# TOWARDS A FINER CLASSIFICATION OF STRONGLY MINIMAL SETS

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ABSTRACT. Let $M$ be strongly minimal and constructed by a 'Hrushovski construction' with a single ternary relation. If the Hrushovski algebraization function $\mu$ is in a certain class $\mathcal{T}$ ($\mu$ triples) we show that for independent $I$ with $|I| > 1$, $\mathrm{dcl}^*(I) = \emptyset$ (* means not in dcl of a proper subset). This implies the only definable truly $n$-ary functions $f$ ($f$ 'depends' on each argument), occur when $n = 1$. We prove for Hrushovski's original construction and for the strongly minimal $k$-Steiner systems of Baldwin and Paolini that the symmetric definable closure, $\mathrm{sdcl}^*(I) = \emptyset$ (Definition 2.7). Thus, no such theory admits elimination of imaginaries. As, we show that in an arbitrary strongly minimal theory, elimination of imaginaries implies $\mathrm{sdcl}^*(I) \neq \emptyset$. In particular, such strongly minimal Steiner systems with line-length at least 4 do not interpret a quasigroup, even though they admit a coordinatization if $k = p^n$. The case structure depends on properties of the Hrushovski $\mu$-function. The proofs depend on our introduction, for appropriate $G \subseteq \mathrm{aut}(M)$ (setwise or pointwise stablizers of finite independent sets), the notion of a $G$-normal substructure $\mathfrak{A}$ of $M$ and of a $G$-decomposition of any finite such $\mathfrak{A}$. These results lead to a finer classification of strongly minimal structures with flat geometry, according to what sorts of definable functions they admit.

The original motives for this paper were to show i) (first author) (Pure) strongly minimal Steiner systems with line length greater than 3 admit neither $\emptyset$-definable quasigroups nor more generally 'non-trivial' $\emptyset$-definable binary functions (Theorem 5.13, Theorem 5.18) and ii) (second author) the original Hrushovski construction does not admit elimination of imaginaries (Theorem 4.4.1).

We proved both of these conjectures. The failure of elimination may not be surprising. Perhaps more surprising is that the proof requires a detailed analysis of actions of two kinds of subgroups of the automorphism group of a model to $T_\mu$ on the algebraic closure of a finite sets and depends on specific values of the $\mu$-function. And given this result, the surprise is the possibility to construct a function in 2 variables in a very specific circumstance (Section 4.2).

The principal innovations of this paper are i) a new characterization of elimination of imaginaries in terms symmetric definable closure (Section 2), ii) decomposing models of flat strongly minimal theories with respect to newly defined subgroups $G_I, G_{\{I\}}$ (Section 3) and using these tools to iii) demonstrate imaginaries cannot be eliminated in certain flat strong minimal sets (Section 4.1-5). Our solutions led us to formulate a finer classification of strongly minimal sets and to explore methods to vary the Hrushovski strongly minimal set construction to exhibit new classes,

make applications in combinatorics, and find automorphism groups that act more transitively.

The question of elimination of imaginaries for *ab initio* stongly minimal sets has largely lain fallow for 25 years since B. Baizhanov [Bai96] asked whether any strongly minimal theory in a finite vocabulary that admits elimination of imaginaries must be an algebraically closed field. [Hru93, p 160] observed that flat geometries obeyed the weaker 'geometric elimination of imaginaries' and this was adequate for studying the geometry. But, as for Zilber, our goal is not to classify the geometries associated with strongly minimal theories but to classify the theories.

**Notation 0.1.**     (1) $\hat{T}_\mu$ denotes a strongly minimal theory constructed with the same $\delta$, same vocabulary of one ternary relation $R$ (required to be a hypergraph), the same $\boldsymbol{L}_0$, and an appropriate $\mu$ as in the main construction in [Hru93].
  (2) $T_\mu^S$ denotes a strongly minimal theory of Steiner systems constructed with the $\delta$ and vocabulary $\{R\}$ as in [BP21]) (Definition 1.2.2).
  (3) $T_\mu$ is used for a strongly minimal theory of either sort; in both cases the geometry is *flat* but not *disintegrated (trivial)*.

We say a theory $T_\mu$ *triples* if for any good pair $C/B$ (Definition 1.1.5.(3)), $\delta(B) \geqslant 2$ and $|C| > 1$ imply $\mu(C/B) \geqslant 3$. This implies that every primitive extension of a 'well-placed' (Definition 3.5) base has at least 3 copies in the generic.

To analyze the elimination of imaginaries in *arbitrary* strongly minimal theories we introduce the notion: $a \in (s)\operatorname{dcl}^*(X)$ (Section 2). The $*$ in $a \in (s)\operatorname{dcl}^*(X)$ means every element of $X$ is used to witness that $a$ is in the (symmetric) definable closure of $X$. It is known ([Pil99], [CF04, 1.6]) that for any strongly minimal set with $\operatorname{acl}(\emptyset)$ infinite, elimination of imaginaries is equivalent to the finite set property. It follows (Lemma 2.12) that if $\operatorname{sdcl}^*(I) = \emptyset$ for some independent set $I$, $T$ fails to eliminate imaginaries. More strongly, $\operatorname{dcl}^*(I) = \emptyset$, implies there is no definable truly $|I|$-ary function. Below, $\emptyset$-definable abbreviates parameter-free definable.

**Theorem 0.2** (Main Results)**.** *Let $T_\mu$ be a strongly minimal theory as in Notation 0.1. Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.*
  (1) *If $T_\mu$ triples then $\operatorname{dcl}^*(I) = \emptyset$ and every definable function is essentially unary (Definition 2.9).*
  (2) *Without the tripling assumption, $\operatorname{sdcl}^*(I) = \emptyset$ and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly $v$-ary function.*
*Consequently, in both cases $T_\mu$ does not admit elimination of imaginaries.[1] Nevertheless the algebraic closure geometry is not disintegrated.*

The crucial tool for this result is a close study of the action of two kinds of subgroups of the automorphism group of a structure $M$: for a finite independent set $I$, the subgroups $G_I$ and $G_{\{I\}}$ of $\operatorname{aut}(M)$ fixing $I$ setwise and pointwise respectively.

Fix a subgroup $G$ of the automorphism group of a generic structure of either sort. The field-theoretic notion of normality in fields uses that the definable closure $\operatorname{dcl}(X)$ is the field (with a vector space structure) generated by a set of parameters $X$

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[1] Note that each of the conditions $\operatorname{dcl}^*(I) = \emptyset$ and $\operatorname{sdcl}^*(I) = \emptyset$ is stronger than failure to eliminate of imaginaries. For example, an affine space does not admit elimination of imaginaries; however $\operatorname{sdcl}^*(I) \neq \emptyset$ because of the addition.

to study the solutions of algebraic equations. In our situation the definable closure may contain infinitely many elements generated by definable unary functions over which we have little control. So, we define the notion of a finite $G$-normal set (Definition 3.1). Then we provide a $G$-invariant tree decomposition of such a set. The pointwise stabilizer is sufficient (using triples) for an induction on the height of decompositions of $G$-normal sets to show the absence of truly binary functions. But stronger result showing (without assuming triples) that $\mathrm{sdcl}^*(I) = \emptyset$ and thus failure of elimination of imaginaries (although possibly with non-trivial binary functions) relies on $G_{\{I\}}$ and is more complex.

Our results indicate a profound distinction at the level of definable functions rather than geometry between the 'field-like' strongly minimal sets and the known counterexamples to Zilber's conjecture, indeed, within the counterexamples. We refine the classification of strongly minimal theories (with flat geometry) by the existence of definable truly $n$-ary functions.

Cases 1) and 2) have distinct theories of the acl-geometry (of saturated models). By [EF12] and[2] [Mer20] the geometries of the countable saturated models of theories in each of 2a) and 2b) are elementarily equivalent (indeed have isomorphic localizations). Within 2) we find for dcl an essentially unary nature similar to that which distinguishes 'trivial acl' in the Zilber classification. The distinction concerns properties of dcl in $M$, not an associated acl-geometry.

We find the following:

**Remark 0.3** (Classes of Theories with flat acl-geometries)**.**

(1) **disintegrated geometry** *For any $A$, $acl(A) = \bigcup_{a \in I} \mathrm{acl}(a)$;*

(2) **strictly flat geometry** *(Definition 1.1.3. 2) $M$ is not* acl-*disintegrated but:*
    (a) *$M$ is* dcl-*disintegrated: $\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$ for independent $I$ (no $\emptyset$-definable truly $n$-ary functions);*
    (b) *$M$ is not* dcl-*disintegrated: For some $n$ there are truly $n$-ary functions:*
        (i) *$M$ is* sdcl-*disintegrated: $\mathrm{sdcl}(I) = \bigcup_{a \in I} \mathrm{sdcl}(a)$ for independent $I$ (no commutative $\emptyset$-definable truly $n$-ary functions);*
        (ii) *$\emptyset$-definable binary functions with domain $M^2$ exist; e.g. quasigroups* [Bal23][3] *and non-commutative counterexamples found here.*

(3) **Further examples: the strongly minimal set is a relativised reduct of an almost strongly minimal structure**[4]
    (a) *ternary rings* [Bal95] *that coordinatize a non-desarguesian plane*[5]. *There are parameter definable binary functions; but the ternary ring is not a composition of the 'addition' and 'multiplication' functions.*
    (b) *2-transitive strongly minimal sets* [Hru93, Proposition 18], [Bal22].
    (c) [MT19] *construct a strongly minimal set which is not flat*[6] *within a geometry of dimension 3. The $\delta$-function is no longer submodular.*

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[2]In a private communication Mermelstein showed the infinite rank case [Pao20] of Steiner has the same geometry as the original strongly minimal example.

[3]The vocabulary in the construction has two ternary relations.

[4]The universe is the algebraic closure of a strongly minimal structure.

[5]Because of the *ad hoc* nature of this construction, the methods of the current paper do not apply. The geometry has not been analyzed.

[6]More precisely, the acl-geometry is 2-ample but not 3-ample.

No theory in class 1) or 2) (and finite vocabulary) admits elimination of imaginaries. However, Verbovskiy [Ver06] has an example with elimination of imaginaries in an infinite vocabulary. Unlike those with locally modular geometries, strongly minimal theories with strictly flat (not disintegrated) geometries, like the field type, have continuum many automorphism of each countable model (Lemma 3.12).

This diversity is obtained by realizing that the 'Hrushovski construction' actually has 5 parameters: $(\sigma, \boldsymbol{L}_0^*, \epsilon, L_0, \mathbf{U})$. $\boldsymbol{L}_0^*$ is a collection of finite structures in a vocabulary $\sigma$, *not necessarily closed* under substructure. $\epsilon$ is a predimension as in Hrushovski with the requirement that it be flat. $\boldsymbol{L}_0$ is a subset of $\boldsymbol{L}_0^*$ defined using $\epsilon$. From such an $\epsilon$, one defines notions of $\leqslant$, primitive extension, and good (minimally simply algebraic) pair. The function $\mu$ counts the number of allowed realizations of a good pair. Hrushovski gave a technical admissibility condition on $\mu$ that ensured the theory is strongly minimal rather than $\omega$-stable of rank $\omega$. Fixing a class $\mathbf{U}$ of functions $\mu$ satisfying variants of this condition provides examples satisfying a wide range of combinatorial and algebraic conditions.

We show the elimination of imaginaries fails when $\boldsymbol{L}_0^*$ is the collection of finite linear spaces. As noted above, B. Baizhanov asked whether any strongly minimal theory in a finite vocabulary, that admits elimination of imaginaries defines a field. We show the most evident counterexamples do not eliminate imaginaries. But, the question of whether this can be extended when the class $\boldsymbol{L}_0^*$ is expanded to arbitrary $\forall\exists$ classes of finite structures seems wide open.

There are a number of applications of these various construction methods in universal algebra and combinatorics. Early on, [Bal95] used the general method to construct an $\aleph_1$-categorical projective plane at the very bottom of the Lenz-Barlotti hierarchy. We now describe some more recent work.

Work by Barbina and Casanovas in model theory [BC19] and by Horsley and Webb in combinatorics [HW21] centers on the constructions of Steiner triple systems that arise from Fraïssé constructions. [HW21] obtain countably homogeneous Steiner systems that omit 'good' classes of finite Steiner systems. In contrast to the model theoretically complex locally finite generics [BC19], the construction techniques here give theories that are strongly minimal, the geometric building blocks of model theoretically tame structures.

[Bal23], building on [Ste56, GW75], shows two results. 1) if the line length $k$ is a prime power then a strongly minimal Steiner system admits a 'coordinatization' by a quasigroup. We show here, as a by-product of our main construction, this coordinatization is not definable in the vocabulary $\{R\}$. 2) Nevertheless, if $k$ is a prime power, there are strongly minimal quasigroups[7] $(Q, *)$ (created in the vocabulary $\{H, R\}$) with $\boldsymbol{L}_0^*$ $\forall\exists$ axiomatizable) which induce Steiner systems.

[Bal22] generalizes the notion of cycle decomposition [CW12] from Steiner 3-systems to that of path graph for Steiner $k$-systems and introduce a uniform method of proof for the following results. For each of the following conditions, there are $2^{\aleph_0}$ families of elementarily equivalent Steiner sytems, each with $\aleph_0$ countable models and one model of each uncountable cardinal satisfying the following condition: i) (extending [CGGW10]) each Steiner triple system is $\infty$-sparse and has a uniform path graph; ii) (extending [CW12] each Steiner $k$-system (for $k = p^n$) is 2-transitive and has a uniform path graph (infinite cycles only) iii) extending [Fuj06] each is anti-Pasch (anti-mitre); iv) Items ii) and iii) have definable quasi-group structure.

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[7]As in [BC19], $H$ is the graph of the quasigroup operation and $R$ is collinearity.

Moreover, by varying $\boldsymbol{L}_0^*$ classes we can demand all models are 2-transitive. Unlike most combinatorial constructions, each example presents a *class of models* with the desired property.

Section 1 outline the general framework of *ab initio* Hrushovski strongly minimal sets and then describes the variant for Steiner system. In Section 2, we introduce subgroups $G_I$ and $G_{\{I\}}$ of aut($M$) (fixing $I$ setwise and pointwise respectively) and explain the connections of definable closure and symmetric definable closure with the elimination of imaginaries. The key tool of $G$-tree-decomposition appears in Section 3 along with the basic properties. There are now two steps in the proof. Section 4.1 proves the non-$\emptyset$-definability of truly $n$-ary functions for triplable theories, using dcl$^*$ and $G_I$. Section 4.2 shows the necessity of the tripling hypothesis. Section 4.3 describes the distinction between flowers and bouquets which is essential for Section 4.4, the proof that sdcl$^*(I) = \emptyset$ for finite independent sets in a standard Hrushovki theory using $G_{\{I\}}$. In Section 5 we adapt that proof to Steiner systems and prove that examples with line length at least 4 from [BP21] do not define quasigroups. We raise some further questions in Section 6.

## 1. Two Contexts

In Subsection 1.1 we give an axiomatic description of the properties of functions $\delta$ and $\mu$ for Hrushovski constructions of $\omega$-stable and strongly minimal sets. In Subsection 1.2 we describe the specific definitions of $\delta$ for the two main cases considered here: the original Hrushovski context and Steiner systems.

In Section 1.1 we denote our classes as $(\boldsymbol{L}, \epsilon)$ with various decorations; these conditions provide a general framework for the study of the family of flat strongly minimal sets announced in the previous paragraph. From Section 1.2 on, we write $(\boldsymbol{K}, \delta)$ to emphasize the restriction to a single ternary relation, particular choices of $\delta$ and to prepare for the variations in [Bal22].

### 1.1. **General Context**

The notions in this section are well-known, under various names. We both fix the notation used here and give some of the definitions in a greater generality needed here but not in e.g., [Hru93]). Fix a countable relational vocabulary $\tau$. We write $\boldsymbol{L}^*$ for the collection of all $\tau$-structures and $\boldsymbol{L}_0^*$ for the finite $\tau$-structures.

All constructions studied here satisfy the properties of 3.4–3.7 of [BS96] and flatness, which follow from flatness of the underlying 'predimension' function $\epsilon$ [BP21, 3.8, 3.10].

**Axiom 1.1.1.** *Let $\boldsymbol{L}_0$ be a countable subset of $\boldsymbol{L}_0^*$ that is closed under isomorphism[8]. Let $\hat{\boldsymbol{L}}_0$ be the collection containing any union of members of $\boldsymbol{L}_0$. Further $\epsilon$ is a map from $L_0^*$ into $\boldsymbol{Z}$. We require that $\boldsymbol{L}_0, \hat{\boldsymbol{L}}_0, \epsilon$ satisfy the following requirements. Let $N \in \hat{\boldsymbol{L}}_0$ and $A, B, C \in \mathbf{L_0}$ be substructures of $N$.*

 *(1) $\epsilon(\emptyset) = 0$*
 *(2) If $B \in \boldsymbol{L}_0$ and $A \subseteq B$ then $\epsilon(A) \geqslant 0$.*
 *(3) If $A$, $B$, and $C$ are disjoint then $\epsilon(C/A) \geqslant \epsilon(C/AB)$, where $\epsilon(C/A) = \epsilon(C \cup A) - \epsilon(A)$.*

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[8]In this paper $\boldsymbol{L}_0$ is closed under substructure. But this condition is relaxed to construct flat strongly minimal quasigroups in [Bal23].

(4) *If $A, B, C$ are disjoint subsets of $N \in \hat{\boldsymbol{L}}_0$ and $\epsilon(A/B) - \epsilon(A/BC) = 0$ then $r(A, ABC, C) = \emptyset$.* [9]

(5) *$\epsilon$ is flat (Definition 1.1.3)*[Hru93, BP21]*).*

(6) **Canonical Amalgamation** *If $A \cap B = C$, $C \leqslant A$ and $A, B, C \in \boldsymbol{L}_0$ there is a direct sum $G = A \oplus_C B$ such that $G \in \boldsymbol{L}_0$. Moreover, $\epsilon(A \oplus_C B) = \epsilon(A) + \epsilon(B) - \epsilon(C)$ and any $D$ with $C \subseteq D \subseteq A \oplus_C B$ is also free. Thus, $B \leqslant G$.*

Disjoint union is the canonical amalgamation for the basic Hrushovski construction and Definition 3.14 of [BP21] gives the appropriate notion satisfying Axiom 1.1.1.5 for linear spaces. Axiom 1.1.1.2 can be rephrased as: $B \subseteq C$ and $A \cap C = \emptyset$ implies $\epsilon(A/B) \geqslant \epsilon(A/C)$; so we can make the following definition.

**Definition 1.1.2.** *Extend $\epsilon$ to $d : \hat{\boldsymbol{L}}_0 \times \boldsymbol{L}_0 \to N$ by for each $N \in \hat{\boldsymbol{L}}_0$, $d(N, A) = \inf\{\epsilon(B) : A \subseteq B \subseteq_\omega N\}$, $d_N(A/B) = d_M(A \cup B) - d_M(B)$. We usually write $d(N, A)$ as $d_N(A)$ and omit the subscript $N$ when clear.*

Hrushovski defined a crucial property of the algebraic closure (pre) geometry of his geometries[10]: *flat.* [BP21] generalized the notion of flatness for a pregeometry to a general predimension function.

