THEORIES IN FINITE MODEL THEORY

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FINITE MODEL THEORY

I. Descriptive Complexity

II. Logic on Finite Models

III. 0-1 laws

I and II concern LOGIC
III provides the ALMOST SURE THEORY

Dawar, Weinstein, Lindell, Grohe, Otto, ... began the development of 'Vaught-style' model theory for finite variable logic.

The developments described are heavily influenced by the work in abstract infinitary model theory carried on by e.g. Grossberg, Hyttinen, Lessmann, Shelah.

The work is primarily due to Hyttinen, Djordjević, and Hedman.

LOGICS AND THEORIES

We work in a finite relational language.

Disiderata: A theory should have many models that are 'very' similar.

That is, we want to have some kind of completeness. We will focus in this talk on:

Complete Theories in L^n

1. n-variable logic
2. pebble games

EXAMPLES

Vector spaces over a finite field (L^3-complete)

For any complete T, T_n, the L^n sentences of T.

If a first order theory is categorical in all powers, T is axiomatized by a single sentence plus an axiom of infinitity. For n large enough to include that sentence, T_n is L^n-complete.

An equivalence relation with class size specified below n.

The n-extension axioms

dense linear order

For any structure A, the L^n-theory of A.

TYPES

Let $M \models T$, $B \subseteq M, a \in M$.

$$\mathrm{tp_{L^n}(a/B, M) = \{\phi(x, b) : M \models \phi(a, b)\}}$$

where:

1. a, x, b etc are finite sequences, $b \in B$
2. $\phi(x, a) \in L^n$

NOTE BENE:

1. Even if A is finite the conjunction of the type need not be in L^n.
2. We need the parameter M.
3. We write $\mathrm{tp_{L^n}(A, M)}$ or $\mathrm{Th}_{L^n}(A, M)$ for $\mathrm{tp_{L^n}(a, \emptyset, M)}$ where a enumerates A.
4. $S_n^k(A, M) = \{\mathrm{tp_{L^n}(a, A, N) : a \in N^k}$ and $\mathrm{M \prec_n N}\}$
5. $S_n^n(T) = S_n^n(\emptyset, M)$ (any $M \models T$).

SIZE OF MODELS AND L^n-SIZE

Dawar et al call the size of an L^n theory the number of L^n-n-types. i.e. $S_n^n(T)$ is finite.

Just observing the construction of the Scott sentence of a finite model, we have:

Fact

1. If an L^n-theory T has a finite model, $S_n^n(T)$ is finite.
2. If $S_n^n(T)$ is finite, then T is axiomatized by a single sentence of L^n.

Note that for dense linear order $S_3^3(T)$ is finite but there are no finite models.

WHAT IS STABILITY THEORY?

Classify complete first order theories by counting types.

T is ω-stable if for every infinite A,

$$|S(A)| \leq |A|.$$

T is stable if for some infinite κ, $|A| = \kappa$ implies $|S(A)| \leq |A|$.

These notions can also be expressed in terms of rank functions and combinatorial properties of formulas. They are crucial tools for computing:

$$n(T, \kappa)$$

the number of nonisomorphic models of T with cardinality κ.

STABILITY in L^n

A complete L^n-theory T is *stable* if there is no L^n-formula $\phi(x, y)$ with the order property

i.e. such that in some model M of T there are sequences a_i, b_j for $i, j < \omega$ such that

$$\phi(a_i, b_j) \text{ iff } i < j.$$

Or working entirely in finite models we could talk about arbitrarily long sequences.

ω-STABILITY in L^n

A complete L^n-theory T is *ω-stable* if for every $M \models T$ and every infinite $A \subseteq M$, and every m

$$|S^m_{L^n}(A)| = |A|$$

.

Theorem. (Djordjević, Hyttinen) A stable theory in k-variable logic is ω-stable.

This is based on old rank computations by Shelah but one must be careful working with incomplete theories.

THE AMALGAMATION PROPERTY

$A \longrightarrow B$ means there is an embedding f from A to B.

Definition. The class $(\mathbf{K_0}, \leq)$ satisfies the *amalgamation property* (**AP**) if for any situation:

$$\begin{tikzcd}[row sep=small] & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

with $\mathrm{Th}_{L^n}(A, M) = \mathrm{Th}_{L^n}(A, N)$ there exists an N_1

such that $\begin{tikzcd}[row sep=small] & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$

and $\mathrm{Th}_{L^n}(A, N_1) = \mathrm{Th}_{L^n}(A, M) = \mathrm{Th}_{L^n}(A, N)$.

OBTAINING AMALGAMATION

Any complete first order theory has the amalgamation property for elementary embeddings.

There are complete L^n theories which do not have the amalgamation property for L^n-elementary embeddings.

