STABILITY AND EMBEDDED FINITE MODELS

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TOPICS

I. Embedded Finite Model Theory

II. Type-definability and extending permutations

III. Indiscernibles and collapse theorems

QUERIES

Let $\mathbf{U}$ be an infinite set, and let S be a finite relational language.

A *query* on $\mathbf{U}$, Q, is a collection of S-structures with domain a finite subset of $\mathbf{U}$.

A *generic query* on $\mathbf{U}$ is a query that is closed under S-isomorphism.

An *abstract query* Q is a collection of isomorphism types of finite S-structures.

Given an abstract query Q and set $\mathbf{U}$, Q defines a generic query $Q_{\mathbf{U}}$ by considering all finite S-structures embedded in $\mathbf{U}$ whose isomorphism type is in Q.

EMBEDDED FINITE MODEL THEORY

Let L be a language disjoint from S, and let M be an L-structure with domain $\mathbf{U}$.

For any (finite) S-structure A with domain contained in $\mathbf{U}$, let $M(A)$ denote the unique $L \cup S$-structure that expands M and agrees with A on the interpretation of the predicates in S.

Such an A is what we mean by an *embedded finite model.*

Given a first-order sentence ϕ in $L \cup S$, the *query defined by* ϕ is the set of finite S-structures A with $A \subset \mathbf{U}$ such that $M(A) \models \phi$.

ϕ is generic if the query defined by it is generic.

A $\mathbf{U}$-query Q is *first-order over* M if there is $\phi \in L \cup S$ that defines Q.

An abstract query Q, is first-order over M if there is a first-order $\phi \in L \cup S$ that defines $Q_{\mathbf{U}}$.

GENERIC COLLAPSE TO EQUALITY

Theorem 1 *Let* $\mathbf{U}$ *be infinite and* M *be a stable* L*-structure with domain* $\mathbf{U}$*. If* Q *is a first-order definable generic query over* M*, then* Q *is a pure first-order query.*

UNSTABLE THEORIES

A model M is *unstable* if it admits a parametrically definable linear-order (on k-tuples). That is, there is a $\phi(\vec{x}, \vec{y})$ such that it is consistent with theory of M that there are $\langle \vec{a}_n : n \in \omega \rangle$ such that

$$\phi(\vec{a}_m, \vec{a}_n) \leftrightarrow m < n.$$

A model M has the independence property if it admits a parametrically definable family of independent sets. That is, there is a $\phi(\vec{x}, \vec{y})$ such that it is consistent with theory of M that there are $\langle \vec{a}_n : n \in \omega \rangle$ and every Boolean combination of the sets $\phi(\vec{x}; \vec{a}_n)$ is consistent.

If M has the independence property then M is unstable.

ORDER QUERIES

An abstract order query is a collection of linearly ordered $S \cup \{<\}$-structures which is closed under $\{S, <\}$-isomorhism.

A *local map* is a partial function on a set $\mathbf{U}$.

A query Q on $\langle \mathbf{U}, < \rangle$ is *locally order generic* if it is preserved by local order-preserving maps on $\langle \mathbf{U}, < \rangle$.

For any infinite linearly ordered structure O and abstract query Q, the query Q_O induced on O is locally order generic. Conversely, any locally order generic query naturally determines an abstract order query.

Let L be a language disjoint from S and let M be an $L \cup \{<\}$-structure with domain $\mathbf{U}$.

An abstract order query Q is *first-order over* M if there is a $\phi \in L \cup \{S, <\}$ that defines Q.

An abstract order query Q is *order definable* if it is defined over the structure $\langle \mathbf{U}, < \rangle$ by an $\{S, <\}$-formula ψ.

GENERIC COLLAPSE TO ORDER

Theorem 2 *Let* $(\mathbf{U}, <)$ *be an infinite linear order and* M *an* $L \cup \{<\}$*-structure that expands* $(\mathbf{U}, <)$*, which does not have the independence property. If* Q *is a first-order definable locally order generic query over* M*, then* Q *is an order-definable query.*

Theorem 3 *Let* $\mathbf{U}$ *be infinite and* M *be an* L*-structure with domain* $\mathbf{U}$ *that does not have the independence property. If* Q *is a first-order definable generic query over* M*, then* Q *is definable over any dense linear order without endpoints.*

A model satisfying the second sentence of Theorem **??** is said to have *generic collapse to order*.