**Definition 1.1.3.**     (1) *Consider a class $(\boldsymbol{L}_0, \epsilon)$, $N \in \boldsymbol{L}_0$ and a sequence $F_1, \dots, F_s$ of subsets of $N$. For $\emptyset \subsetneq T \subseteq \{1, \dots, s\} = I$, we let $F_T = \bigcap_{i \in T} F_i$ and $F_\emptyset = \bigcup_{1 \leqslant i \leqslant s} F_i$. We say that $\epsilon$ is flat if for all such $F_1, \dots, F_s$ we have:*

$$(*) \quad \epsilon\left( \bigcup_{1 \leqslant i \leqslant s} F_i \right) \leqslant \sum_{\emptyset \neq T} (-1)^{|T|+1} \epsilon(F_T).$$

(2) *Suppose $(A, \mathrm{cl})$ is a pregeometry on a structure $M$ with dimension function $d$ and $F_1, \dots, F_s$ are finite-dimensional $d$-closed subsets of $A$. Then $(A, \mathrm{cl})$ is flat if $d$ satisfies equation $(*)$.*

(3) *$(A, \mathrm{cl})$ is strictly flat if it is flat but not distintegrated ($\mathrm{acl}(ab) \neq \mathrm{acl}(a) \cup \mathrm{acl}(b)$).*

What Hrushovski called *self-sufficient* closure is in the background.

**Definition 1.1.4.** (1) *We say $A \subseteq N$ is strong in $N$ and write $A \leq N$ if $\epsilon(A) \leq \epsilon(C)$ for any $C$ with $A \subseteq C \subseteq_\omega N$, where $\subseteq_\omega$ stands for 'be a finite subset'.*

(2) *For any $A \subseteq B \in L_0$, the intrinisic (self-sufficient) closure of $A$, denoted $\mathrm{icl}_B(A)$, is the smallest superset of $A$ that is strong in $B$.*

Note that $A \leq N$ if and only if $d_N(A) = \epsilon(A)$. It is well-known that when $\epsilon$ is integer valued then $\mathrm{icl}(A)$ is finite if $A$ is. The following definitions describe the pairs $B \subseteq A$ such that in the generic model $M$, $A$ will be contained in the algebraic closure of $B$.

**Definition 1.1.5.**     (1) *$A$ is a primitive extension[11] of $B$ if $B \leqslant A \cup B$, $A \cap B = \emptyset$, and there is no $A_0$ with $\emptyset \subsetneq A_0 \subsetneq A$ such that $B \leqslant B \cup A_0 \leqslant B \cup A$.*

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[9]See Definition 1.2.1 for details concerning $r$. In the general case, we count each of a sequence of predicates $R_i$ separately. We make appropriate modification for linear spaces when they are considered.

[10]A pregeometry/matroid becomes a geometry by modding out $\mathrm{cl}(\emptyset)$.

[11]In [Hru93], 0-primitive is called *simply algebraic* and good [Zie13] is called *minimally simply algebraic.*

A is a k-primitive extension *if, in addition, $\epsilon(A/B) = k$. We stress that in this definition, while B may be empty, A cannot be.*

*Sometimes primitive is used with $B \subsetneq A$ and the primitive as $A - B$. In that case we sometimes write $\hat{A}$ for $A - B$ when the disjointness is essential.*

(2) *We say that the 0-primitive pair $A/B$ is* good[12] *if every element of B is in some relation with an element of A.*

(3) *If A is 0-primitive over D and $B \subseteq A$ is such that $A/B$ is good, then we say that B is a* base *for A over D if B is a minimal subset of D such that $A/B$ is 0-primitive.*

**Remark 1.1.6.** In the Hrushovski case, the definition of $\delta$ makes clear that the base defined by minimality in Definition 1.1.5.3 is also the maximal subset of $D$ that is related to an element of $A$. This fails in the Steiner case; see Lemma 5.4.

**Definition 1.1.7.** *Good pairs were defined in Definition 1.1.5.*

(1) **Adequacy condition:** *Fix a function $\mu$ assigning to every isomorphism type $\boldsymbol{\beta}$ of a good pair $C/B$ in $\boldsymbol{L}_0$ a number $\mu(\beta) = \mu(B, C) = \mu(C/B) \geqslant \delta(B)$.*

(2) *For any good pair $(A/B)$ with $B \subseteq M$ and $M \in \hat{\boldsymbol{L}}_0$, $\chi_M(A/B)$ denotes the maximal number of disjoint copies of A over B in M. A priori, $\chi_M(A/B)$ may be 0.*

(3) *Let $\boldsymbol{L}_\mu$ be the class of structures M in $\boldsymbol{L}_0$ such that if $(B, C)$ is a good pair then $\chi_M(B, C) \leqslant \mu(B, C)$.*

(4) *$\hat{\boldsymbol{L}}_\mu$ is the universal class generated by $\boldsymbol{L}_\mu$.*

(5) *[d-closed] For $M \in \hat{\boldsymbol{L}}_\mu$ and $X \subseteq M$, X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (equivalently, for all $Y \subseteq_\omega M - X$, $d(Y/X) > 0$).*

(6) *Let $\boldsymbol{L}_d^\mu$ consist of those $M \in \hat{\boldsymbol{L}}_\mu$ such that $M \leqslant N$ and $N \in \hat{\boldsymbol{L}}_\mu$ imply M is d-closed in N.*

The restriction on $\mu$ in Definition 1.1.7.1 comes from [Hru93]. It appeared as a useful condition to guarantee the amalgamation. Surprisingly, we find the following slight strengthening plays a central role in preventing the definability of *any* truly binary functions.

**Definition 1.1.8.** *We say $T_\mu$ triples if $\delta(B) \geqslant 2$ implies $\mu(C/B) \geqslant 3$ for any good pair $C/B$ with $|C| > 1$.*

We can show that any element of $\hat{\boldsymbol{L}}_\mu$ (not just $\boldsymbol{L}_\mu$) can be amalgamated (possibly with identifications) over a (necessarily finite) *strong* substructure $D$ of $F$ with a strong extension of $D$ to a member $E$ of $\boldsymbol{L}_\mu$. This yields the following conclusions; they are largely the same as [Hru93]; in order to treat line length 3, [BP21] make the adequacy requirement Definition 1.1.7.1 apply only when $|B| \geqslant 3$ and add $\mu(\boldsymbol{\beta}) \geqslant 1$, if $\beta = \boldsymbol{\alpha}$ (Definition 1.2.6).

Recall that a *generic model* for a class $(\boldsymbol{L}_0, \leqslant)$ is an $M$ such that if $A \leqslant M$, $A \in \boldsymbol{L}_0$ and if $A, A' \leqslant M$ are isomorphic, the isomorphism extends to an automorphism of $M$.

**Conclusion 1.1.9.** *Suppose $L_\mu$ satisfies the properties described in Axiom 1.1.1 and Definition 1.1.7:*

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[12]In the Hrushovski case, it is equivalent to say if there is no $B' \subsetneq B$ such that $(A/B')$ is 0-primitive. But for linear spaces these conditions are no longer equivalent.

If $D \leqslant F \in \hat{\boldsymbol{L}}_\mu$ and $D \leqslant E \in \boldsymbol{L}_\mu$ then there is $Q \in \hat{\boldsymbol{L}}_\mu$ that embeds (possibly with identifications) both $F$ and $E$ over $D$. Moreover, if $F \in \boldsymbol{L}_d^\mu$, then $F = Q$. In particular, $(\boldsymbol{L}_\mu, \leqslant)$ has the amalgamation property, and there is a generic structure $\mathcal{G}_\mu \in \hat{\boldsymbol{L}}_\mu$ for $(\boldsymbol{L}_\mu, \leqslant)$.

The more refined conclusion of model completeness is shown in [Hol99] and for the linear space case in [BP21].

**Conclusion 1.1.10.** *Under the hypotheses of Conclusion 1.1.9 there is a collection $\Sigma_\mu$ of $\pi_2$ sentences which*

(1) *axiomatize the complete theory $T_\mu$ of the class $\boldsymbol{L}_d^\mu$, $d$-closed models in $\hat{\boldsymbol{L}}_0$.*
(2) *$T_\mu$ is model complete and strongly minimal.*
(3) *The* acl*-geometry of $T_\mu$ is flat (e.g. 1.1.3. 2)*

### 1.2. 3-hypergraphs and Linear Spaces

We now describe the main examples for this paper of the context axiomatized in Subsection 1.1. We replace $\boldsymbol{L}$ and $\epsilon$ by $\boldsymbol{K}$ and $\delta$ to indicate that properties here may depend on the specific definition of the class.

**Definition 1.2.1** (Context). (1) *The vocabulary $\tau$ contains a single ternary relation $R$. We require that $R$ is a predicate of 3-elements sets (distinct in any order).*

(2) *Let $A, B, C$ each be a subset of $D \in \boldsymbol{K}^*$. We write $R(A, B, C)$ for the collection of tuples $\mathbf{x}$ such that $D \models R(\mathbf{x})$ and $\mathbf{x}$ intersects each of $A, B, C$. (The letters may be repeated to indicate only two sets are represented.) We write $R(A, B)$ for $R(A, A \cup B, B)$. We write $r(A, B, C)$ for the number[13] of tuples (up to permutation) in $R(A, B, C)$. Finally for $A \in \boldsymbol{K}_0^*$, $r(A) = r(A, A, A)$.*

We restrict to integer valued $\delta$ which is essential (but not sufficient) to guarantee $\omega$-stability. The crucial distinction between [Hru93] and linear spaces is restricting the class of finite structures by more than the assertion that $R$ is a ternary predicate of sets (3-hypergraph).

**Definition 1.2.2.** *The choices here for $\delta$*

- *(3-hypergraph) For a finite $\tau$-structure $A$, $\delta(A) = |A| - r(A)$.*
- *(linear space)*
  (1) *A linear space is a $\tau$-structure such that 2-points determine a unique line. We interpret $R$ as collinearity. By convention two unrelated elements constitute a* trivial line.
  (2) *For $B, \ell$ subsets of $A$, we say $\ell \in L(B)$ (is supported by $B$) if $\ell$ is a maximal $R$-clique contained in $A$ and $|\ell \cap B| \geqslant 2$.*
  (3) *Let*
  $$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$
  (4) *Then $\boldsymbol{K}_0$ is the collection of finite linear spaces $A$ such that for any $A' \subseteq A$, $\delta(A') \geqslant 0$.*

Note, that as opposed to Section 1.1, we have restricted $\boldsymbol{K}_0$ both by (a different) $\delta$ and by the linear space axiom.

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[13]For the Steiner system case count lines and compute $\delta$ as in Section 5.

**Notation 1.2.3.** *Let $\delta(B/A)$ denote $\delta(A \cup B) - \delta(A)$. Suppose $A \cap B = C$ and $A, B, C \in \boldsymbol{L}_0$:*

(1) *We say $A$ and $B$ are $\delta$-independent if $\delta(A \cup B) = \delta(A) + \delta(B) - \delta(C)$.*

(2) *We say $A$ and $B$ are* fully independent *over $C$ if there are no relations involving elements from each of $A - C$ and $B - C$ and possibly $C$, i.e. $R(A - C, A \cup B, B - C) = \emptyset$.*

Just rewriting the definition, we have $\delta(A \cup B/C) = \delta(A/B \cup C) + \delta(B/C)$.

**Remark 1.2.4.** *Note that when $\delta$ just counts relations as in [Hru93], $\delta$-independence implies full independence. The situation is more complicated for linear spaces; see Definition 5.6 and Remark 5.12.*

The following useful tool is easy from the definition ([BP21, 4.7]).

**Lemma 1.2.5.** *If $C$ is $0$-primitive over $\mathfrak{D}$ with base $B \subseteq \mathfrak{D}$ and $|C - B| \geqslant 2$, then every point $b \in B$ satisfies $R(c_1, c_2, b)$ for some $c_1, c_2 \in C$ and every point in $C$ satisfies at least two instances of $R$.*

We single out an isomorphism type of a good pair (a 1-element extension, $\boldsymbol{\alpha}$) that plays a special role in the general proof that elimination of imaginaries fails. $T_\mu$ is a Steiner $k$-system if $\mu(\boldsymbol{\alpha}) = k - 2$ [BP21].

**Notation 1.2.6** (Line length)**.** *We write $\boldsymbol{\alpha}$ for the isomorphism type of the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$.*

*By Lemma 5.18 of [BP21], lines in models of the theory $T_\mu$ of a Steiner system have length $k$ if and only if $\mu(\boldsymbol{\alpha}) = k - 2$.*

## 2. Elimination of imaginaries and (Symmetric) Definable closure

In this section we study arbitrary strongly minimal sets and lay out the connections among elimination of imaginaries, definable closure and a new notion *symmetric definable closure*. Recall that [She78, III.6] introduced extensions of structures by *imaginary elements* and [Poi83] discovered the importance of a theory not needing to add them.

**Definition 2.1.** *Elimination of Imaginaries:*

(1) *[Poi83] A theory $T$ admits* elimination of imaginaries *if for every model $M$ of $T$, for every formula $\varphi(\mathbf{x}, \mathbf{y})$ and for every $\boldsymbol{a} \in M^n$ there exists $\mathbf{b} \in M^m$ such that*

$$\{f \in \mathrm{aut}(M) \mid f|\mathbf{b} = id_{\mathbf{b}}\} = \{f \in \mathrm{aut}(M) \mid f(\varphi(M, \boldsymbol{a})) = \varphi(M, \boldsymbol{a})\}$$

(2) *[Poi85, Theorem 16.15] A theory $T$ admits weak elimination of imaginaries if and only if for every formula $\phi(\mathbf{x}; \boldsymbol{a})$ there exists a formula $\psi_{\boldsymbol{a}}(\mathbf{x}; \mathbf{y})$ such that there are only finitely many parameters $\mathbf{b}_1, \ldots, \mathbf{b}_n$ such that each of $\psi_{\boldsymbol{a}}(\mathbf{x}; \mathbf{b}_1), \ldots, \psi_{\boldsymbol{a}}(\mathbf{x}; \mathbf{b}_n)$ is equivalent to $\phi(\mathbf{x}; \boldsymbol{a})$.*

Weak elimination arose in [Poi85]; we use the precise version of [Tsu93, Def. 2.3].

**Definition 2.2.** A finite set $F = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ of tuples from $M$ is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over $A$ if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut}(M/A).$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples $F$ is coded by some set $S$ over $\emptyset$.

Part 1) of the next result appears in [Pil99]. Part 2) is [CF04, 1.6]. [Yon22, Yon23] clarifies the relation among variants of elimination of imaginaries and shows Pillay's assumption of infinite algebraic closure is essential.

**Fact 2.3.**     (1) *Every strongly minimal theory such that* $\mathrm{acl}(\emptyset)$ *is infinite has weak elimination of imaginaries.*
     (2) *If $T$ admits weak elimination of imaginaries then $T$ satisfies the finite set property if and only if $T$ admits elimination of imaginaries.*

Immediately from Fact 2.3, since in almost all[14] in the examples studied here $\mathrm{acl}(\emptyset)$ is infinite [BP21, Fact 5.26], if $T$ admits elimination of imaginaries there is an $\emptyset$-definable truly binary function (given by the coding of a pair of independent points).

Note that elimination of imaginaries immediately yields finite coding, so in proving that elimination of imaginaries fails it suffices to prove finite coding fails. Section 4.2 exhibits a strongly minimal theory with a truly binary function that still fails to (by Section 4.1) eliminate imaginaries, as it is not commutative.

*Below $X$ denotes an arbitrary subset of a structure $M$ and $I$ denotes an $v$-element independent set $\{a_1, \ldots, a_v\}$ with $I \leqslant M$.*

We work with two groups of automorphisms; Section 3 treats properties that hold of both of them so the group is denoted $G$. Section 4.1 uses $G_I$ and $G_{\{I\}}$ is needed in Section 4.4.

**Notation 2.4.** *Let $G_{\{I\}}$ be the set of automorphisms of $M$ that fix $I$ setwise and $G_I$ be the set of automorphisms of $M$ that fix $I$ pointwise.*

**Definition 2.5.** *For $G$ either $G_I$ or $G_{\{I\}}$, $D$ is said to be $G$-invariant if $D$ contains the $G$ orbits of each of its elements, equivalently, $g(D) = D$ whenever $g \in G$.*

We introduce the (minimal) definable closure $\mathrm{dcl}^*$ of a set $X$ to distinguish points which depend on *all* elements of $X$. Recall that for any first order theory $T$, if $X \subseteq M \models T$, then $c \in \mathrm{dcl}(X)$ means $c$ is the unique solution of a formula with parameters in $X$. This implies the orbit of $c$ under $\mathrm{aut}_X(M)$ consists of just $c$ and the converse holds in any $\omega$-homogenous model. All the models considered here are $\omega$-homogeneous (since $\aleph_1$-categorical [BL71]).

**Notation 2.6.** *By $b \in \mathrm{dcl}^*(X)$ we mean $b \in \mathrm{dcl}(X)$, but $b \notin \mathrm{dcl}(U)$ for any proper subset $U$ of $X$ (and analogously for $\mathrm{acl}^*$). Note that $\mathrm{dcl}^*(X)$ consists of the subset of $\mathrm{dcl}(X)$ of elements not fixed by $G_T$ for any $T \subsetneq X$.*

The notion of symmetric definable closure, $\mathrm{sdcl}(I)$, captures one direction of finite coding.

**Notation 2.7.** *The* symmetric definable closure *of $X$, $\mathrm{sdcl}(X)$, is those $a$ that are fixed[15] by every $g \in G_{\{X\}}$. $b \in \mathrm{sdcl}^*(X)$ means $b \in \mathrm{sdcl}(X)$ but $b \notin \mathrm{sdcl}(U)$ for any proper subset $U$ of $X$.*

Note $\mathrm{sdcl}^*(X) \subseteq \mathrm{dcl}^*(X) \subseteq \mathrm{dcl}(X)$. However, $\mathrm{sdcl}(X)$ may not be contained in $\mathrm{dcl}^*(X)$.[16]

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[14][Bal22] gives general conditions that imply $\mathrm{acl}(\emptyset)$ is infinite or empty.

[15]This suffices in our context since all models are homogenous; more generally, it is easy to give a 'there exists a formula' definition.

[16]For a simple example, consider the theory of $(\mathbf{Z}, S, 0)$. Then $dcl^*(a, b) = \emptyset$ for $a, b$ in distinct $\mathbf{Z}$-chains but $0 \in \mathrm{sdcl}(\emptyset) \subseteq \mathrm{sdcl}(X)$, for any $X$ with at least two elements.

