conditions

1. For arbitrary rings: definable ideals
2. The Zariski topology – fundamental tool of algebraic geometry
3. and many more

Methodology: Definable descending chain conditions

1. Stability conditions imply definable descending chain conditions.
2. Definable descending chain conditions imply mathematical consequences.

Example: descending chain conditions in Ring Theory

Artin-Wedderburn Theorem

If the Jacobson radical of R is 0 and R satisfies the descending chain condition on left ideals then R is a direct sum of matrix rings.

Stable version

If the Jacobson radical of R is 0 and R is stable then R is a direct sum of matrix rings.

This allows the extension of the idea to suitable classes of groups.

Methodology: Dimension

The 'dimension theory' of vector spaces or algebraically closed fields can be generalized.

1. A first order theory is categorial in one/all uncountable cardinalities iff each model is controlled by definable subset that a very good dimension function.
2. A first order theory is stable iff every model has a good dimension function.
3. A first order theory is superstable iff each model locally has very good dimension functions.

Foundations for various areas of mathematics

1. algebraically closed fields (ω-stable) and the notion of 'generic' in geometry
2. 0-minimality and real exponentiation
3. stability and definable chain conditions
4. quasiminimality and complex exponentiation
5. non-standard analysis

Further model theoretic tools

1. Local dimension
2. orthogonality
3. geometrical stability
4. canonical bases and the elimination of imaginaries
5. Zariski Geometries

In another direction

Many mathematics educators use the phrase ‘variable quantity’.
I regard this as an oxymoron.
Can anyone suggest (privately) sources on this issue.