APPLICATIONS

Corollary 4 *Let* $\mathbf{U}$ *be infinite and* M *be an* L*-structure with domain* $\mathbf{U}$ *that does not have the independence property. If* Q *is a first-order definable generic query over* M*, then the abstract ordered query defined by* Q *(i.e the class of linearly ordered expansions of isomorphism types in* Q*) is definable by a formula in* $\{S, <\}$*.*

Corollary 5 *Let* S *contain a single binary relation symbol. If* M *does not have the independence property, then the following queries are not first-order definable over* M*: parity, transitive or deterministic transitive closure of a graph; maximal matching in a bipartite graph; Eulerian cycle.*

PREDICATES FOR INDISCERNIBLES

Let M model the L-theory T and I be either an indiscernible set in M or an order-indiscernible sequence in M with dense order type. In the latter case, Δ is $\{=,<\}$, and in the former case Δ is $\{=\}$.

- L^+ is $L \cup \Delta$ plus a unary predicate P.
- M^+ is M expanded to interpret P by I and $<$ by the ordering of the indiscernible sequence.
- T^+ denotes the theory of (M, I).

PSEUDOSMALL MODELS

The structure M^+ is *small* if further

- M is $|I|^+$-saturated.
- if $\Delta = \{= \ . \ <\}$, then the linear-order is complete.

A structure for L^+ that is elementarily equivalent to a small (M, I) is *pseudo-small*.

The theory T^+ of a small model is called a *pseudo-small theory*.

DEFINABILITY IN L^+

Theorem 6 *For every L^+-formula $\phi(\vec{x}, \vec{y})$ there is a Δ-formula $\psi(\vec{w}, \vec{y})$ such that for every $\vec{m}$ there is a $\vec{c}_{\vec{m}} \in I$ such that*

$$\forall \vec{y} \in P[\psi(\vec{c}_{\vec{m}}, \vec{y}) \leftrightarrow \phi(\vec{m}, \vec{y})].$$

That is, all L^+-types over I are Δ-definable.

This result is accomplished by a long induction from the following base.

Assumption 7 *For every basic formula $\phi(\vec{x}, \vec{y})$ there is a quantifier free Δ-formula $\psi(\vec{w}, \vec{y})$ such that for every model (M, I) of T^+ if I is a set of (Δ, L)-indiscernibles in M then for every $\vec{m} \in M$ there is a $\vec{c}_{\vec{m}} \in I$ such that*

$$\forall \vec{y} \in P[\psi(\vec{c}_{\vec{m}}, \vec{y}) \equiv \phi(\vec{m}, \vec{y})].$$

This assumption is an immediate consequence of stability; in the NIP case, we prove it.

QUANTIFIER REDUCTION

Theorem 8 (The Main Reduction Theorem) *Let T be a stable theory (a theory without the independence property) and let T^+ be a pseudo-small extension of T. For each formula $\phi(\vec{x}) \in L^+$, there is an equivalent P-bounded formula $\phi'(\vec{x})$. That is,*

$$T^+ \models \forall \vec{x}\ [\phi(\vec{x}) \leftrightarrow \phi'(\vec{x})].$$

This result is proved by induction on quantifier rank by bootstapping with the definability of types result.

BACK AND FORTHS OVER g

We write written $A \approx_{L,p} B$ if two L-structures A and B are back-and-forth equivalent.

Notation 9 *Let I be a subset of N and let L^* be the expansion of L^+ by constants $\{c_\alpha : \alpha < |I|\}$. Fix a permutation g of I and an enumeration $\{a_\alpha : \alpha < |I|\}$ of I. Then N_1^g is the expansion of N where c_α is interpreted as a_α and N_2^g is the expansion of N where c_α is interpreted as $g(a_\alpha)$. We will omit the g if it is fixed in context.*

Note that $N_1^g \approx_{L^*,p} N_2^g$ implies that the partial isomorphisms in the back-and-forth system can be taken to all extend g.