**Remark 2.8.** We will give in Example 4.2.1 a theory $\hat{T}_\mu$ where for some $B$ with $\delta(B) = 2$, there is a good pair $A/B$ with $\mu(A/B) = 2$, and $\hat{T}_\mu$ admits an independent set $I = \{a, b\} \subseteq M \models \hat{T}_\mu$ with $\mathrm{dcl}^*(I) \neq \emptyset$. Given a $v$-element independent set $I$ with $v \geq 2$, we will show in Section 4.1 assuming $\mu(A/B) \geqslant 3$ (whenever $\delta(B) = 2$), that $\mathrm{dcl}^*(I) = \emptyset$ and show in Section 4.4 that, even if some $\mu(A/B)$ might be 2, $\mathrm{sdcl}^*(I)$ is empty.

**Definition 2.9** (non-trivial functions). *Let $T$ be a strongly minimal theory.*

*1* (Essentially unary) *An $\emptyset$-definable function $f(x_0 \ldots x_{n-1})$ is called* essentially unary *if there is an $\emptyset$-definable function $g(u)$ such that for some $i$, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.*

*2* (truly $n$-ary)
    (a) *Let $\mathbf{x} = \langle x_0 \ldots x_{n-1} \rangle$: a function $f(\mathbf{x})$ truly depends on $x_i$ if for any independent sequence $\boldsymbol{a}$ and some (hence any) independent[17] $\boldsymbol{a}'$ which disagrees with $\boldsymbol{a}'$ only in the $i$th place $f(\boldsymbol{a}) \neq f(\boldsymbol{a}')$.*
    (b) *$f$ is truly $n$-ary if $f$ truly depends on all its arguments and $f(\boldsymbol{a})$ is not a component of $\boldsymbol{a}$ for all but a set of Morley rank $< n$ of tuples $\boldsymbol{a} \in M^n$ .*

**Lemma 2.10.** *For a strongly minimal $T$ with $\mathrm{acl}(\emptyset)$ infinite, the following conditions are equivalent:*

    *(1) for any $n > 1$ and any independent set $I = \{a_1, a_2, \ldots a_n\}$, $\mathrm{dcl}^*(I) = \emptyset$;*
    *(2) every $\emptyset$-definable $n$-ary function ($n > 0$) is essentially unary;*
    *(3) for each $n > 1$ there is no $\emptyset$-definable truly $n$-ary function in any $M \models T$.*

*Proof.* 1) implies 2). Fix $I$ as in the statement and let $f$ be an $\emptyset$-definable $n$-ary function. Then 1) implies that for some $i$, say, 1, there is an $\emptyset$-definable function $g(u)$, with $g(a_1) = f(\boldsymbol{a}) = d$. Let $p^1$ denote the generic type over $\emptyset$ realized by each $a_i$ and $p^n$ the type of the $n$-tuple. There are parameter-free formulas $\psi(u, y)$ and $\chi(u, \mathbf{v}, w)$ ($\mathrm{lg}(\mathbf{v}) = n - 1$) such that $p^1(u)$ entails $\psi(u, y)$ defines $g$ (i.e. $y = g(u)$) and $p^n$ implies $\chi$ defines $f$ (i.e. $w = f(u, \mathbf{v})$).

Now we have $\phi(a_1, d) \wedge \chi(\boldsymbol{a}, d)$. But since $a_1$ is independent from $(a_2, \ldots, a_n)$, for cofinitely many $c \in M$, we have $\exists y (\phi(c, y) \wedge \chi(c, a_2 \ldots a_n, y))$. For each $k$ with $1 \leqslant k < n$ let

$$A_{k+1}^c = \{x : (\exists^\infty x_1 \ldots \exists^\infty x_k)\ g(x) = f(x_1, x_2 \ldots x_k, x, a_{k+2} \ldots a_n)\}.$$

By induction, since $a_{k+1} \in A_{k+1}^c$ and is independent from the parameters defining $A_{k+1}^c$, each $A_{k+1}^c$ is cofinite, so, its complement $A_{k+1}$ is finite. Thus, the subset on which $f(\mathbf{x}) \neq g(x_1)$ is contained in $\bigcup_{1 \leqslant i \leqslant n} A_i \times M^{n-1}$, which has Morley rank at most $n - 1$. Thus, $f$ is essentially unary witnessed by $g$.

2) implies 3): Suppose $f$ is such a definable truly $n$-ary function, let $\boldsymbol{a}$ enumerate an independent set $I$. By 2) there are $i, g$ with $g(a_i) = f(\boldsymbol{a})$ and this holds on any independent sequence. For some $j \neq i$, let $\boldsymbol{a}'$ be obtained from $\boldsymbol{a}$ by replacing $a_j$ by an $a_j'$ such that $\boldsymbol{a}'$ is independent. Then $f(\boldsymbol{a}) = g(a_i) = f(\boldsymbol{a}')$ so $f$ is not truly $n$-ary since it doesn't depend on $x_i$.

3) implies 1): Suppose 1) fails. Fix the least $n \geq 2$ such that $\mathrm{dcl}^*(I) \neq \emptyset$ for some independent set $I = \{a_1, a_2, \ldots a_n\}$. Let $d \in \mathrm{dcl}^*(I)$. By the definition of

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[17]This definition is more restrictive than the standard (e.g. [Grä79, p. 35]) as for our definition, in a ring the polynomial $xy + z$ does not depend on $y$ while usually one is allowed to substitute 0 to witness dependence.

dcl, there exists $\varphi(x, \mathbf{y})$ such that $\models \exists! x \varphi(x, \boldsymbol{a}) \wedge \varphi(d, \boldsymbol{a})$, so $\varphi$ defines some $n$-ary function $f$. We now show $f$ is truly $n$-ary. If $f(\boldsymbol{a})$ is a component of $\boldsymbol{a}$, then $d = a_i$ for some $i$ and $d \in \mathrm{dcl}(a_i)$, contradicting $d \in dcl^*(I)$. If $f$ is not truly $n$-ary, there exists $i$ such that for any independent sequences $\mathbf{b}$ and $\mathbf{b}'$ which disagree in the $i$th place it holds that $f(\mathbf{b}) = f(\mathbf{b}')$. We choose $a_i'$ so that $\boldsymbol{a} a_i'$ is independent, then $\boldsymbol{a}_i' = (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$ is independent, too. Then $f(\boldsymbol{a}) = f(\boldsymbol{a}_i')$. Let $\psi(x', \boldsymbol{a})$ denote $f(\boldsymbol{a}) = f(a_1, \ldots, a_{i-1}, x', a_{i+1}, \ldots, a_n)$. Since $\boldsymbol{a} a_i'$ is independent and $a_i'$ satisfies $\psi(x', \boldsymbol{a})$, this formula has cofinitely many solutions. Then the formula

$$\exists x (\exists^\infty x')(f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, x', a_{i+1}, \ldots, a_n) \wedge$$
$$\wedge\, y = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n))$$

defines $d$, so $d \in \mathrm{dcl}^*(I - \{a_i\})$, for a contradiction. $\blacksquare$

The strongly minimal affine space $(Q, R^4)$ with $R(x, y, z, t) = x + y = z + t$ fails weak elimination of imaginaries; weak elimination is obtained if a point is named.

**Remark 2.11.** We cannot isolate 'non-triviality' by simply saying there is no definable $n$-ary function, nor even none which depends on all its variables. The insight is that if $a \in \mathrm{acl}(B)$ by a formula $\phi(v, \mathbf{b})$ which has $k$ solutions, any solution is in the definable closure of $B$ and the other $k-1$. Steiner systems with line length $k$ give a stark example. Consider the function of $k-1$ variables which projects on the first entry unless the $k-1$ arguments are a clique (partial line) and gives the last element of the line in that case. This function satisfies the 'depends' hypothesis but not the projection hypothesis (Definition 2.9.2). So, although $\emptyset$-definable, it is not a *truly $(k-1)$-variable* function.

The main aim of this paper is to study the definable closure in Hrushovski's example in order to get to know which algebraic operations can be defined in those examples. As a by-product, we obtain the application in Lemma 2.12 of our results to elimination of imaginaries.

**Lemma 2.12.** *Let $I = \{a_0, a_1, \ldots, a_{v-1}\}$ be an independent set with $I \leqslant M$ and $M$ be a model of a strongly minimal theory constructed as in Section 1.1.*

  (1)  *For any finite $X$, if $\mathrm{sdcl}(X) = \emptyset$ then $X$ is not finitely coded.*
  (2)  *If $\mathrm{sdcl}^*(I) = \emptyset$ then $I$ is not finitely coded.*
  (3)  *If $\mathrm{dcl}^*(I) = \emptyset$ then $I$ is not finitely coded and there is no parameter-free definable truly $n$-ary function for $n = |I|$.*

*Proof.* 1) is immediate from Definition 2.2 and 3) follows immediately from 2) since $\mathrm{sdcl}^*(I) \subseteq \mathrm{dcl}^*(I)$. 2) requires some effort. Suppose $t \in T$ where $T$ is a finite code for $I$. If $t \in \mathrm{sdcl}(I) - \mathrm{sdcl}^*(I)$, then either $t \in \mathrm{dcl}(\emptyset)$ or $t \in \mathrm{dcl}(J)$ for some $\emptyset \subsetneq J \subsetneq I$. In the first alternative, if $T$ is a finite code for $I$ so is $T - \{t\}$. And since $I$ is independent, it cannot be coded by the empty set. So we must consider the second case. But if $t \in \mathrm{dcl}(I) - \mathrm{dcl}^*(I)$ is in, say, $\mathrm{dcl}^*(J)$, a permutation switching $a_i$ and $a_j$ for some $a_j \in J$, $a_i \in I - J$ and fixing the other $a_k$ takes $t$ to some $t' \neq t$. Thus $t \notin \mathrm{sdcl}(I)$. $\blacksquare$

We use $G_I$ to prove hypotheses (1) and $G_{\{I\}}$ for (2) of Theorem 2.13. The 'proof' below just indicates the organization of the argument that follows.

**Theorem 2.13.** *Let $T_\mu$ be a Hrushovski construction as in Theorem 4.1.2 or a strongly minimal Steiner system as in Theorem 5.2.*

> *(1) If $T_\mu$ triples ($\delta(B) = 2$ and $|C| > 1$ imply $\mu(C/B) \geqslant 3$), then $\mathrm{dcl}^*(I) = \emptyset$.*
> *(2) In any case $\mathrm{sdcl}^*(I) = \emptyset$.*

*Consequently, $T_\mu$ does not admit elimination of imaginaries.*

*Proof.* 1) By Lemma 2.12 and Theorem 4.1.2, $I$ is not finitely coded. So by Fact 2.3, $T$ does not admit elimination of imaginaries. And by Lemma 2.10, $\mathrm{dcl}^*(I) = \emptyset$ implies there is no $\emptyset$-definable truly $n$-ary function. 2) Theorem 4.4.1 (Theorem 5.19 for the Steiner case) provides $\mathrm{sdcl}^*(I) = \emptyset$ without the extra (triplable) hypothesis. ∎

Importantly, if the $A$ in a good pair $A/B$ is fixed setwise by $G$ then so is $B$.

**Observation 2.14.** *Assume $\boldsymbol{L}_0$ consists of all structures $A$ in $\boldsymbol{L}_0^*$ such $\emptyset \leqslant A$. Let $\mathfrak{A}, B, C, I \subseteq M$, $M \models T_\mu$ and $G = G_{\{I\}}$ or $G_I$. Suppose $C$ is 0-primitive over $\mathfrak{A}$ and based on $B \subseteq \mathfrak{A}$. If the automorphism $f \in G$ fixes $\mathfrak{A}$ setwise, and fixes $C$ setwise, then it fixes $B$ setwise.*

Note that the first assumption on $L_0$ in Observation 2.14 *fails* for the Steiner system case (and also for Proposition 18 of [Hru93]). We find a suitable substitute in Lemmas 5.4 and 5.11.

*Proof.* By Definition 1.1.5.3, $B$ is uniquely determined by $C$ and $\mathfrak{A}$. Thus, if $f \in G$ satisfies $B \neq f(B)$, we have a contradiction since $f(C) = C$ and so some element of $C$ is $R$-related to an element not in $B$. ∎

## 3. $G$-Decomposition

We continue with the hypotheses of Sections 1.2. Our original goal was to show $\mathrm{dcl}^*(I) = \emptyset$ for $I = \{a_1, \ldots, a_v\}$ with $d(I) = v \geq 2$. For this we introduced the notion of a $G_I$-decomposition to analyze the algebraic closure of a finite set. However we needed an additional hypothesis on $\mu$ to show $\mathrm{dcl}^*(I) = \emptyset$. In order to eliminate that hypothesis, we consider decompositions with respect to two subgroups of $\mathrm{aut}(M)$: $G = G_I$ or $G = G_{\{I\}}$ that fix $I$ pointwise or setwise, respectively. Using the decomposition associated to group we inductively show the appropriate definable closure is empty. We give a joint account of the decomposition but by changing the group prove Theorem 0.2 1) (for $\mathrm{dcl}^*$) or 2) (for $\mathrm{sdcl}^*$).

**Definition 3.1.** *Let $M$ be the generic model of $T_\mu$, $I = \{a_1, \ldots, a_v\}$ be independent with $I \leq M$, and let $G \in \{G_I, G_{\{I\}}\}$. A subset $\mathfrak{A}$ is $G$-normal if it is finite, contains $I$, $G$-invariant ($G$ fixes $\mathfrak{A}$ setwise), and is strong in $M$.*

We need the following easy observation to prove Lemma 3.3; finite $G$-normal sets exist. The forward implication in Observation 3.2 holds for any first order theory. As in Notation 2.6 the conditions are equivalent here since all models are $\omega$-homogeneous.

**Observation 3.2.** *Let $A \subseteq M$. 1) implies 2) and 3); all are equivalent in an $\omega$-homogenous model.*

> *(1) $c \in \mathrm{acl}(A)$*
> *(2) The orbit of $c$ under $\mathrm{aut}_A(M)$ ($A$ fixed pointwise) is finite.*
> *(3) If $A$ is finite then the orbit of $c$ under $\mathrm{aut}_{\{A\}}(M)$ ($A$ fixed setwise) is finite.*

**Lemma 3.3.** *Each finite $U \subseteq \mathrm{acl}(I) \subseteq M$ is contained in a finite $G$-normal set. If $U$ is a finite $G$-normal set, then $U \subseteq \mathrm{acl}(I)$.*

*Proof.* Given any finite set $U$, let $G(U) = \{g(u) : u \in U, g \in G\}$, and $\mathfrak{A}_U^G = \mathrm{icl}(I \cup G(U))$. Then $\mathfrak{A}_U^G$ is $G$-normal. For this, note $G(U)$ is finite by Lemma 3.2. The intrinsic closure of a set is unique, so $\mathfrak{A}_U$ is fixed setwise for either $G$. The second part of the lemma is immediate since $\mathrm{icl}(X) \subseteq \mathrm{acl}(X)$.    ∎

Note that for given $I$ and $U$ in $M$, both the set $\mathfrak{A}_U^G$ and the height (Definition 3.8) of the $G$-decomposition depend on the choice of $G$.

We need the following result from [Ver06, 4.2] to carry out the decomposition.

**Lemma 3.4.** *Suppose $A_1 \subset A_2 \subset A_3$ are such that $C_i = A_{i+1} \setminus A_i$ is $0$-primitive over $A_i$, for $i = 1, 2$. If $C_2$ is $0$-primitive over $A_1$, then $C_1$ is $0$-primitive over $A_1 \cup C_2$.*

*Proof.* Let $D \subseteq C_1$. Then

$$\delta(D/A_1 \cup C_2) = \delta(D \cup C_2/A_1) - \delta(C_2/A_1) = \delta(D \cup C_2/A_1)$$
$$= \delta(D/A_1) + \delta(C_2/A_1 \cup D) = \delta(D/A_1)$$

The first three equalities follow easily from the definition of $\delta(X/Y)$ and the conditions of the lemma. The last equality follows from: $0 = \delta(C_2/A_2) \leq \delta(C_2/A_1 \cup D) \leq \delta(C_2/A_1) = 0$. So $C_1$ is $0$-primitive over $A_1 \cup C_2$.    ∎

The next definitions and theorems provide the tools for the decompositions. Roughly speaking, capital Roman letters (A, B, C) denote specific components of the decomposition; Gothic letters $\mathfrak{A}$, $\mathfrak{D}$ range over initial segments of the decomposition. In particular, this means that each of $\mathfrak{A}$, $\mathfrak{D}$ contains $I$ and is closed in $M$.

**Definition 3.5.**    (1) *We call a good pair $A/B$ well-placed by $\mathfrak{D}$, if $B \subseteq \mathfrak{D} \leqslant M$ and $A$ is $0$-primitive over $\mathfrak{D}$, and*
   (2) *$A/B$ is well-realized in $Y$ if $\chi_Y(A/B) = \mu(A/B)$. If $Y = M$ we omit it and write simply $A/B$ is well-realized.*

Lemma 3.7 is crucial for the general decomposition Construction 3.8.

**Definition 3.6.** *Let $A$ be a subset of $\mathfrak{D}$. We say $A$ splits over $\mathfrak{D}$ if both $A \cap \mathfrak{D}$ and $A - \mathfrak{D}$ are non-empty.*

**Lemma 3.7.** *Let $M$ be the countable generic model for $T_\mu$. Suppose $A/B$ is well-placed by $\mathfrak{D}$ and $\mathfrak{D}$ is $G$-invariant.*

   (1) *Then $\chi_M(A/B) = \mu(A/B)$.*
   (2) *For each $i < \mu(A/B)$ there is a partial isomorphism $h_i$ fixing $B$ pointwise with domain $B \cup A$ and satisfying either i) $h_i(A) \cap \mathfrak{D} = \emptyset$ or ii) $h_i(A) \subsetneq \mathfrak{D}$. Moreover, by Definition 1.1.7.3 the $h_i(A)$ are disjoint over $B$. In case i) there is $g \in G$ that fixes $IB$ (and indeed $\mathfrak{D}$) pointwise and takes $A$ to $h_i(A)$. That is, $G$ acts transitively on the copies of $A$ that are disjoint from $\mathfrak{D}$.*

While the proof uses that $M$ is generic, the conclusion passes to any model of $T_\mu$, because models are algebraically closed.

*Proof.* 1) Part 1 is the translation to this notation of a result proved for the Hrushovski case in [Ver02, 2.25]; for the Steiner case, it is [BP21, 5.14]. Since the article [Ver02] is difficult to access, we repeat the proof here.

Let $\mathfrak{D} \leq M$ and $A$ be a 0-primitive extension of $\mathfrak{D}$ with the base $B$. If $\chi_{\mathfrak{D}}(A/B) = \mu(A/B)$ we are done, so we assume that $\chi_{\mathfrak{D}}(A/B) < \mu(A/B)$. Let $A_1$ be an isomorphic copy of $A$ over $\mathfrak{D}$ and let $E$ be the canonical amalgam of $\mathfrak{D} \cup A$ and $\mathfrak{D} \cup A_1$ over $\mathfrak{D}$. By [Hru93, Lemma 3], $E \in \boldsymbol{L}_\mu$ and there is an embedding $g : E \to M$ such that $g{\restriction}\mathfrak{D} \cup A = id_{\mathfrak{D} \cup A}$ and $g(E) \leq M$. Then $\chi_E(A/B) = \chi_{\mathfrak{D}}(A/B) + 1$ and we proceed by induction.