E.g. Thomas has constructed for $n \geq 4$ theories T_n which have nonisomorphic models of the same finite cardinality and no larger models. He has similar examples with infinite models.

Further examples were constructed by Poizat, Cherlin and unintentionally by Gurevich and Shelah - multipedes.

AMALGAMATION QUESTION I

Question Is there an L^n-theory which is complete in first order logic but does not have the L^n-amalgamation property?

Rosen has observed that, if not, the work reported on below yields that no $\aleph_0$-categorical, stable theory is finitely axiomatizable.

An L^n-theory T has the *independence property* if for some L^n formula ϕ for every $n < \omega$ there exists a model M_n and elements $a_i \in M_n$ for $i < n$ such that for every $X \subset n$ each type

$$p_X = \{\phi(v, a_i) : i \in X\}$$

is consistent.

AMALGAMATION QUESTION II

Note that in the absence of amalgamation the p_X do not have to be simultaneously realizable.

Problem Construct a complete theory in L^4 which has the independence property in the above sense but such that ϕ does not satisfy the m-extension axiom for sufficiently large m.

In the absence of amalgamation, Hyttinen proves that a natural notion of ω-stability is equivalent to the conjunction of 'not the order property' and 'not the independence property'.

He calls it stability rather than ω-stability but this seems a misnomer.

L^n-SATURATED MODELS

A structure M is L^n-κ-saturated if for every $A \subset M$ with $|A| < \kappa$, every L^n-elementary extension N of M, if $a \in N$, $\mathrm{tp}_{L^n}(a, A, N)$ is realized in M.

Theorem (Hyttinen, Djordjević) If T has the L^n-amalgamation property then for any κ, every model of T can be L^n-embedded in an L^n-κ-saturated model.

Note that if T is the L^3-theory of $(Z, +, 1)$, there is no countable countably L^3-saturated model.

EXAMPLE

Consider the structure (M, E) where:

1. E is an equivalence relation.
2. There is one class of each finite cardinality.
3. There are infinitely many infinite classes.

M is L-saturated but not L^n saturated for any n.

Fix $m > n$. Let $a_0, \ldots a_{m-1}$ enumerate one equivalence class.

The L^n-type $p = \{xEa_1, x \neq a_i : i < m\}$ is omitted in M.

Note that p is inconsistent with the first order theory of $(M, \overline{a})$ but consistent with the first order theory of M.

FINITE MODELS

Theorem. (Djordjević)

1. If a stable L^k-theory T with the amalgamation property has only finitely many L^k-types then T can be extended to an L-theory which is ω-categorical and ω-stable.
2. It follows that T has arbitrarily large finite models.

PROOF OUTLINE

1. Let T^c be the theory of an L^n-saturated model.
2. Prove T^c is $\aleph_0$-categorical and stable.
3. Using ω-stablity of T, show that T^c is ω-stable.
4. The Zilber/ Cherlin-Harringon-Lachlan theorem asserts:
An ω-stable, $\aleph_0$-categorical T has the *strong finite model property*:
If $M \models T$ and $M \models \psi$ then some finite substructure of N of M models ψ.
5. Apply this result to ψ - the sentence axiomatizing the L^n theory T.

(In fact this yields arbitrarily large finite models.)

TRIVIALITY

A dependence relation is *trivial* if whenever a depends on a set B, a depends on a single element of B.

Theorem. (Hyttinen) Let T be a complete theory in L^n which has amalgamation and with $S_n^n(T)$ finite. Suppose T is stable and forking is trivial. Then, for every m there is a finite m-saturated model M of T.

(Hytinnen assumed T has arbitrarily large finite models, but this follows from Djordjević.)

INFINITE MODELS

Hedman has studied which complete first order theories admit a finite-variable axiomatization. This work has deep connections with 'involved groups'; a classical question of Abraham Robinson becomes, Are algebraically closed fields finite variable axiomatizable?

Theorem. (Hedman)

1. Eliminating functions for relations does not affect finite variable axiomatizability.
2. (in the air) If T is $\aleph_0$-categorical, T is finitely variable axiomatizable if and only T is finitely axiomatizable.
3. If T is a strongly minimal theory with trivial algebraic dependence then T is axiomatized by sentences in some L^n in a language with (infinitely many) constants.

REFERENCES

k-variable logic: Poizat JSL-82, Thomas JSL 86, Gurevich-Shelah JSL 1996

Contact Hyttinen (Helsinki), Djordjević (Upsalla), Hedman (UIC) for papers in preparation.

Dawar,Weinstein, Grohe -see home page via FMT homepage

http://speedy.informatik.rwth-aachen.de/WWW/FMT.html

I have notes, links and papers at

http://www.math.uic.edu/ jbaldwin