EXTENDING MAPS ON INDISCERNIBLES

Theorem 10 *Let N be a model that is stable (does not have the independence property) and I be a set (sequence) of (order) indiscernibles. Suppose that (N, I) is pseudo-small and that (N, I) is ω_1-saturated.*

For any permutation g on I (order-preserving permutation of I),

$$N_1^g \approx_{L^*,p} N_2^g.$$

PROOF OF EXTENDABILITY

For simplicity I consider only the stable case: $\Delta = \{=\}$.

We shall show the collection of pairs of finite sequences $\langle \vec{a}, \vec{b} \rangle$ such that $\langle N_1, \vec{a} \rangle$ is L^*-elementary equivalent to $\langle N_2, \vec{b} \rangle$ form a back-and-forth system. So suppose $\langle N_1, \vec{a} \rangle$ is L^*-elementarily equivalent to $\langle N_2, \vec{b} \rangle$, and a' is an element of N_1. We shall find b' such that $\langle N_1, \vec{a}, a' \rangle$ is L^*-elementarily equivalent to $\langle N_2, \vec{b}, b' \rangle$.

It suffices to find a b' such that for each $\rho(x, \vec{w}, \vec{v})$ in L^+, the g-image of the $\rho(x, \vec{w}, \vec{v})$-type of $a', \vec{a}$ over I is the same as the $\rho(x, \vec{w}, \vec{v})$-type of $b', \vec{b}$ over I.

By Corollary **??**, we know that there is a quantifier-free Δ-formula $\gamma_\rho(\vec{v}, \vec{v}')$ such that the ρ-type of $a', \vec{a}$ over I is definable by an instance of γ_ρ. That is, there is $\vec{i}_\rho \in I$ such that for each $\vec{c} \in I$ $\gamma_\rho(\vec{c}, \vec{i}_\rho)$ holds if and only if $\rho(a', \vec{a}, \vec{c})$. Let $\vec{j}_\rho$ be $g(\vec{i}_\rho)$.

Since $\langle N_1, \vec{a}\rangle$ is L^*-elementarily equivalent to $\langle N_2, \vec{b}\rangle$, and

$$\langle N_1, \vec{a}\rangle \models \exists x \forall \vec{v} \in P(\gamma_\rho(\vec{v}, \vec{i}_\rho) \leftrightarrow \rho(x, \vec{w}, \vec{v})),$$

we have

$$\langle N_1, \vec{b}\rangle \models \exists x \forall \vec{v} \in P(\gamma_\rho(\vec{v}, \vec{j}_\rho) \leftrightarrow \rho(x, \vec{w}, \vec{v})).$$

Let I_0 be the union of the ranges of all the sequences $\vec{i}_\rho$ and let $q(x)$ be the type over $I_0 \cup \{\vec{a}\}$ containing the formulas

$$\forall \vec{v} \in P(\gamma_\rho(\vec{v}, \vec{i}_\rho) \leftrightarrow \rho(x, \vec{a}, \vec{v}))$$

for all L^+-formulas ρ.

Then I_0 is countable and the type $g_1(q)$ (where $g_1 = g \cup \langle \vec{a}, \vec{b} \rangle$) over the countable set $J_0 \cup \{\vec{b}\}$, where $J_0 = g_1(I_0)$, is also consistent. So by ω_1-saturation of (N, I) we can choose b' to realize $g_1(q)$ and complete this part of the back-and-forth. The other direction of the back-and-forth is done by a symmetric argument, completing the theorem. $\square_{\textbf{??}}$

CONCENTRATION LEMMA

For any two S-structures A and B we write $A =_{n,S,\Delta} B$ or say the structures are n, S, Δ *equivalent* if the two structures agree on the first n sentences of the language with vocabulary $\{S \cup \Delta\}$.

Lemma 11 *Let I be a set (Δ, L) indiscernibles in a structure M with universe* $\mathbf{U}$. *Suppose there are finite $\Delta \cup S$-interpretations $\vec{A}_n$, $\vec{B}_n$ with* $\operatorname{dom} A_n, \operatorname{dom} B_n \subseteq I$ *with* $(I, \Delta, \vec{A}_n) =_{n,S} (I, \Delta, \vec{B}_n)$ *and* $(M, \vec{A}_n) \models \phi$ *and* $(M, \vec{B}_n) \models \neg\phi$.