2) By 1) there are partial isomorphisms $h_i$ for $i < \mu(A/B)$, fixing $B$, but not necessarily $I$, giving structures $A_i = h_i(A)$ isomorphic to $A$ over $B$. Note that $A_i$ cannot split over $\mathfrak{D}$ and $I \leqslant M$ (the last holds because $I$ is independent). Moreover, we have $BA \approx BA_i$, $\mathfrak{D} \leqslant M$, and there are no relations between $A$ and $\mathfrak{D} - B$. So, if $A_i \cap \mathfrak{D} = \emptyset$ there must be no relations between $A_i$ and $\mathfrak{D} - B$. Else, $\delta(\mathfrak{D}A_i) < \delta(\mathfrak{D})$. As $\mathfrak{D}A \approx \mathfrak{D}A_i$, there is an automorphism of $M$ taking $A$ to $A_i$ and fixing $\mathfrak{D}$ and in particular $I$. ∎

The following definition and description of the decomposition of a $G$-normal set is intended to be evident (modulo the references). The next diagram gives an overall view; Example 3.13 gives a closer view.

**Construction 3.8.** Let $\mathfrak{A}$ be $G$-normal. We can linearly decompose $\mathfrak{A}$ as the union of $X_n$, $n \leq r$, where $X_0 = I$ and $X_{n+1}$ is 0-primitive over $X_n$ and good over $Y_{n+1} \subseteq X_n$ for $n < r$. This is a cumulative decomposition: $X_n \subseteq X_{n+1}$.

Since we aim to prove that $\mathrm{dcl}^*(I) \cap X_r = \emptyset$ $(\mathrm{sdcl}^*(I) \cap X_r = \emptyset)$ by induction on $n$, it would be convenient to assume that $X_n$ is $G$-invariant for each $n < r$. But it is not true. In order to reach an induction on $G$-invariant sets, we create, by grouping the images of various partial isomorphisms of the $\hat{X}_{n+1} = (X_{n+1} - X_n)$ over $Y_{n+1}$, $G$-invariant strata $\mathfrak{A}^{m+1}$ of components that are independent over $\mathfrak{A}^m$. The new tree decomposition creates strata $\langle \mathfrak{A}^m : m < m_0 \rangle$; $m_0$ is called the *height* of the decomposition.

We define the new decomposition of $\mathfrak{A}$ into strata $\mathfrak{A}^m$ by inductively assigning to each $\hat{X}_{n+1}$ an integer $S(\hat{X}_{n+1}, Y_{n+1})$, the *strata* of $\hat{X}_{n+1}$, the least $m + 1 \leqslant n$ such that $Y_{n+1} \subseteq \mathfrak{A}^m$ and renaming $\hat{X}_{n+1} = (X_{n+1} - X_n)$ as an $A_{x,y}^{m+1}$ for an appropriate $x, y$ (more detail below). The $Y_{n+1}$ may be omitted when clear from context. By fiat, $S(X_0, \emptyset) = 0$.

i) $\mathfrak{A}^0$:
   (a) Let $D_0 = \mathfrak{A} \cap \mathrm{acl}(\emptyset)$.
   (b) For $i = 1, 2, \ldots, v$, let $D_i = \mathfrak{A} \cap \mathrm{acl}(a_i)$ and let $\mathfrak{A}^0 = D_1 \cup D_2 \cup \cdots \cup D_v$. Note that $D_i \cap D_j = D_0$ for any $1 \leq i < j \leq v$, $\mathfrak{A}^0$ is finite, $\mathfrak{A}^0 \leqslant M$, and so $\delta(\mathfrak{A}^0) = v$. Moreover, since $d(a_i) = 1$ the $D_i$ are fully independent over $D_0$.

As we continue the construction we will rearrange the components $\hat{X}_n$ into a quasi-order by introducing sets $\mathfrak{A}^m$ such that each component in $\mathfrak{A}^m$ is based on a subset of $\mathfrak{A}^{m-1}$. At the $n$th stage of construction, considering $(\hat{X}_{n+1}, Y_{n+1})$, $\hat{X}_{n+1}$ is added to $\mathfrak{A}^{S(\hat{X}_{n+1}, Y_{n+1})}$ and given an appropriate name as described below. Each $\mathfrak{A}^m$ is divided into $q_m$ subsets $\mathfrak{A}_i^m$, where $\mathfrak{A}_i^m$ consists of $\ell_i^m$, disjoint off $\mathfrak{A}^{m-1}$, sets $A_{i,f}^m$ which are 0-primitive over $\mathfrak{A}^{m-1}$ and pairwise isomorphic over $\mathfrak{A}^{m-1}$, and each $A_{i,f}^m$ is based on the same set $B_i^m \subseteq \mathfrak{A}^{m-1}$.

$B$ is the base of both $X_k$ and $X_m$

$X_k \cup B \cong_B X_m \cup B$

$X_k$

$X_m$

Since $B \subseteq \mathfrak{A}^1$,

$X_k$ and $X_m$ were placed into $\mathfrak{A}^2$ and
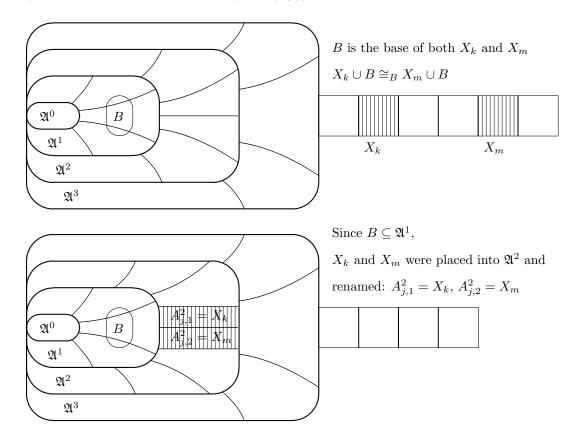
renamed: $A^2_{j,1} = X_k$, $A^2_{j,2} = X_m$

FIGURE 1. From a linear to a tree-decomposition: One Step

We call the $A^m_{i,f}$ *petals*. Lemma 3.7.(2) ensures that $G$ acts 1-transitively on $\{A^m_{i,f} : 1 \leq f \leq \ell^m_i\}$. We describe further petals of $A^m_{i,1}/B^m_i$ in the next few paragraphs.

We now give a precise definition of $\mathfrak{A}^{m+1}$. We set $\mathfrak{A}^{-1} = \emptyset$ to allow uniform treatment for all $m \geqslant 0$. Note that new petals may be added to $\mathfrak{A}^{m+1}$ at later stages in the construction.

ii) $\mathfrak{A}^{m+1}$: Suppose $S(X_n) = m > 1$. We consider the good pair $\hat{X}_{n+1}/Y_{n+1}$ with $Y_{n+1} \subseteq X_n$. If $Y_{n+1} = Y_{n'}$ for some $n' \leq n$ with $S(X_{n'}) = m$ then $Y_{n+1}$ has already been denoted $B^{m+1}_t$ for some $t \leqslant j$. If $\hat{X}_{n+1} \approx_{Y_{n'}} X_{n'}$, set $\hat{X}_{n+1}$ as $A^{m+1}_{t,k} \in \mathfrak{A}^{m+1}_t$, where $k$ is the least index not previously used with $t$.

If $X_n \not\approx_{Y_{n'}} X_{n'}$ and $Y_{n+1} \neq Y_{n'}$ for any $n' < n$ with $S(X_{n'}) = m$, set $\hat{X}_{n+1}$ as $A^{m+1}_{u,1}$ and set $Y_{n+1}$ as $B^{m+1}_u$ for the next available $u$. Then $Y_{n+1} \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1}) \neq \emptyset$, $\mathfrak{A}^m \leqslant M$, $\mathfrak{A}^m \hat{X}_{n+1} \leqslant M$. It is possible that $(Y_{n+1}, \hat{X}_{n+1}) \approx (Y_{n'}, \hat{X}_{n'})$ for some smaller $n'$.

By Lemma 3.7.2, there are partial isomorphisms[18] $h_i$ for $i \leqslant \mu(\hat{X}_{n+1}/Y_{n+1})$ that fix $Y_n$ and the $h_i(\hat{X}_{n+1})$ are independent (and so disjoint) over $Y_n$. Note

_____

[18]It is essential here that each $(X_n/Y_n)$ is well-placed (Definition 3.5).

that some of these $h_i$ may not extend to automorphisms of $M$ and if so by Lemma 3.7.2 they map $X_{n+1}$ into $\mathfrak{A}^m$. Suppose that $\ell_{j+1}^{m+1}$ of these partial isomorphisms extend to automorphisms $h_i$ of $M$ that fix $I$ and so $h_i(\hat{X}_n) \cap \mathfrak{A}^m = \emptyset$ for $1 \leqslant i \leqslant \ell_{j+1}^{m+1}$.

We have relabeled the $h_i(\hat{X}_{n+1})$ as $A_{j+1,f}^{m+1}$, for $1 \leqslant f \leqslant \ell_j^{m+1}$ and added them to $\mathfrak{A}_j^{m+1}$ forming $\mathfrak{A}_{j+1}^{m+1} = \bigcup_{1 \leqslant f \leqslant \ell_{j+1}^{m+1}} A_{j+1,f}^{m+1}$, which is thus $G$-invariant.

Since $\mathfrak{A}$ is $G$-normal, each of the $h_i(\hat{X}_{n+1})$ is an $\hat{X}_{n'}$ for some $n' \geqslant n+1$.

Let $\mu_{j+1}^{m+1}$ denote $\mu(\hat{X}_{n+1}/Y_{n+1}) = \mu(A_{j+1}^{m+1}/B_{j+1}^{m+1})$. The other $\mu_{j+1}^{m+1} - \ell_{j+1}^{m+1}$ images are subsets of $\mathfrak{A}^m$ and are labeled as $C_{j+1,k}^{m+1}$ for $1 \leqslant k \leqslant \nu_{j+1}^{m+1} = \mu_{j+1}^{m+1} - \ell_{j+1}^m$.

Each of the $A_{i,f}^{m+1}$ for $1 \leq f \leq \ell_{j+1}^{m+1}$ and the $C_{j,k}^{m+1}$ for $1 \leqslant k \leqslant \nu_{j+1}^{m+1}$ is a petal.

Note that $\hat{X}_{n+1} = X_{n+1} - X_n$ is based on $Y_{n+1}$, which we have designated[19] as $B_{j+1}^{m+1} \subseteq \mathfrak{A}^m$; by the minimality of $m$, $B_{j+1}^{m+1}$ intersects $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ non-trivially. Thus as we construct $\mathfrak{A}_j^{m+1}$, we are moving $\ell_{j+1}^{m+1}$ components down so they are directly above their base. This is possible by Lemma 3.4. We sometimes call the $A_{j,i}^{m+1}$ which have the same base $B_j^{m+1}$ a cluster $\mathfrak{A}_j^{m+1}$.

At the conclusion of the construction for each $m < m_0$, for some $t_{m+1} < r$, there will be $t_{m+1}$ $(t_{m+1} = \Sigma_{i<q_m} \ell_i^{m+1})$ distinct $\hat{X}_{n+1}$, labeled as $A_{j,f}^{m+1}$ with $S(X_n) = m$; the $A_{j,f}^{m+1}$ are independent over $\mathfrak{A}^m$. Then $\mathfrak{A}^{m+1} = \mathfrak{A}^m \cup \bigcup_{j<q_m} \mathfrak{A}_j^{m+1}$ and the union is a partition of $\mathfrak{A}^{m+1} - \mathfrak{A}^m$. While $\bigcup_{i \leq r} X_i$ is a chain, the $\mathfrak{A}^m$ form a tree with the petals $A_{j,f}^{m+1}$ partitioning each level. More locally $B_j^{m+1} \cup \bigcup_{f=1}^{\ell_j^{m+1}} A_{j,f}^{m+1}$ looks like a flower with the base $B_j^{m+1}$ and two collections of petals. $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ is a collection of petals $\bigcup_{1 \leq j \leq q_m, 1 \leq f \leq \ell_j^{m+1}} A_{j,f}^{m+1}$ on the stem $\mathfrak{A}^m$. But for each $j$, the flower over $B_j$ also contains the $C_{j,k}^{m+1} \subseteq \mathfrak{A}^m$ for $k$ and $j$ with $1 \leq k \leq \nu_j^{m+1}$ and $1 \leq j \leq q_m$. Further, $\mathfrak{A} = \bigcup_{m \leqslant m_0} \mathfrak{A}^m$.

Note that any two petals on the same strata, say on $\mathfrak{A}^{m+1}$, are $\delta$-independent over $\mathfrak{A}^m$ and in the case of Hrushovski's construction are fully independent. For Steiner systems we obtain that if these petals do not belong to the same linear cluster (Definition 5.6) then they are fully independent.

**Remark 3.9.** *Note that a $G$-decompostion depends on, and is determined by, the original linear decomposition.*

The following observation is key to the proof of the ensuing Lemma 3.11 and Lemma 3.21.

**Observation 3.10.** [Ver02, Note 2.8] We say that a 3-hypergraph $A$ is *disconnected over $B$* if there is a partition of $A$ into $A_1 \cup A_2$ such that for every $a \in A_1$ and $b \in A_2$ there is no $d \in A \cup B$ such that $R(a, b, d)$. It is easy to see that if $A$ is 0-primitive over $B$, then $A$ is connected over $B$. As, $\delta(A_1/A_2B) = 0$, if $A$ is disconnected over $B$; but then $A_2$ is 0-primitive over $B$, contrary to the minimality of $A$.

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[19]We use a single subscript because, while we are considering several copies of the $\hat{X}_n$, there is a fixed base.

Here are the basic properties of $\mathfrak{A}^1$ showing $\mathrm{dcl}^*(I) \cap \mathfrak{A}^1 = \emptyset$; the situation is simpler than the $m > 1$ case as there are no maps of $\hat{X}_n$ into $\mathfrak{A}^m$ over $Y_n$ when $m = 0$.

**Lemma 3.11** ($\mathfrak{A}^1$). *Let $\mathfrak{A}$ be $G$-normal and decomposed as $\langle \mathfrak{A}^n : n < m_0 \rangle$. Then for any $i$ and $f < \ell_i^1$, $A_{i,f}^1$, and $B_i^1 \subseteq \mathfrak{A}^0$ the following hold: $d(B_i^1) \geq 2$, $\mathfrak{A}^0 \leqslant M$, $\ell_i^1 = \mu_i^1 = \mu(B_i^1, A_{i,1}^1)$, and by $G$-decomposition, there is no copy of $A_{i,f}^1$ over $B_i^1$ in $\mathfrak{A}^0$. So, no $A_{i,f}^1$ is invariant under $G$.*

*Proof.* Note that for each $i$, $d(B_i^1) \geq 2$; otherwise there exists $b \in B_i^1$ such that $d(b) = d(B_i^1) = 1$. Since $b \in \mathfrak{A}^0$, $b \in \mathrm{acl}(a_k)$ for some $k \in \{1, 2, \ldots, v\}$; this implies that $B_i^1 \subseteq \mathrm{acl}(a_k)$ and thus $A_{i,j}^1 \subseteq \mathrm{acl}(a_k)$; the last assertion contradicts $A_{i,j}^1 \cap \mathfrak{A}^0 = \emptyset$. As noted in Construction 3.8.i), $I \subset \mathfrak{A}^0$ and $\delta(\mathfrak{A}^0) = v$ so $d(\mathfrak{A}^0) = v$ and $\mathfrak{A}^0 \leqslant M$.

We show there cannot be a copy, $C^1 = C_{i,x}^1$ for some $i, x$, of $A_{i,f}^1$ with base $B_i^1$ embedded in $\mathfrak{A}^0$. Since $d(B_i^1) \geq 2$, $B_i^1$ intersects at least two $D_k - D_0$ and $D_j - D_0$ for some $k \neq j$. Note that $C^1 \subseteq \mathfrak{A}^0$ is not a singleton $c$, because otherwise $M \models R(c, d_k, d_j)$ for some $d_k \in B_i^1 \cap D_k - D_0$ and $d_j \in B_i^1 \cap D_j - D_0$, contradicting full independence of the $D_i$'s over $D_0$. By Lemma 1.2.5 $C^1$ should intersect both $D_k - D_0$ and $D_j - D_0$. If not, there would be an $R(c_1, c_2, d)$ with $c_1, c_2 \in D_k$ and $d \in D_j \cap B_i^1$; this can't happen as the $D_i$ for $i > 0$ are fully independent over $D_0$ (Construction 3.8 $\mathfrak{A}^0$.b). But then $C^1$ is disconnected, contrary to Observation 3.10. Thus $\ell_i^1 = \mu_i^1 \geqslant \delta(B_i^1) \geqslant d(B_i^1) \geq 2$.                              ∎
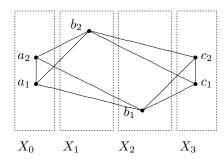
We pause to note a distinction between the flat geometries and the locally modular ones. [Bal73] showed that the finite dimensional models of an $\aleph_1$-categorical theory had either countably many or $2^{\aleph_0}$ automorphisms, with vector space-like strongly minimal sets on the first side and algebraically closed fields on the other. We now note:

**Corollary 3.12.** *If $T_\mu$ is constructed by a Hrushovski construction (including Steiner systems) with a flat geometry, each finite dimensional model $M_n$ has $2^{\aleph_0}$ automorphisms.*

*Proof.* Suppose $M$ is prime over the algebraically independent set $X$ with $n$ elements. There are countably many distinct good pairs $(A_n/X)$ (Remark 4.1.11); each has multiplicity at least 2, and we can define automorphisms of $M$ that fix or permute the realizations $A_n$ at will to give $2^{\aleph_0}$ automorphisms.                              ∎

The following example shows the situation gets much more complicated with the second strata.

**Example 3.13.** This example illustrates i) the shift from a chain to a strata decomposition, ii) $\mathrm{acl}^*(\mathfrak{A}^0)$ may properly extend $\mathrm{dcl}(\mathfrak{A}^0)$ and iii) that some $A_{1,i}^2$ may intersect $\mathfrak{A}^0$. Let $M$ be any model of $\hat{T}_\mu$ with $\mu(\boldsymbol{\alpha}) = 2$. Suppose $I = \{a_1, a_2\}$ and let $R$ hold of the triples $a_1 a_2 b_1, a_1 a_2 b_2, c_1 c_2 b_1, c_1 c_2 b_2$ and the entire six point diagram be strong in $M$.
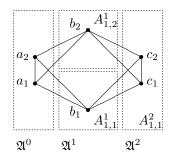
FIGURE 2. Chain



FIGURE 3. Decomposition

Figure 2 shows a chain decomposition; Figure 3 illustrates the downward embedding in a strata decomposition (as both $A_{1,1}^1$ and $A_{1,2}^1$ are based in $\mathfrak{A}^0$).

Further $\mathrm{sdcl}^*(\mathfrak{A}^0) \cap \mathfrak{A}^2 = \mathrm{dcl}^*(\mathfrak{A}^0) \cap \mathfrak{A}^2 = \emptyset$. Theorem 4.4.1 implies a stronger result that $\mathrm{sdcl}^*(\mathfrak{A}^0) = \emptyset$. However, we will show in Section 4.2 that for an independent pair $I$, there may be elements in $\mathrm{dcl}^*(I) - \mathrm{sdcl}^*(I)$. $X_1$ and $X_2$ demonstrate that there may be components $X_n$ and $X_{n'}$, both in strata $m$, such that $(Y_n, \hat{X}_n) \approx_I (Y_{n'}, \hat{X}_{n'})$; we provide the tool to study this situation in Definition 3.14.

Finally, in Figure 4 $A_{1,1}^2 \cap \mathfrak{A}^1 = \emptyset$ while $C_{1,1}^2 = \mathfrak{A}^0 = B_1^1$ although both are based on and isomorphic over $B_1^2$, as it shown in Figure 4.
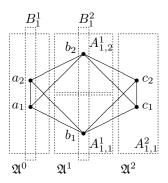


FIGURE 4. $A_{2,2}^2 = \mathfrak{A}^0$

Suppose further that $\mu(A_{1,1}^2/B_{1,1}^2) = 2$. Then this is a $G$-decomposition of $\mathfrak{A}^0 \cup \{c_2\}$ for either $G$. This shows that (in the presence of certain good pairs with $\mu(A/B) = 2$) we cannot avoid $G$-invariant petals.

**Definition 3.14** ($J_j^{m+1}$). *Let $\mathfrak{A}$ be $G$-normal and decomposed by $\langle \mathfrak{A}^n : n \leqslant m_0 \rangle$. We let $J_{G,j}^{m+1}$ consist of all indices $j'$ that $g(B_j^{m+1}) = B_{j'}^{m+1}$ for some $g \in G$. Thus we have an equivalence relation on the $j$'s with $1 \leq j \leq q_m$ enumerating the bases $B_j^{m+1}$; $j \sim j'$ if $B_{j'}^{m+1} = g(B_j^{m+1})$ for some $g \in G$.*

*If $G$ is fixed we omit it in $J_{G,j}^{m+1}$ and write simply $J_j^{m+1}$. Note that $j \sim j'$ implies $J_{j'}^{m+1} = J_j^{m+1}$, $A_j^{m+1} \approx A_{j'}^{m+1}$, and $\mu_j^{m+1} = \mu_{j'}^{m+1}$.*

Immediately,

**Observation 3.15.** *Let everything be as in Definition 3.14. Then $B_j^{m+1}$ is $G$-invariant if and only if $|J_{G,j}^{m+1}| = 1$.*

*Thus, $\mathfrak{A}^m$ will consist of $\Sigma_j |J_{G,j}^m| \cdot \ell_j^m$ petals.*

We summarise in Notation 3.16, which also depends on the choice of $G$. In Section 4.1 we are using $G_I$. In Section 4.4, we employ $G_{\{I\}}$.

**Notation 3.16.** *The height of $\mathfrak{A}$ is the maximal index, $m_0 \leqslant r$ of a non-empty strata.*

*$q_m$ denotes the number of bases $B_j^{m+1}$ that support elements of strata $\mathfrak{A}^{m+1}$.*

*And, for fixed $G$, (Definition 3.14), $|J_j^{m+1}|$ is the number of those $B_{j'}^{m+1}$ ($j' \in J_j^{m+1}$) that are isomorphic to $B_j^{m+1}$ over $I$ by some $g \in G$.*

*For each $m, j$, $\ell_j^{m+1}$ is the number of conjugates of $A_j^{m+1}$ over $I \cup B_j^{m+1}$ under $G$. Since $\mathfrak{A}^m$ is $G$-invariant, $\ell_j^{m+1}$ is the number of $B_j^{m+1}$-copies of $A_{j,1}^{m+1}$ that are not embedded in $\mathfrak{A}^m$.*

*We denote by $\nu_j^{m+1}$ the number of $B_j^{m+1}$-copies of $A_{j,1}^{m+1}$, labeled as $C_{j,q}^{m+1}$, that are embedded in $\mathfrak{A}^m$.*

*Finally, $\mathfrak{A}^m = \bigcup_{i \leqslant m} \mathfrak{A}^i$.*

**Lemma 3.17.** *Let $\mathfrak{A}$ be $G$-normal and decomposed by $\langle \mathfrak{A}^n : n < m_0 \rangle$. For any positive $m \leq m_0$ and $j$ it holds that $\ell_j^m + \nu_j^m = \mu(A_{j,1}^m / B_j^m)$.*

*Proof.* Fact 3.7.(2) implies that $\chi_M(A_{j,1}^m / B_j^m) = \mu(A_{j,1}^m / B_j^m)$. Let $C$ be a copy over $B_j^m$ of $A_{j,1}^m$. Since $B_j^m \subseteq \mathfrak{A}^{m-1} \leq M$, the definition of a 0-primitive extension implies that either $C \subseteq \mathfrak{A}^{m-1}$ or $C \cap \mathfrak{A}^{m-1} = \emptyset$.                                   ∎

The following notion is central for analyzing the position of a $G$-invariant petal in $\mathfrak{A}$. As, a $G_I$-invariant singleton is in $\mathrm{dcl}^*(I)$; our goal is to show there are no such singletons.

**Definition 3.18.** *We say $A_{j,1}^{m+1}$ determines $A_{i,f}^m$ if $A_{i,f}^m$ is the unique petal based in $\mathfrak{A}^{m-1}$ that intersects $B_j^{m+1} - \mathfrak{A}^{m-1}$.*

$A_{j,1}^{m+1}$ is $G$-invariant and determines $A_{i,f}^m$ then $A_{i,f}^m$ is $G$-invariant, so we normally denote the determined petal by $A_{i,1}^m$. We now see that a $G$-invariant singleton determines a petal that contains $B_j^{m+1}$. The following lemmas show that when $|A_{j,1}^{m+1}| > 1$, under appropriate inductive hypotheses, $B_j^{m+1}$ is 'almost' contained in $A_{i,1}^m$ (Lemma 3.23.1).

**Lemma 3.19.** *Let $m \geqslant 1$ and $B = B_j^{m+1}$ be the base of $A_{j,1}^{m+1}$ over $\mathfrak{A}^m$. If $|A_{j,1}^{m+1}| = 1$ and $A_{j,1}^{m+1}$ is $G$-invariant then*

*(1) $A_{j,1}^{m+1}$ determines some $A_{i,f}^m$;*
*(2) and if $B$ does not contain a $G$-invariant singleton, $B \subseteq A_{i,f}^m$.*

*Proof.* (1) By Observation 2.14 $B$ is $G$-invariant. Assume the contrary, that $A_{j,1}^{m+1} = \{c\}$, but $B$ intersects at least two petals $A_{i,f}^m$ and $A_{i',f'}^m$. Observe that if the singleton $c$ is primitive over $\mathfrak{A}^m$, then for some $b_1, b_2 \in \mathfrak{A}^m$, $((b_1, b_2), c)$ realizes the good pair $\boldsymbol{\alpha}$. So, $B = \{b_1, b_2\}$ and $M \models R(b_1, b_2, c)$. By construction $B \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1}) \neq \emptyset$, so at least one of $b_1, b_2$ is in $A_{i,f}^m$ for some $f$ and $i$, say $b_1$.

Let $C^1 = \{c_1\}$ be an isomorphic over $B$ copy of $A_{j,1}^{m+1}$ with $C^1 \subseteq \mathfrak{A}^m$. As there is no relation $R(b_1, b_2, c_1)$ with the $b_i$ in $A_{i,f}^m$ and $A_{i',f'}^m$ (since they are fully

independent over $\mathfrak{A}^{m-1}$), $B - \mathfrak{A}^{m-1} \subseteq A_{i,f}^m$. Since $B$ is $G$-invariant, $A_{i,f}^m$ is $G$-invariant.

(2) Suppose for contradiction $b_2 \in \mathfrak{A}^{m-1}$. Then since $B$ and $\mathfrak{A}^{m-1}$ are each $G$-invariant both $b_1$ and $b_2$ are fixed by $G$ violating the additional assumption for case (2). ∎

Now we investigate the various images contained in $\mathfrak{A}^m$ of $A_{j,1}^{m+1}$. To simplify notation we continue the special notations in Lemma 3.19 and add some more

**Notation 3.20.** We write $(A/B)$ for the good pair $(A_{j,1}^{m+1}/B_j^{m+1})$ and $\mu_j^{m+1}$ for $\mu(A_{j,1}^{m+1}/B_j^{m+1})$. Let $C^d$, for $1 \leqslant d \leqslant \nu = \nu_j^{m+1} = \mu_j^{m+1} - 1$ (since $\ell_j^{m+1} = 1$) enumerate the isomorphic images over $B = B_j^{m+1}$ of $A_{j,1}^{m+1}$ that lie in $\mathfrak{A}^m$. Let $C_+^d = C^d \cap \mathfrak{A}^{m-1}$, $C_-^d = C^d - \mathfrak{A}^{m-1}$, $B_+ = B \cap \mathfrak{A}^{m-1}$, and $B_- = B - \mathfrak{A}^{m-1}$.

The next diagram illustrates Notation 3.20, where the petal $A = A_{j,1}^{m+1}$ is a $G$-invariant subset of $\mathfrak{A}^{m+1} - \mathfrak{A}^m$, its base is $B = B_j^{m+1}$, which is a subset of $\mathfrak{A}^m$; $A$ has two copies $C^1$ and $C^2$ over $B$, which is a subset of $\mathfrak{A}^m$. In the diagram the $m$-th strata is $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. So as not to overlap the images of the sets $B$, $C^1$, and $C^2$, we drew only two petals $A_{i,1}^m$ and $A_{i,2}^m$ from $\mathfrak{A}^m$ and separated them with a dotted line.
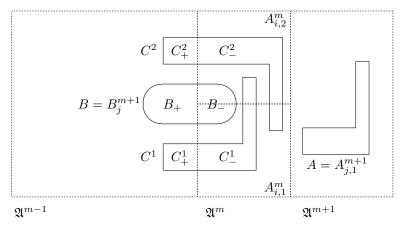


FIGURE 5. Illustrating Notation 3.20

With this notation we continue to set the stage; now, we assume both $|A_{j,1}^{m+1}| > 1$ and $|A_{i,f}^m| > 1$ for $i$, $f$ with $A_{i,f}^m \cap B \neq \emptyset$. The second assumption follows from the first, when $\mu_j^{m+1} \geqslant 3$, by Lemma 3.22 but will be an issue in Section 4.4. Recall Definition 1.2.1 of $R(X, Y)$ and $R(X, Y, Z)$.

**Lemma 3.21.** *Assume that $A_{j,1}^{m+1}$ is $G$-invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$. Then, for any $d$ with $1 \leqslant d \leqslant \nu = \nu_j^{m+1}$:*

(1) *For any $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$, $C^d \cap A_{i,f}^m \neq \emptyset$, i.e., $C_-^d \neq \emptyset$.*
(2) *Using Notations 3.20 and 1.2.1, $R(B_-, C_+^d) = \emptyset$ and $R(B_-, C_+^d, B_+) = \emptyset$. Thus, $\delta(B_-/B_+ \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d) = \delta(B_-/B_+)$.*
(3) *If $C^d \cap \mathfrak{A}^{m-1} = \emptyset$, that is $C_+^d = \emptyset$, then there is a unique petal $A_{i,f}^m$ that contains both $C^d$ and $B_-$. So, $A_{i,f}^m$ is $G$-invariant.*

*Proof.* (1) Lemma 1.2.5 implies for any $b \in B_-$, and any $d \leqslant \nu$ there must be $c_1, c_2 \in C^d$ with $R(c_1, c_2, b)$. Since $b \in B_-$ and by construction, $b \in B_- \cap A_{i,f}^m$ for some $i$ and $f$. If both $c_1$ and $c_2 \in \mathfrak{A}^{m-1}$, then $|A_{i,f}^m| = |\{b\}| = 1$, a contradiction. So at least one of $c_1, c_2$, say $c_1$, must be in $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. Since the petals on the same strata $m$ are freely joined over $\mathfrak{A}^{m-1}$, $c_1$ must be in $A_{i,f}^m$.

(2) Since $|A_{i,f}^m| > 1$, for any $b \in B_-$ there do not exist $x_1, x_2 \in \mathfrak{A}^{m-1}$ such that $M \models R(b, x_1, x_2)$. Hence $r(B_-, B_+ \cup \bigcup_{d=1}^{\nu} C_+^d) = r(B_-, B_+) + \sum_{d=1}^{\nu} r(B_-, C_+^d)$. The conditions $|A_{j,1}^{m+1}| > 1$ and $C^d \cong_B A_{j,1}^{m+1}$ imply for any $c \in C^d$ there are no $b_1, b_2 \in B$ such that $M \models R(b_1, b_2, c)$. Consequently, $r(B_-, C_+^d) = 0$.

(3) Assume that $C^d \cap \mathfrak{A}^{m-1} = \emptyset$. Assume also that $C^d \cap A_{s,t}^m \neq \emptyset$ for some $s$ and $t$, but $C^d \nsubseteq A_{s,t}^m$. Since petals $A_{u,v}^m$ are free over $\mathfrak{A}^{m-1}$ we obtain that $C^d$ is disconnected over $B$, contradicting Observation 3.10. So, there is a unique petal $A_{i,f}^m$ that contains $C^d$. Obviously, then $B_- \subseteq A_{i,f}^m$. The assumption $A_{j,1}^{m+1}$ is $G$-invariant implies (Observation 2.14) that $B$ is $G$-invariant. So, $B_-$, and thus, $A_{i,f}^m$ are $G$-invariant. $\blacksquare$

We prove a consequence of: $\mu(A_{j,1}^{m+1}/B_j^{m+1}) \geqslant 3$ and $A_{j,1}^{m+1}$ is $G$-invariant.

**Lemma 3.22.** *If $\mu(A_{j,1}^{m+1}/B_j^{m+1}) \geqslant 3$ and $A_{j,1}^{m+1}$ is $G$-invariant, then $\ell_j^{m+1} + 1 < \mu_j^{m+1}$ and $|A_{j,1}^{m+1}| > 1$ together imply $|A_{i,f}^m| > 1$ for any $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$.*

*Proof.* Let $b \in B \cap A_{i,f}^m$. For $A_{i,f}^m$ there is a unique base[20] $B_i^m$ by Lemma 2.14. Since $\mu_i^{m+1} = \ell_i^{m+1} + \nu_i^{m+1}$, $\ell_i^{m+1} = 1$ and $\mu_j^{m+1} \geqslant 3$ imply $\ell_j^{m+1} + 1 < \mu_j^{m+1}$. By the same observation as in Lemma 3.19.2, if $|A_{i,f}^m| = 1$, then $A_{i,f}^m = \{b\}$ and $B_i^m$ is a pair $(c_1, c_2) \in \mathfrak{A}^{m-1}$ that satisfy $R(c_1, c_2, b)$. On the other hand, $b$ satisfies $R(\alpha_1, \alpha_2, b)$ for some $\alpha_1, \alpha_2 \in A_{j,1}^{m+1}$ by Lemma 1.2.5. But, $|A_{i,f}^m| = 1$ implies there is no pair $x, y$ from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ satisfying $R(x, y, b)$. Since $\ell_j^{m+1} + 1 < \mu_j^{m+1}$ and $\mu_j^{m+1} \geqslant 3$ there must be at least two disjoint embeddings of $A_{j,1}^{m+1}$ in $\mathfrak{A}^m$, this implies that some $d \in \mathfrak{A}^{m-1} - B_i^m$ is in relation with $b$; this contradicts that $A_{i,f}^m$ is related only to elements of the doubleton $B_i^m$. $\blacksquare$

The hypothesis $C_+^d \neq 0$ of the next lemma is verified, using the inductive hypotheses, in the proof of Lemma 4.1.7 before Lemma 3.23 is applied.

**Lemma 3.23.** *Let $m \geqslant 1$. Assume that $\delta(B) \geq 2$, $A_{j,1}^{m+1}$ is $G$-invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$. Further, assume that $C_+^d \neq \emptyset$ for each $d$. Then*
*A) If $\mu_j^{m+1} \geq 3$, then $A_{j,1}^{m+1}$ determines an $A_{i,f}^m$ and $\delta(B_+) \leq 1$.*
*B) If $\mu_j^{m+1} = 2$, then $2 \geqslant \sum_{i \in \mathcal{I}} \ell_i^m + \delta(B_+)$, where $\mathcal{I} = \{i : (\exists t) A_{i,t}^m \cap B \neq \emptyset\}$, and thus $\delta(B_+) \leq 1$.*

*Proof.* Most of the proof is the same for both *A)* and *B)*; we split near the end. Let $\mathcal{I} = \{i : A_{i,t}^m \cap B \neq \emptyset$ for some $t\}$. By Observation 2.14 $B$ is $G$-invariant, so for each $i \in \mathcal{I}$ if $A_{i,t}^m \cap B \neq \emptyset$ for some $t$, then it holds for any $t$, because $G$ acts transitively on the set of petals $\{A_{i,t}^m : 1 \leq t \leq \ell_i^m\}$, as, by construction, they have the same base $B_i^m$ and they are on the same strata.

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[20]We will indicate a slight modification of the proof for the Steiner case in Lemma 5.14.

Let $\nu = \mu(B_j^{m+1}, A_{j,1}^{m+1}) - 1$. The conditions $|A_{j,1}^{m+1}| > 1$ and $|A_{i,f}^m| > 1$ for each $i$ and $f$ such that $A_{i,f}^m \cap B_j^{m+1} \neq \emptyset$ imply that we may apply Lemma 3.21.2 below.