Then there is a model M' of $Th(M)$, a set of Δ-indiscernibles I' in M' with $(M, I) \equiv_{L^+} (M', I')$, two infinite S-interpretations $\vec{T}$ and $\vec{U}$ with domains in I', and a map f on I' such that f is an isomorphism from $(I', \vec{T}, \Delta)$ to $(I', \vec{U}, \Delta)$ but $M'(\vec{T}) \models \phi$ and $M'(\vec{U}) \models \neg\phi$.

TWO REMARKS

Remark 12 *If M' and M are elementarily equivalent L-structures and the abstract query Q is defined by ϕ over M, then Q is also defined by ϕ over M'.*

Remark 13 *Since the relativization of any Δ-sentence to $\operatorname{dom} \vec{A}$ can be expressed as a sentence about $(\mathbf{U}, \vec{A}_n)$, for any A_n, B_n with $(\mathbf{U}, \vec{A}_n) =_{n,S,\Delta} (\mathbf{U}, \vec{B}_n)$ there are A'_n, B'_n with finite domains contained in $\mathbf{U}$ such that $A'_n =_{n,S,\Delta} B'_n$. The converse also holds.*

PROOF OF GENERIC COLLAPSE

Theorem 14 *Let $M \models T$ where T is stable. Every generic query Q which is $L \cup S$-definable by a formula ϕ in M is S-definable in M.*

Proof. If Q is not defined by any formula θ in S, there are finite relations $\vec{A}_n$, $\vec{B}_n$ with $\operatorname{dom} A_n, \operatorname{dom} B_n \subseteq \mathbf{U}$ with $(\mathbf{U}, \vec{A}_n) =_{n,S} (\mathbf{U}, \vec{B}_n)$ and $M(A_n) \models \phi$ and $M(B_n) \models \neg\phi$. By Remark **??** we have (possibly rechoosing the A_n, B_n) that $A_n =_{n,S} B_n$.

Fix a countable set of indiscernibles $I = \{i_0, i_1, \ldots\}$ in M. By Remark **??**, we can replace M by an $|I|^+$ saturated model and thus assume (M, I) is pseudo-small.

Let $\vec{A}'_n$ ($\vec{B}'_n$) be the image of $\vec{A}_n$ ($\vec{B}_n$) under an injective map taking the domain of $\vec{A}_n$ ($\vec{B}_n$) to finite subsets of I. Then we have $\vec{A}'_n =_{n,S} \vec{B}'_n$, $M(\vec{A}'_n) \models \phi$ and $M(\vec{B}'_n) \models \neg\phi$.

The first statement follows from the fact that we are taking isomorphic images of all relations in S, and the second from the genericity of ϕ. Reversing the use of Remark **??**, we may assume, $(I, \vec{A}'_n) =_{n,S} (I, \vec{B}'_n)$.

We now have the hypotheses of the concentration lemma with $\Delta = \{x = y\}$ (for $(I, \vec{A}'_n), (I, \vec{B}'_n)$); we apply that result to get a model (M', I'). Let the infinite S-structures $\vec{T}, \vec{U}$ and the function f on I' witness the conditions on M' guaranteed in the concentration lemma.

Let N^* be an ω_1-saturated model of $\mathrm{Th}(M', I', \vec{T}, \vec{S}, f)$. So in N^* we have $(I, \vec{U})$ and $(I, \vec{T})$ are S-isomorphic but $\phi(\vec{\mathbf{T}})$ and $\neg\phi(\vec{\mathbf{U}})$ hold; moreover, $N^+ = N^*|L^+$ is pseudo-small. Expand N^+ to models N_1 and N_2 for the language L^* as defined in Notation **??**.

Now, we apply Theorem **??**: this tells us exactly that there is a family of partial-isomorphisms between N_1 and N_2 that each extend f. Since f is an S-isomorphism and $\vec{U}, \vec{T}$ are contained in I, the back and forth system shows, in fact, that $(N_1, \vec{T})$ and $(N_2, \vec{U})$ are $L^* \cup S$-elementarily equivalent.

Now we have a contradiction, since the reducts of these models to $L \cup S$ disagree on the sentence ϕ. $\square_{\mathbf{??}}$