First, we show $\delta(B_-/B_+) \geqslant \nu \cdot \sum_{i \in \mathcal{I}} \ell_i^m$. For this, using notation 3.20, by Lemma 3.21.1 and the last hypothesis, both $C_-^d$ and $C_+^d$ are nonempty. Since, we noted that for each $i \in \mathcal{I}$ and each $t$ with $1 \leq t \leq \ell_i^m$, $A_{i,t}^m \cap B \neq \emptyset$, by Lemma 3.21.1 again, $C_-^d \cap A_{i,t}^m \neq \emptyset$. So, invoking the definition of good pair we have $\delta(C_-^d \cap A_{i,f}^m / B \cup C_+^d) < 0$. Indeed, $0 > \delta(C \cap A_{i,f}^m / B \cup (C - A_{i,f}^m) = \delta(C_-^d \cap A_{i,f}^m / B \cup C_+^d)$ because all petals on the $m$-strata are $\delta$-independent over $\mathfrak{A}^{m-1}$. Consequently, by submodularity of $\delta$ for any $i \in \mathcal{I}$ and $f = 1, \ldots, \ell_i^m$, taking into account for the first equality that any two petals on the same strata, namely on $\mathfrak{A}^m$, are $\delta$-independent over $\mathfrak{A}^{m-1}$, we can conclude:

$$(1) \qquad \delta(C_-^d / B \cup C_+^d) = \sum_{i \in \mathcal{I}} \sum_{1 \leqslant f \leqslant \ell_i^m} \delta(C_-^d \cap A_{i,f}^m / B \cup C_+^d) \leq - \sum_{i \in \mathcal{I}} \ell_i^m.$$

Since $\mathfrak{A}^{m-1} \leqslant M$ and by monotonicity

$$(2) \qquad 0 \leqslant \delta(\bigcup_{1 \leqslant d \leqslant \nu} C_-^d \cup B_- / \mathfrak{A}^{m-1}) \leqslant \delta(\bigcup_{1 \leqslant d \leqslant \nu} C_-^d \cup B_- / \bigcup_{1 \leqslant d \leqslant \nu} C_+^d \cup B_+).$$

But by the definition of $\delta(A/C)$, we can rewrite the last term to obtain

$$(3) \qquad 0 \leqslant \delta(\bigcup_{1 \leqslant d \leqslant \nu} C_-^d / B \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d) + \delta(B_- / (B_+ \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d)).$$

The last term of the right hand side of equation 3 equals $\delta(B_-/B_+)$ by Lemma 3.21.2. And, the first term satisfies

$$(4) \quad \delta(\bigcup_{1 \leqslant d \leqslant \nu} C_-^d / B \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d) \leq \sum_{i \in \mathcal{I}} \sum_{\substack{1 \leqslant f \leqslant \ell_i^m \\ 1 \leqslant d \leqslant \nu}} \delta((C_-^d \cap A_{i,f}^m) / B \cup C_+^d) \leqslant -\nu \sum_{i \in \mathcal{I}} \ell_i^m$$

since each $C_-^d$ contributes at most $-\sum_{i \in \mathcal{I}} \ell_i^m$ for each $d = 1, \ldots, \nu$ by equation (1). Substituting our evaluations of the two terms on the right hand side of equation (3) (one from Lemma 3.21.2) and transposing, we have

$$(5) \qquad \delta(B_-/B_+) \geqslant \nu \sum_{i \in \mathcal{I}} \ell_i^m.$$

Now, $B = B_+ \cup B_-$ implies $\delta(B) = \delta(B_-/B_+) + \delta(B_+)$. So

$$(6) \qquad \delta(B) \geqslant \nu \sum_{i \in \mathcal{I}} \ell_i^m + \delta(B_+).$$

A) Assume that $\mu_j^{m+1} \geq 3$; then $\nu \geqslant 2$. If $\delta(B) = 2$, then $\nu = 2$, $\mathcal{I} = \{i\}$ for some $i$, $\ell_i^m = 1$, and $A_{i,1}^m$ is $G$-invariant.

If $\delta(B) \geqslant 3$, divide equation (6) by $\delta(B)$. Then, since $\nu \geqslant \delta(B) - 1$, substituting in equation 6, we obtain

$$(7) \qquad 1 = \frac{\delta(B)}{\delta(B)} \geqslant \frac{(\delta(B) - 1) \sum_{i \in \mathcal{I}} \ell_i^m}{\delta(B)} + \frac{\delta(B_+)}{\delta(B)}$$

Since $\delta(B) \geqslant 3$, $\frac{\delta(B)-1}{\delta(B)} \geqslant \frac{2}{3}$ so

$$(8) \qquad\qquad 1 \geqslant \frac{2\sum_{i\in\mathcal{I}} \ell_i^m}{3} + \frac{\delta(B_+)}{\delta(B)}$$

Equation (8) implies $\mathcal{I} = \{i\}$ for some $i$, $\ell_i^m = 1$, and $A_{i,1}^m$ is $G$-invariant. Equation (6) implies

$$\delta(B) \geqslant \nu + \delta(B_+) \geq (\delta(B) - 1) + \delta(B_+).$$

So, $\delta(B_+) \leq 1$.

B) Now we assume that $\mu_j^{m+1} = 2$. Then, invoking the first hypothesis, $\delta(B) = 2$. In this case, since $\nu = 1$, equation (6) implies $2 \geqslant \sum_{i\in\mathcal{I}} \ell_i^m + \delta(B_+)$ and so $\delta(B_+) \leqslant 1$.                                                                        ∎

## 4. dcl IN HRUSHOVSKI'S FIRST EXAMPLE $\hat{T}_\mu$

In this section we describe $\mathrm{dcl}^*(I)$ in the main example $\hat{T}_\mu$ (Definition 0.1) in [Hru93]. We find that the values of $\mu$ for good pairs with $\delta(B) = 2$ distinguish whether $\mathrm{dcl}^*(I)$ may be empty. Subsection 4.1 deals with $\mathrm{dcl}^*$ under a stronger hypothesis on $\mu$ to conclude $\mathrm{dcl}^*(I) = \emptyset$. Subsection 4.2 provides an example that when $\mu(B, C) = 2$ for certain good pairs, there is a theory where $\mathrm{dcl}^*(I) \neq \emptyset$. However, Subsection 4.4 sdcl$^*$ shows that such examples still satisfy $\mathrm{sdcl}^*(I) = \emptyset$ and fail elimination of imaginaries. That proof uses a deeper study of flowers and bouquets from Subsection 4.3.

The very raw idea is that if some petal $A$ on the $(i+1)$-th strata is $G$-invariant, and $\mu(A, B) \geqslant 3$ then it determines a petal on the $i$-th strata, which is also $G$-invariant, for each positive $i$; but Lemma 3.11 implies that no petal on the 1st strata is $G$-invariant, for a contradiction. The hypothesis that $\mu(B, C) \geq 3$ for any good pair $(B, C)$ with $\delta(B) = 2$ implies that this idea works and $\mathrm{dcl}^*(I) = \emptyset$. However in the general case $\mathrm{dcl}^*(I)$ may not be empty. We consider in Section 4.4 a more complicated construction in order to prove that $\mathrm{sdcl}^*(I) = \emptyset$.

Note, however that the family of theories described in Proposition 18 of [Hru93] (Steiner quasigroups) as well as the Steiner *triple* system of [BP21] have a truly binary function defined by $R$. The definition of $\boldsymbol{K}_0$ from $\delta$ in Hrushovki's Proposition 18 is non-standard; in the linear space case, $\mu(\boldsymbol{\alpha}) = 1$, for $\alpha$ the good pair of a line (Definition 1.2.6). Section 5 adapts our main results for Steiner systems.

### 4.1. $G_I$: No truly $n$-ary definable functions

We slightly vary Hrushovski's original example by adding a further adequacy requirement (Definition 1.1.7.1).

**Definition 4.1.1.** *We say that a function $\mu$ (or theory $T_\mu$) bounding good pairs triples if for all good pairs $(A/B)$ with $|A| > 1$, $\mu(A/B) \geqslant \delta(B)$:*

$$\delta(B) = 2 \Rightarrow \mu(A/B) \geq 3.$$

In this section, using this triples condition, we $G$-decompose a finite set using $G_I$ (fixing $I$ pointwise) and show $\hat{T}_\mu$ is essentially unary and so fails to eliminate imaginaries. We give a more refined argument using $G_{\{I\}}$ in Section 4.4 showing that even with truly $n$-ary function (i.e. dropping the 'triples' hypothesis), $\hat{T}_\mu$ must fail to eliminate imaginaries.

**Theorem 4.1.2.** *Assume that $\hat{T}_\mu$ triples. Let $I$ be a finite independent set that contains at least 2 elements. Fix a $G$-normal $\mathfrak{A} \leqslant M \models \hat{T}_\mu$ with height $m_0$. For every $m \leqslant m_0$, $\mathfrak{A}^m \cap \mathrm{dcl}^*(I) = \emptyset$.*

*Thus, $\mathrm{dcl}^*(I) \cap \mathfrak{A} = \emptyset$; so there is no truly n-ary $\emptyset$-definable function (Definition 2.9) for each $n \geq 2$ and $\hat{T}_\mu$ does not admit elimination of imaginaries. As a corollary, we obtain that $\mathrm{dcl}(J) = \bigcup_{a \in J} \mathrm{dcl}(a)$ for any independent set $J \subseteq M$.*

By the definition of $\mathfrak{A}^0$, $\mathfrak{A}^0 \cap \mathrm{dcl}^*(I) = \emptyset$. It suffices to show by induction on $m \geqslant 1$ that for each $e \in \mathfrak{A}^m$, $d(G_I(e)) \geqslant 2$. As, if $G_I(e) = \{e\}$ then $d(G_I(e)) = 1$. But we must begin with $m = 1$ since elements $e \in \mathfrak{A}^0$ may have $d(G_I(e)) = 1$. We obtain the conclusion by proving Lemma 4.1.3 by double induction. Note that the truth of $\dim_m$ and $\mathrm{moves}_m$ each depend on the choice of $G$ as $G_I$. Once we have this decomposition, satisfying moves for $G_I$-normal $\mathfrak{A} \leqslant M$ of any height, we can conclude $\mathrm{dcl}^*_M(I) = \emptyset$.

**Lemma 4.1.3.** *Assume that $\hat{T}_\mu$ triples. For $m \geqslant 1$,*

(1) $\dim_m$: *$d(E) \geq 2$ for any $G_I$-invariant set $E \subseteq \mathfrak{A}^m$, which is not a subset of $\mathfrak{A}^0$.*

(2) $\mathrm{moves}_m$: *No $A^m_{f,k}$ is $G_I$-invariant.*

The remainder of this section is devoted to the proof of Lemma 4.1.3. If $m_0$ is the height of $\mathfrak{A}$, then $\dim_{m_0}$ gives Theorem 4.1.2.

For each $m < m_0$, each $\mathfrak{A}^{m+1}$, and each $j$, $k$ such that $A^{m+1}_{j,k} \subseteq \mathfrak{A}^{m+1}$, Observation 2.14 implies that if $A^{m+1}_{j,k}$ is $G_I$-invariant, then $B^{m+1}_j$ is $G_I$-invariant. However the converse is false. The main part of the construction in Section 3 was to describe the family of $G_I$-conjugates $A^{m+1}_{j,k}$ over $B^{m+1}_j$ of $A^{m+1}_{j,1}$. We now take into account that the $B^{m+1}_j$ need not be $G_I$-invariant.

How do we use a joint induction? The $A^{m+1}_{j,i}$ are disjoint. If either $|J^{m+1}_j| > 1$ (Definition 3.14) or $\ell^m_j > 1$, $A^{m+1}_{j,i}$ moves and so no element of $A^{m+1}_{j,i}$ is definable over $I$. But, in Section 4.2, we show that when some $\mu^{m+1}_j = 2$, it may be that $|J^{m+1}_j| = 1$. So, in this section we add an hypothesis implying each relevant $\mu^{m+1}_j \geqslant 3$. In order to prove by induction on $m$ that no $A^m_{j,i}$ is $G_I$-invariant (Conclusion 4.1.7), we need the dual hypothesis $dim_m$. So, the simultaneous induction is organized as follows:

$$\dim_m \wedge \mathrm{moves}_{m+1} \quad \Rightarrow \quad \dim_{m+1} \text{ (Lemma 4.1.4)}$$
$$\dim_m \wedge \mathrm{moves}_m \quad \Rightarrow \quad \mathrm{moves}_{m+1} \text{ (Conclusion 4.1.7)}$$

In the following Corollary 4.1.5, we slightly modify the proof of Lemma 4.1.4 to ground the induction by showing $\dim_1$ and $\mathrm{moves}_1$.

We use without further notice the fact that for any $A \subseteq B$, $d(A) \leqslant d(B)$, e.g. $d(X) \leqslant d(G_I(X))$. Note that Construction 3.8 shows that any base $B$ arising in the construction of $\mathfrak{A}^{m+1}$ satisfies $1 \leqslant d(B) \leqslant v$, where $v = |I|$.

Lemma 4.1.4 is formulated for $G$; it is applied for $G_I$ in Section 4.1 while in Section 4.4 we use $\mathrm{sdim}_m$ instead of $\dim^G_m$ with $G = G_{\{I\}}$. The 'moves hypothesis' in Lemma 4.1.4 will follow from the inductive hypothesis in the main proof.

**Lemma 4.1.4.** *Fix $m$ with $1 \leqslant m \leqslant m_0$. If[21] $\dim^G_m$ and $G$ moves $A^{m+1}_{j,k}$ then $\dim^G_{m+1}$. That is, for each $e \in A^{m+1}_{j,k}$, $d(G(e)) \geq 2$.*

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[21]We do not use the adequacy hypothesis (Definition 4.1.1) in proving this lemma.

*Proof.* Fix $m \leqslant m_0$, $j < q_m$, $k$ and $e$ with $e \in A_{j,k}^{m+1}$. We show $d(G(e)) \geq 2$. Let $E = G(e)$. Clearly $d(E) \leq v$, since $e \in \mathrm{acl}(I)$. By Lemma 3.7.2, $E \cap A_{j,k}^{m+1} \neq \emptyset$ for each $k \leq \ell_j^{m+1}$. And for each $j' \in J_j^{m+1}$ the map taking $B_j^{m+1}$ to $B_{j'}^{m+1}$ and Construction 3.8, guarantee each $E \cap A_{j',k'}^{m+1} \neq \emptyset$ for $k' \leq \ell_{j'}^{m+1} = \ell_j^{m+1}$.

Note that $\mathrm{icl}(E) \subseteq \mathfrak{A}^{m+1}$, because $\mathfrak{A}^{m+1} \leq M$. The proof now breaks into three cases.

(1) If all the bases $B_{j'}^{m+1}$ are subsets of $\mathrm{icl}(E)$ (for $j' \in J_j^{m+1}$), the hypothesis $\dim_m$ and the monotonicity of $d$ implies

$$2 \leq d(\bigcup_{g \in G_I} g(B_j^{m+1})) \leq d(\mathrm{icl}(E)) \leq v$$

and we finish.

(2) Suppose no $B_{j'}^{m+1}$ (with $j' \in J_j^{m+1}$) is a subset of $\mathrm{icl}(E)$. For each $j' \in J_j^{m+1}$ there is a $t'$ such that $A_{j',t'}^{m+1} \cap \mathrm{icl}(E) \neq \emptyset$. And since $B_{j'}^{m+1} \nsubseteq \mathrm{icl}(E)$, Definition 1.1.5.3 of base[22] implies $\delta(\mathrm{icl}(E) \cap A_{j',t}^{m+1}/\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq 1$. Thus,

$$v \geqslant d(E) = \delta(\mathrm{icl}(E)) = \delta((\mathrm{icl}(E) - \mathfrak{A}^m)/\mathrm{icl}(E) \cap \mathfrak{A}^m) + \delta(\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq$$

$$\geq \delta((\mathrm{icl}(E) - \mathfrak{A}^m)/\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq \sum_{j' \in J_j^{m+1}} \sum_{t=1}^{\ell_{j'}^{m+1}} \delta(\mathrm{icl}(E) \cap A_{j',t}^{m+1}/\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq 2.$$

The double summation is at least 2 because either $|J_j^{m+1}| \geqslant 2$ and there are 2 non-zero outer summands or $J_j^{m+1} = \{j\}$ and since $G_I$ moves $A_{j,1}^{m+1}$,

$$\sum_{t=1}^{\ell_j^{m+1}} \delta(\mathrm{icl}(E) \cap A_{j,t}^{m+1}/\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq 2.$$

(3) Suppose one of the $B_{j'}^{m+1}$ (with $j' \in J_j^{m+1}$) is a subset of $\mathrm{icl}(E)$ and another one $B_{j''}^{m+1}$ is not. Clearly,

$$d(E) = \delta(\mathrm{icl}(E)) = \delta(\mathrm{icl}(E) \cap \mathfrak{A}^m) + \delta(\mathrm{icl}(E) - \mathfrak{A}^m/\mathrm{icl}(E) \cap \mathfrak{A}^m).$$

Say $B_{j_1}^{m+1} \subseteq \mathrm{icl}(E)$ and $B_{j_2}^{m+1} \nsubseteq \mathrm{icl}(E)$. The first summand is at least 1, because $B_{j_1}^{m+1}$ is a subset of $\mathrm{icl}(E) \cap \mathfrak{A}^m$, but is not a subset of $\mathrm{acl}(\emptyset)$; as, otherwise $A_{j_1,1}^{m+1}$ is a subset of $\mathrm{acl}(\emptyset)$. The second summand is also at least one, because

$$\delta(\mathrm{icl}(E) - \mathfrak{A}^m/\mathrm{icl}(E) \cap \mathfrak{A}^m) = \sum_{j' \in J_{j_2}^{m+1}} \sum_{t=1}^{\ell_{j'}^{m+1}} \delta(\mathrm{icl}(E) \cap A_{j',t}^{m+1}/\mathrm{icl}(E) \cap \mathfrak{A}^m)$$

and for some $t \leqslant \ell_{j_2}^{m+1}$, $\delta(\mathrm{icl}(E) \cap A_{j_2,t}^{m+1}/\mathrm{icl}(E) \cap \mathfrak{A}^m) \geq 1$, because $B_{j_2}^{m+1} \nsubseteq \mathrm{icl}(E)$. ∎

We establish the $m = 1$ step of Lemma 4.1.3 by emulating the proof of Lemma 4.1.4; we can't really apply the result as $\dim_0$ was not defined.

**Corollary 4.1.5.** *Both* $\dim_1$ *and* $\mathrm{moves}_1$ *hold. More precisely, for any* $e \in \mathfrak{A}^1 - \mathfrak{A}^0$, $d(G_I(e)) \geq 2$.

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[22]That is, the base $X$ of $C/A$ is the least $X$ such that $\delta(A'/X) = 0$ for every $A' \subseteq A$.

*Proof.* By Lemma 3.11, we have moves$_1$ and for each $j$, $d(B_j^1) \geq 2$. Now follow the proof of Lemma 4.1.4, noting that it only uses that $d(B_j^1) \geq 2$. ∎

**Lemma 4.1.6.** *If* moves$_m$, dim$_m$ *and* $A_{j,1}^{m+1}$ *is* $G_I$*-invariant, then*

(1) *For* $B = B_j^{m+1}$, $d(B) \geq 2$.
(2) $|A_{j,1}^{m+1}| > 1$.
(3) *If in addition,* $\mu_j^{m+1} \geqslant 3$, *then* $|A_{i,f}^m| > 1$ *for any* $i$, $f$ *such that* $A_{i,f}^m \cap B \neq \emptyset$.

*Proof.* By Lemma 4.1.5, we may assume $m \geqslant 1$. (1) By Observation 2.14, $A_{j,1}^{m+1}$ is $G_I$-invariant implies $B_j^{m+1}$ is $G_I$-invariant. Then, since $B \nsubseteq \mathfrak{A}^0$, dim$_m$ implies that $d(B) \geq 2$.

(2) By 1) we may apply Lemma 3.19.2 to conclude that if $A_{j,1}^{m+1} = \{c\}$, then $A_{i,1}^m$ is $G_I$-invariant for some $i$, contradicting moves$_m$.

(3) Directly follows from Lemma 3.22. ∎

With the next result we can complete the induction. The hypothesis that each $\mu_j^{m+1} \geqslant 3$ is essential for the induction.

**Conclusion 4.1.7.** *Under the hypotheses of Theorem 4.1.2,* move$_m$ *and* dim$_m$ *imply* move$_{m+1}$.

*Proof.* Assume for contradiction that $A_{j,1}^{m+1}$ is $G_I$-invariant. The assumption dim$_m$ and Lemma 4.1.6 imply both $d(B) \geq 2$, so $\delta(B) \geqslant 2$, and that the hypotheses of Lemma 3.21 hold. And Lemma 3.21 gives the remaining hypotheses of Lemma 3.23. Indeed, if $C_+^d = \emptyset$ for some $d$, we obtain that $A_{i,f}^m$ is $G_I$-invariant for some $i, f$ by Lemma 3.21.3; that contradicts moves$_m$. Now, by Lemma 3.23 we obtain that $A_{i,1}^m$ is $G_I$-invariant for some $i$; that contradicts moves$_m$. ∎

Completing this induction gives Lemma 4.1.3 and so Theorem 4.1.2, asserting there are no $\emptyset$-definable truly $n$-ary functions. Now we generalize the result by allowing parameters (Corollary 4.1.10).

**Lemma 4.1.8.** *Let $I$ be a finite independent set and $J \subset I$. Let a tuple $\mathbf{c} \in \mathrm{acl}(J)$ and $d \in \mathrm{acl}(I)$. Let $\mathfrak{A}_I$ be the $G_I$-normal closure of $I \cup \{d, \mathbf{c}\}$, and $\mathfrak{A}_J$ the $G_J$-normal closure of $J \cup \{\mathbf{c}\}$. Then there exists a tree decomposition $\mathcal{T}_I$ of $\mathfrak{A}_I$ such that the collection $\mathcal{T}_J$ of petals $A_{f,j}^m$ that are based in $\mathfrak{A}_J$ are a downwardly closed subset of $\mathcal{T}_I$, whose union is $\mathfrak{A}_J$. (By downwardly closed we mean that if $A_{f,j}^m \subseteq \mathfrak{A}_J$ then $B_f^j \subseteq \mathfrak{A}_J$.)*

*Proof.* We note $G_J(\mathbf{c}) = G_I(\mathbf{c})$ is the finite collection of automorphic images of $\mathbf{c}$ over $J$, as for any automorphism $\rho$ fixing $J$ pointwise $\rho \upharpoonright \mathfrak{A}_J$ can be extended to one fixing $I$. By this equality and by monotonicity of the intrinsic closure we obtain

$$\mathfrak{A}_J = \mathrm{icl}(J \cup G_J(\mathbf{c})) = \mathrm{icl}(J \cup G_I(\mathbf{c})) \subseteq \mathrm{icl}(I \cup G_I(d, \mathbf{c})) = \mathfrak{A}_I$$

Thus, $\mathfrak{A}_J \subseteq \mathfrak{A}_I$. Let $L_J = \langle X_i : i \leq r_J \rangle$, where $X_0 \leq X_1 \leq \cdots \leq X_{r_J}$, be a linear decomposition of $\mathfrak{A}_J$; that is, $X_{i+1}$ is a 0-primitive extension of $X_i$ for each $i < r_j$ and $X_{r_J} = \mathfrak{A}_J$. Since $I - J$ is independent over $J$, $I - J$ is independent over $\mathfrak{A}_J$, moreover $I - J$ and $\mathfrak{A}_J$ are fully independent. Then

$$I = X_0' \leq X_1 \cup (I - J) \leq \cdots \leq X_{r_J} \cup (I - J)$$

is a linear chain of 0-primitive extensions and for each $i \leqslant r_J$ the base of $X_{i+1}$ is a subset of $\mathfrak{A}_J$. Thus, it can be considered as an initial segment of a linear decomposition $L_I$ of $\mathfrak{A}_I$ by Lemma 3.4. Let $\mathcal{T}_K$ be the tree decomposition of $\mathfrak{A}_K$ determined by $L_K$ for $K \in \{I, J\}$. We now show that if a petal in $\mathfrak{A}_I$ intersects $\mathfrak{A}_J$, it is one of $X_{i+1} - X_i$ and a subset of $\mathfrak{A}_J$. Clearly, $\{\mathrm{acl}(a_i) \cap \mathfrak{A}_I : a_i \in J\} \subseteq \{\mathrm{acl}(a_i) \cap \mathfrak{A}_I : a_i \in I\}$ and $\{\mathrm{acl}(a_i) \cap \mathfrak{A}_I : a_i \in J\} \subseteq \mathfrak{A}_J^0$ by Construction 3.8. So we only have to show the result for petals of $\mathcal{T}_I$ of the form $A_{f,j}^{m+1}$. Note that any $A_{f,j}^{m+1}$ that nontrivially intersects $\mathfrak{A}_J$ is contained in $\mathfrak{A}_J$, since $\mathfrak{A}_I^m \cup \mathfrak{A}_J \leqslant M$ (We start with closed subsets $I$ and $J$ and obtain $\mathfrak{A}_I^m \cup \mathfrak{A}_J$ by add a sequence of 0-primitive extensions) and $A_{f,j}^{m+1}$ is 0-primitive over $\mathfrak{A}_I^m$.

We need to show $A_{f,j}^{m+1} \in \mathcal{T}_J$.

**Level: 1** If $m = 0$ and $B_f^{m+1} \nsubseteq \bigcup_{a \in J} \mathrm{acl}(a)$, then $A_{f,j}^{m+1} \nsubseteq \mathfrak{A}_J$ by Construction 3.8. So, $A_{f,j}^{m+1} \subseteq \mathfrak{A}_J^1$ implies $B_f^{m+1} \subseteq \mathfrak{A}_J$.

**Level: $m + 1$** Since $A_{f,j}^{m+1} \subseteq \mathfrak{A}_J$ and is some $\hat{X}_i$ of the given initial segment of the linear decomposition, $B_f^{m+1} \subseteq \mathfrak{A}_J$; so $A_{f,j}^{m+1} \in \mathcal{T}_J$.

By induction we have Lemma 4.1.8. Since $\mathfrak{A}_J$ is $G_J$-normal, $\bigcup \mathcal{T}_J = \mathfrak{A}_J$.    ∎

We have the following immediate corollary. Let $r$ be the height (i.e. the largest index $k$ of an $A_{f,i}^k$ with $A_{f,i}^k \subseteq \mathfrak{A}_J$) of $\mathfrak{A}_J$.

**Lemma 4.1.9.** *For any $m \leqslant r$, and any $A_{p,q}^{m+s+1} \subseteq \mathfrak{A}_J - \mathfrak{A}_I^m$, and for any $A_{f,j}^{m+1} \subseteq \mathfrak{A}_I^{m+1} - \mathfrak{A}_J$, $r(A_{p,q}^{m+s+1}, A_{f,j}^{m+1}) = 0$.*

*Proof.* If $s = 0$ the result is clear since the petals over $\mathfrak{A}_I^m$ are fully independent. If $0 < s < r - m$, we have shown in Lemma 4.1.8 for any $m \leqslant r$, any $p, q$ with $A_{p,q}^{m+s+1} \subseteq \mathfrak{A}_J - \mathfrak{A}_I^m$, that $B_p^{m+s+1} \subseteq \mathfrak{A}_J^{m+s}$. So for any $s \geqslant 1$, $r(A_{p,q}^{m+s+1}, \mathfrak{A}_I^{m+s} - \mathfrak{A}_J) = 0$. In particular, $r(A_{p,q}^n, A_{f,j}^{m+1}) = 0$ for any $A_{f,j}^{m+1} \subseteq \mathfrak{A}_I - \mathfrak{A}_J$, and any $A_{p,q}^n \subseteq \mathfrak{A}_j$ with $m + 1 \leqslant n \leqslant r$.    ∎

**Corollary 4.1.10.** *Assume that $\hat{T}_\mu$ triples. Then, for $n > 1$, no truly $n$-ary function is definable in $\hat{T}_\mu$ even with parameters.*

*Proof.* Let $M \models \hat{T}_\mu$ and suppose $\phi(y, \mathbf{x}, \mathbf{c})$ defines a truly $n$-ary function $y = g(\mathbf{x})$ on $M^n$. Taking $M$ saturated, we can choose $a_1, a_2, \ldots, a_n$ independent over $\mathbf{c}$. Fix $\mathbf{c}'$ maximal independent inside $\mathbf{c}$. Then $I = \{a_1, \ldots a_n\} \cup \{x \in \mathbf{c}'\}$ is independent. Let $\mathfrak{A} = \mathfrak{A}_I$ be the $G_I$-normal closure of $I \cup \{d, \mathbf{c}\}$, where $d = g(\boldsymbol{a})$.

For each $q \in \{1, \ldots, n\}$ we define $I_q = \{a_q\} \cup \{x \in \mathbf{c}'\}$. Clearly, $I_q$ is independent as a subset of the independent set $I$. Let $\mathfrak{A}_q$ be the $G_{I_q}$-normal closure of $\{\mathbf{c}\}$, that is, $\mathrm{icl}(I_q \cup G_{I_q}(\mathbf{c}))$. We apply Lemma 4.1.8 with $J = I_q$.

We now consider two cases. First, assume that $d$ is not in $\bigcup_{q=1}^n \mathfrak{A}_q$. Let $d \in A_{f,j}^{m+1}$, where $A_{f,j}^{m+1}$ is a petal in the $G_I$-decomposition of $\mathfrak{A}$. Applying moves$_{m+1}$ to $\mathfrak{A}$ we obtain that there exists $A_{f,j'}^{m+1}$ with $j \neq j'$. By strata decomposition there exists $\tau \in \mathrm{aut}(M/\mathfrak{A}^m)$ such that $\tau(A_{f,j}^{m+1}) = A_{f,j'}^{m+1}$.

Since $q$ is arbitrary, Lemma 4.1.9 implies $\sum_{q=1}^n r(A_{f,j}^{m+1}, \mathfrak{A}_q - \mathfrak{A}^m) = 0$. But we need more, namely, $r(A_{f,j}^{m+1}, \bigcup_{q=1}^n \mathfrak{A}_q - \mathfrak{A}^m) = 0$. Assume the contrary, that there exist $a \in A_{f,j}^{m+1}$, $b \in \mathfrak{A}_q - \mathfrak{A}^m$, and $c \in \mathfrak{A}_p - (\mathfrak{A}^m \cup \mathfrak{A}_q)$ with $R(a, b, c)$. Not both $b$ and $c$ can be on the same strata, because they are in different petals but petals on the same strata are fully independent. Then one of them, say $b$, is in a lower

strata, say, in $\mathfrak{A}^{m+k}$ for some $k \geq 1$. Then $\{a,b\} \subseteq \mathfrak{A}^{m+k}$ and $(\{a,b\},c)$ is a good pair. By $G$-decomposition we obtain that $\{c\} = A_{u,v}^{m+k+1}$ for some $u$ and $v$. As $A_{u,v}^{m+k+1} \subseteq \mathfrak{A}_p$, by Lemma 4.1.8 its base $B_u^{m+k+1} = \{a,b\}$ is in $\mathfrak{A}_p$. So, both $b$ and $c$ are in $\mathfrak{A}_p$, but $r(A_{u,v}^{m+k+1}, \mathfrak{A}_p - \mathfrak{A}^m) = 0$ by Lemma 4.1.9, for a contradiction.

Now we finish the proof of the first case. Let $\rho = \tau \upharpoonright (\mathfrak{A}^m \cup A_{f,j}^{m+1}) \cup id_{\bigcup_{q=1}^n \mathfrak{A}_q - \mathfrak{A}^m}$. Recall, that $\tau \upharpoonright \mathfrak{A}^m$ is the identity, so $\rho$ moves only $A_{f,j}^{m+1}$ to $A_{f,j'}^{m+1}$ and fixes $\mathfrak{A}^m \cup \bigcup_{q=1}^n \mathfrak{A}_q$. By our last remark, $r(A_{f,j'}^{m+1}, \mathfrak{A}^m \cup \bigcup_{q=1}^n \mathfrak{A}_q) = r(A_{f,j}^{m+1}, B_f^{m+1})$. Taking into account $A_{f,j}^{m+1} \cup B_f^{m+1} \cong_{B_f^{m+1}} A_{f,j'}^{m+1} \cup B_f^{m+1}$, we obtain $\rho$ is a partial isomorphism and can be extended to an isomorphism $\rho'$ of $M$, as its domain is closed in $M$. So, $\rho'$ fixes $\boldsymbol{a}$ and $\mathbf{c}$, but, by choice of case, moves $d$, for a contradiction.

Theorem 4.1.10 by doing the second case: $d \in \bigcup_{q=1}^n \mathfrak{A}_q$, say, $d \in \mathfrak{A}_1$. Let $a_n'$ be independent over $\boldsymbol{ac}$. Let $\rho \in \mathrm{aut}(M/\mathbf{c})$ fix $a_q$ for $q < n$ and move $a_n$ to $a_n'$. Then $\rho \in G_{I_1}$, so $\rho(d) = d$ and $d = g(a_1, \ldots, a_{n-1}, a_n')$, that contradicts $f$ is truly $n$-ary. $\blacksquare$

**Remark 4.1.11.** Note that depending on whether $\hat{T}_\mu$ triples, there may or may not be a truly $n$-ary function. [EF12, Theorem 3.1] show that for any $\hat{T}_\mu$ the geometry (i.e. of the countable saturated model $M$), with any finite set $X$ with $|\mathrm{acl}_M(X)|$ infinite named, is isomorphic to that of the $\omega$-stable version of the construction[23]. Since $\mathrm{acl}(\emptyset)$ is easily made infinite e.g. [BP21, Lemma 5.26], it is easy to construct examples with the same geometry.

We show now that for our general situation $\mathrm{acl}_M(X)$ is infinite for any finite $X$. For this, it is sufficient to show that for any finite set $B$ there are infinitely many pairwise non-isomorphic good pairs $A/B$. Let $B = \{b_1, \ldots, b_k\}$. Let $n \geq \max\{3, k\}$ and $A_n = \{a_1, \ldots, a_n\}$. Let $\langle c_1, \ldots, c_n \rangle$ be a sequence over $B$, that contains each element of $B$. We put $R(a_i, a_{i+1}, c_i)$ for each $i \in \{1, \ldots, n-1\}$ and $R(a_n, a_1, c_n)$.

In contrast when $B = \emptyset$ the Hrushovski adequacy condition can be satisfied when $\mu(A/\emptyset) = 0$ for any $A$ primitive over $\emptyset$. And it is not hard to show that amalgamation still holds [Bal23]. But, in such a case the geometries are not elementarily equivalent as the formula $(\forall x)D_1(x)$ holds in the pregeometry with $\mathrm{acl}(\emptyset) = \emptyset$ where $D_n$ is a predicate that holds of $n$ independent elements.

*Thus the varied behavior of our examples show the coarseness of classifying only by geometry.*

## 4.2. **Counterexample**

Let $M$ be any model of $\hat{T}_\mu$ with $\mu(\boldsymbol{\alpha}) = 2$. The following example satisfies $\mathrm{dcl}^* I \neq \emptyset$ for $I = \{a_1, a_2\}$. This shows the assumption that $\mu(A/B) \geq 3$ for any good pair $(A/B)$, where $\delta(B) = 2$, is essential to show $\mathrm{dcl}^* I = \emptyset$ (Theorem 4.1.2).

We sketch the motivation for the example. Recall, that in the decomposition of $\mathfrak{A}$ into strata we have the 0-strata $\mathfrak{A}^0$, that is obtained as $\mathfrak{A}^0 = \bigcup_{i=1}^2 (\mathfrak{A} \cap \mathrm{acl}(a_i))$. We have used the hypothesis, $\mu(A/B) \geq 3$ for any good pair $(A/B)$ with $\delta(B) = 2$, twice in the proof of Lemma 3.23. Recall Example 3.13, which shows that it is possible that $\mathfrak{A}$ is $G_{\{I\}}$-invariant and $A_{1,1}^2$ is $G_{\{I\}}$-invariant. However, in that example the elements $c_1$ and $c_2$ are indiscernible over $\{a_1, a_2\}$. This happens because the elements $a_1$ and $a_2$ are indiscernible over $\{b_1, b_2\}$ and $\{a_1, a_2\} \cup \{b_1, b_2\} \cong_{\{b_1, b_2\}}$

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[23]Mermelstein (personal communication) has shown this result extends to Steiner systems.

$\{c_1, c_2\} \cup \{b_1, b_2\}$. Below we modify $A_{1,1}^1$ so that $a_1$ and $a_2$ are no longer indiscernible over $A_{1,1}^1 \cup A_{1,2}^1$.

Let $tp(a_1/A_{1,1}^1) \neq tp(a_2/A_{1,1}^1)$: for instance the number of relations of $a_1$ with $A_{1,1}^1$ and the number of relations of $a_2$ with $A_{1,1}^1$ are different. Then $A_{1,1}^2$ contains a copy of $a_1$, or a copy of $a_2$, or both. But if $a_1$ and $a_2$ are distinguishable over $B_1^1$ inside $\mathfrak{A}^1$ (realize different types over $B_1^1$), then their copies $\alpha_1, \alpha_2$ in $A_{1,1}^2$ are distinguishable, too. But then the $\alpha_i$ can belong to $\mathrm{dcl}^*(I)$.

The following construction describes an accessible case of the general strategy described in Remark 4.2.3.

**Example 4.2.1.** We consider the following example with universe $\{a_1, a_2\} \cup \{c_1, c_2, c_3\}$ and then 9 more points with $d, \delta, \gamma$ replacing $c$. We define the following relations:

  0) $\mathfrak{A}^0 = I = \{a_1, a_2\}$.
  1,1) $A_{1,1}^1 = \{c_1, c_2, c_3\}$ with $R(a_1, c_1, c_3)$, $R(a_2, c_1, c_2)$, and $R(a_2, c_2, c_3)$.
  1,2) $A_{1,2}^1 = \{d_1, d_2, d_3\}$ with $R(a_1, d_1, d_3)$, $R(a_2, d_1, d_2)$, and $R(a_2, d_2, d_3)$.
  2,1) $A_{1,1}^2 = \{\alpha_1, \alpha_2, \gamma_1, \delta_1, \gamma_3, \delta_3\}$ with the following relations: $R(\alpha_1, \gamma_1, \gamma_3)$, $R(\alpha_2, \gamma_1, c_2)$, $R(\alpha_2, c_2, \gamma_3)$, and $R(\alpha_1, \delta_1, \delta_3)$, $R(\alpha_2, \delta_1, d_2)$, $R(\alpha_2, d_2, \delta_3)$.

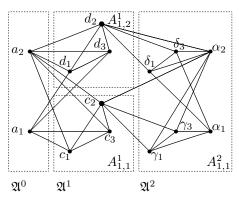Set $\mu(A_1^1) = \mu(A_1^2) = 2$. In the diagrams, we represent a triple satisfying $R$ by a triangle.



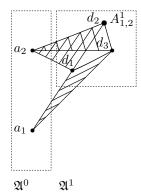FIGURE 6.  $\mathrm{dcl}^*(I) \neq \emptyset$                FIGURE 7.  $\mathfrak{A}^1$

Figure 7 shows by shaded triangles the $R$-triples in $I \cup \{d_1, d_2, d_3\}$. The petals $A_{1,1}^1$ and $A_{1,2}^1$ are isomorphic over $I$.

Clearly, $G_I(c_2) = \{c_2, d_2\}$, because there is no relation of either of these elements with $a_1$ and there are two relations of each one with $a_2$. By mapping the point with the Greek label to the corresponding Roman one, we show that $A_{1,1}^2 = \{\alpha_1, \alpha_2, \gamma_1, \gamma_3, \delta_1, \delta_3\}$ is isomorphic to $\mathfrak{A}^1 - B_1^2 = \{a_1, a_2, c_1, c_3, d_1, d_3\}$ over $B_1^2 = \{c_2, d_2\}$.

It is routine to check that $\mathfrak{A}^1 - B_1^2$ is 0-primitive over $B_1^2$. Obviously, $A_{1,1}^2$ is $G_I$-invariant. The element $\alpha_2$ is a unique element in $A_{1,1}^2$ which is in 4 relations in $\mathfrak{A}^2$, so $\alpha_2 \in \mathrm{dcl}^*(I)$.

**Remark 4.2.2.**      (1) In Example 4.2.1 moves$_1$ and dim$_1$ one hold; but $\mu_{1,1}^2 = 2$ so we cannot apply Lemma 3.23.A to conclude moves$_2$. In fact, $A_{1,1}^2$ is $G_I$-invariant.

(2) Note that this example is not a linear space (Section 5); if it satisfied the linear space axiom each of $A_{1,1}^1$ and $A_{1,2}^1$ would be a clique.

(3) $\alpha_2$ is in $\mathrm{dcl}^*(I)$ but not in $\mathrm{sdcl}^*(I)$, because an automorphism which swaps $a_1$ and $a_2$ cannot preserve $A_{1,1}^1 \cup A_{1,2}^1$, since in $\mathfrak{A}^1$, $a_1$ is in two relations and $a_2$ is in four relations. Thus this structure is $G_I$-invariant but not $G_{\{I\}}$-invariant. In order to build $\mathfrak{A}^{G_{\{I\}}}$ we add new copies of $A_{1,1}^1$ and $A_{1,2}^1$:

2,1) $A_{2,1}^1 = \{c_1', c_2', c_3'\}$ with $R(a_2, c_1', c_3')$, $R(a_1, c_1', c_2')$, and $R(a_1, c_2', c_3')$.

2,2) $A_{2,2}^1 = \{d_1', d_2', d_3'\}$ with $R(a_2, d_1', d_3')$, $R(a_1, d_1', d_2')$, and $R(a_1, d_2', d_3')$.

Now there is an $f \in G_{\{I\}}$ with $f(a_0) = a_1$, $f(a_1) = a_0$ that maps $A_{1,i}^1$ to $A_{2,i}^1$. One can construct an $A_{2,1}^2$ containing $\alpha_2'$ that is the image of $\alpha_2$ under $f$.

(4) Note that $\{A_{1,1}^1, A_{1,2}^1\}$ is called a flower of $A_{1,1}^1$ over its base $B_1^1$ in Definition 4.3.1. Also note that $\{A_{2,1}^1, A_{2,2}^1\}$ is another flower of $A_{1,1}^1$ over its base $B_1^1$. The difference is that if we arrange $B_1^1$ for the first considered flower as $\langle a_1, a_2 \rangle$, then for the second flower the arrangement of $B_1^1$ must be $\langle a_2, a_1 \rangle$.

(5) In Definition 4.3.3, we call the collection $\{\{A_{1,1}^1, A_{1,2}^1\}, \{A_{2,1}^1, A_{2,2}^1\}\}$ of all of these flowers a bouquet.

We now explain here the methodology and motivation for constructing a set with non-empty $\mathrm{dcl}^*(I)$. It may be useful for further examples.

**Remark 4.2.3.** Let $E_i$ be a subset of $\mathrm{acl}(a_i)$, for $i \in \{1, 2\}$ such that $\delta(E_1) = \delta(E_2) = 1$ (that is, $E_i \leq M$). The most simple case is $E_i = \{a_i\}$. Let $B_1^1 = E_1 \cup E_2$. Then $\delta(B_1^1) = 2$. Let $A_{1,1}^1$ be any set that is good over $B_1^1$. We put $\mu(A_{1,1}^1/B_1^1) = 2$. So, $\mathfrak{A}^1 = \mathrm{icl}(I \cup E_1 \cup E_2) \cup A_{1,1}^1 \cup A_{1,2}^1$.

We choose one element $b_i$ from $A_{1,i}^1$ for $i = 1, 2$. Let $B_1^2 = \{b_1, b_2\}$. Then $\delta(B_1^2) = \delta(\mathfrak{A}^1) = 2$ and there is a chain $B_1^2 = X_0 \leq X_1 \leq \cdots \leq X_r = \mathfrak{A}^1$ such that $X_{i+1}$ is a 0-primitive extension of $X_i$. So, $X_1$ is a 0-primitive extension of $B_1^2$ and is a subset of $\mathfrak{A}^1$. We must choose $A_{1,1}^1$ and $B_1^2$ so that $X_1$ is good over $B_1^2$. This is not true in general[24], but Example 4.2.1 shows it can be done.

We are going to find $A_{1,2}^2$ inside $\mathfrak{A}^1$ in order to make $A_{1,1}^2$ $G_I$-invariant. Suppose there are $m$ copies of $X^1$ over $B_1^2$ that are inside $\mathfrak{A}^1$; put $\mu(X_1/B_1^2) = m + 1$. Let $A_{1,1}^2$ be the (m+1)-th copy of $X_1$ over $B_1^2$. Obviously, $A_{1,1}^2$ is not in $\mathfrak{A}^1$. We put $\mathfrak{A}^2 = \mathfrak{A}^1 \cup A_{1,1}^2$. If $B_1^2$ is fixed pointwise by $G_I$ (that is, $b_1$ is definable in $A_{1,1}^1$ over $I$ and $b_2$ its copy in $A_{1,2}^1$), then $\ell_1^2 = 1$ and $|J_1^2| = 1$.

The following is not essential to achieving Remark 4.2.3 but is mandated by the construction.

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[24]Here is a counterexample. To begin with we find a 0-primitive extension of a one element set, $\{b\}$. We consider $A'$ as four points $c_1, c_2, d_1, d_2$ satisfying $R(c_1, c_2, b)$, $R(d_1, d_2, b)$, $R(c_1, d_1, b), R(c_2, d_2, b)$. There are 5 points 4 edges and any subset has larger $\delta$.

We would like to make this structure 0-primitive over $a_1, a_2$. It needs one more trick. Replace $A'$ by $A$ by adding a point $c_3$ to $A'$ and replacing the edge $R(c_1, c_2, b)$ by two edges $R(c_1, c_2, c_3), R(c_2, c_3, b)$. Then, $A$ is 0-primitive over $\{b\}$. Now consider two new elements $A_0 = \{a_1, a_2\}$; we want $A$ 0-primitive over $A_0$. For this let the new relations be $R(c_1, c_2, a_1), R(d_1, d_2, a_2)$.

So while the discussion here is fine for motivating the example it doesn't suffice to show that $\mathfrak{A}^1$ with pair $A_{1,1}^1, A_{1,1}^1$ must contain an $A_{1,1}^2$ good over one point from each.

**Claim 4.2.4.** *The intersection $X_1 \cap B_1^1$ is not empty.*

*Proof.* This follows from the fact that $A_{1,1}^1$ and $A_{1,2}^1$ are free over $\mathfrak{A}^0$. Indeed, assume that $X_1 \cap B_1^1 = \emptyset$. Then

$$0 = \delta(X_1/B_1^2) = \delta(X_1 \cap A_{1,1}^1/B_1^2) + \delta(X_1 \cap A_{1,2}^1/B_1^2) + \delta(X_1 - (A_{1,1}^1 \cup A_{1,2}^1)/B_1^2)$$

because there are no relations between $A_{1,1}^1$, $A_{1,2}^1$ and $\mathfrak{A}^0 - B_1^1$. Then each of these predimensions is equal to 0; that contradicts the definition of a good pair. ∎

### 4.3. **Bouquets and Flowers**

In Remark 4.2.2.3, we noted that to make a $G_{\{I\}}$-normal structure we required not only an image of a 0-primitive $A$ with base $B$ but an image $\pi(A)$ for a $\pi$ in $G_{\{I\}}$ that fixes $B$ setwise but not pointwise. The analysis of the case where there are good pairs $A/B$ with $\delta(B) = 2$ and $\mu(A/B) = 2$ requires a much finer analysis of the second realization of $A/B$. We introduce here some further notation to describe the situation and illustrate them in Example 4.3.6.

**Definition 4.3.1** (flower). *Let $A/B$ be a good pair. A* flower *$\mathcal{F}$ of $A/B$ in a set $\mathfrak{D}$ is the set of all images of isomorphisms of $A$ over $B$ into $\mathfrak{D}$ which fix $B$ pointwise. The elements of the flower are called* petals[25].

*Suppose a flower $\mathcal{F}$ of $A/B$ is a subset of $\mathfrak{A}$. A* certificate $\mathcal{C}$ *of $A/B$ (witnessing $\mathfrak{A} \in \boldsymbol{K}_\mu$) is a* maximal disjoint *set of $\chi_{\mathfrak{A}}(A/B) = \mu(A/B)$ images of isomorphisms of $A$ over $B$ into $\mathfrak{A}$ that fix $B$ pointwise.*

When $A/B$ is well-placed, a flower $\mathcal{F}$ contains at least one certificate $\mathcal{C}$ for $\chi_M(A/B) = \mu(A/B)$ and, since each intersection decreases $\delta$, $|\mathcal{F}| \leqslant \mu(A/B) + \delta(B)$. Moreover any pair of petals from distinct certificates (or flowers) that intersect are in $\mathrm{icl}(B) \subseteq \mathfrak{A}^m$, for the least $m$ such that $B \subseteq \mathfrak{A}^m$.

Of course each petal $C \in \mathcal{F}$ is isomorphic to $A$ over $B$. $\mathfrak{D}$ will usually be fixed in context as either the generic $M$ or a $G_{\{I\}}$-decomposable $\mathfrak{A}$ (e.g. an $\mathfrak{A}^m$). There are only finitely many certificates of $A/B$ in $M$; an upper bound is $\binom{\mu(A/B)}{\mu(A/B)+\delta(B)}$.

Note that in the description of the class $\boldsymbol{L}_\mu$ one put the upper bound on the cardinality of a certificate of a good pair $A/B$—it does not exceed $\mu(A/B)$.

When we write two structures $C$ and $D$ are equal we mean they have both the same domain and each symbol in the vocabulary has the same interpretation in each. For a substructure $X$ of $M$, $diag_X(\mathbf{x})$ denotes the diagram of $X$, with respect to a *fixed enumeration* $\mathbf{x}$ of the domain of $X$.

**Notation 4.3.2.** *Let $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$ enumerate $B$ and $\gamma \in \mathrm{aut}(B)$; write $\mathbf{b}^\gamma$ for $\langle \gamma(b_1), \ldots, \gamma(b_n) \rangle$.*

*Any sequence $\mathbf{c}$ that satisfies $diag_{A \cup B}(\mathbf{x}, \mathbf{b}^\gamma)$ determines an enumeration of a* petal *of the flower of $A/B$. The set enumerated by this sequence is a petal $F_i^\gamma$. Each $F_i^\gamma$ may have multiple enumerations that satisfy the fixed diagram. A flower $\mathcal{F}^\gamma$ of $A/B$ is a maximal set $\{F_i^\gamma : i < r^\gamma\}$ of such petals.*

Note that for fixed $\gamma$ there may be different certificates. Any two such certificates must have at least one pair of intersecting petals (by maximality). But distinct flowers $\mathcal{F}, \mathcal{F}'$ generated by $A/B$ and $A'/B$ with $A, A'$ non-isomorphic over $B$ cannot

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[25]In Construction 3.8 there were $\mu(A/B)$ petals, the $A_{i,f}^m$ and $C_{i,k}^m$. We no longer assume that $A/B$ is well-placed and we allow the petals to intersect so we have less control over the number of petals; in particular it will vary with $\mathfrak{D}$.

have a common petal. If $f$ and $g$ map $A$ and $A'$ to some $A''$ while fixing $B$ pointwise, then $g^{-1} \circ f$ is an isomorphism from $A$ to $A'$ fixing $B$.

However, a different problem appears when we allow automorphisms that fix the base setwise but not pointwise. We must do this when considering $G_{\{I\}}$ since $I$ itself can be the base.

For simplicity of reading we denote $G_{\{I\}}$ by $G^*$ and $G^*_{\{B\}}$ $(G^*_B)$ denotes the elements of $G^*$ that fix $B$ *setwise* (*pointwise*).

We now have a subclass of the 0-primitive extensions $A$ where $A/B$ is well-placed by $\mathfrak{A}^m$: the orbit of *the flower of $A/B$* under $G^*_{\{B\}}$.

**Definition 4.3.3** (bouquet). *Let $A/B$ be a good pair. The bouquet $\mathcal{B}$ of $A/B$ is the collection of all images $\{\pi(F_i) : F_i \in \mathcal{F}\}$ of each flower $\mathcal{F}$ as $\pi$ ranges through elements of $G^*_{\{B\}}$.*

Can two flowers in a bouquet contain a common petal? When does a bouquet contain more than one flower? Lemma 4.3.4 and 4.3.5 answer these questions.

**Lemma 4.3.4.** *Let $A/B$ be a good pair and $\{\{F_i^\gamma : i < r_\gamma\} : \gamma \in G^*_{\{B\}}\}$ list its bouquet $\mathcal{B}$. If $i \neq j$ then $F_i^\gamma \neq F_j^\delta$ for each $\gamma, \delta \in G^*_{\{B\}}$ unless $\mathcal{F}_\gamma = \{F_t^\gamma : t < r_\gamma\}$ and $\mathcal{F}_\delta = \{F_t^\delta : t < r_\delta\}$ are the same flower.*

*Proof.* Assume that $F_i^\gamma = F_j^\delta$ for some $\gamma, \delta \in \mathrm{aut}(G^*_{\{B\}})$. We will show $\mathcal{F}_\delta = \mathcal{F}_\gamma$.

Let $\langle f_1, \ldots, f_k \rangle$ be an enumeration of $F_i^\gamma$, such that

$$M \models diag_{A \cup B}(f_1, \ldots, f_k, \gamma(b_1), \ldots, \gamma(b_n))$$

Since $|F_i^\gamma| = |F_j^\delta|$ there is $\varepsilon \in S_k$ such that $\langle f_1, \ldots, f_k \rangle = \langle f_{\varepsilon(1)}, \ldots, f_{\varepsilon(k)} \rangle$ and

$$M \models diag_{A \cup B}(f_{\varepsilon(1)}, \ldots, f_{\varepsilon(k)}, \delta(b_1), \ldots, \delta(b_n))$$

Let $s < r_\gamma$ and let $\langle d_1, \ldots, d_k \rangle$ enumerate the petal $C_s^\gamma$ of the flower $\mathcal{F}_\gamma$ of $A$ over $B$. That is,

$$M \models diag_{A \cup B}(d_1, \ldots, d_k, \gamma(b_1), \ldots, \gamma(b_n)).$$

By the property of $\varepsilon$ noted above and the definition of diagram, we have

$$M \models diag_{A \cup B}(d_{\varepsilon(1)}, \ldots, d_{\varepsilon(k)}, \delta(b_1), \ldots, \delta(b_n))$$

but $D = \{d_{\varepsilon(1)}, \ldots, d_{\varepsilon(k)}\}$ is a petal of the flower $\mathcal{F}_\delta$. Obviously, $F_s^\gamma = D$. So, each petal of the flower $\{F_t^\gamma : t < r_\gamma\}$ is also a petal of $\{F_t^\delta : t < r_\delta\}$.

The inverse inclusion is similar. Hence, the flowers are equal. ∎

We can now conclude:

**Lemma 4.3.5.** *Assume that $A/B$ is well-placed by some $\mathfrak{D} \supseteq B$ and $A$ is $G_{\{I\}}$-invariant. Then*

    *(1) the bouquet of $A/B$ consists of a single flower;*
    *(2) the bouquet of $A/B$ is $G_{\{I\}}$-invariant.*

*Proof.* 1) Assume to the contrary that the bouquet of $A_{j,1}^{m+1}$ over $B_j^{m+1}$ consists of at least two flowers. Let $\pi \in G_{\{I\}}$ be an automorphism which moves one flower of the bouquet of $A_{j,1}^{m+1}$ over $B_j^{m+1}$ to another one. Since $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant, $\pi(A_{j,1}^{m+1}) = A_{j,1}^{m+1}$; so, these two flowers have a common petal. By Lemma 4.3.4 these flowers are equal, for a contradiction.

2) By Lemma 3.19.1, $B$ is $G_{\{I\}}$-invariant, so each $g \in G_{\{I\}}$ fixes $B$ setwise. But then the $C/B$-bouquet is just the $G_{\{I\}}$-orbit of the unique flower $\mathcal{F}$ of $A/B$, namely $\mathcal{F}$.                                                                                                    ∎

We now give several examples to clarify the relationship among these concepts.

**Example 4.3.6.** 1) Two certificates in the same flower: Let $A/B$ be 0-primitive and $C_j^i$ for $i < 3, j < 2$ be isomorphic with $A$ over $B$. For each $i$, $|C_0^i \cap C_1^i| = 1$; these are the only intersections. Let $\mathfrak{D} = B \cup \bigcup_{i<3, j<2} C_j^i$. $A/B$ is well-placed by $\mathfrak{D}$. $\{A\} \cup \{C_j^i : i < 3, j < 2\}$ is the flower of $A/B$. But each of $\{A\} \cup \{C_j^i : i < 3\}$ for $j = 0$ and $j = 1$ is a certificate (Actually, there are 8 certificates.)

2) Two flowers in the same bouquet: Let $B = \{b_1, b_2\}$ and $C_i = \{c_1^i, c_2^i, c_3^i\}$ with $R(b_1, c_1^i, c_2^i)$, $R(b_2, c_2^i, c_3^i)$, $R(b_2, c_3^i, c_1^i)$, for $i = 1, 2$, and let $\mu(C_i/B) = 2$.

Let $D_i = \{d_1^i, d_2^i, d_3^i\}$ with $R(b_2, d_1^i, d_2^i)$, $R(b_1, d_2^i, d_3^i)$, $R(b_1, d_3^i, d_1^i)$, for $i = 1, 2$. There is a $\pi \in G_{\{I\}}^*$ that swaps $b_1$ and $b_2$ and takes $\{C_1, C_2\}$ to $\{D_1, D_2\}$. Recall that flowers are given by maps that fix $B$ pointwise. Note $C_1$ and $D_1$ are in the same orbit under $G_{\{B\}}^*$ but not $G_B^*$.

There are two flowers: $\{C_1, C_2\}$ over $\langle b_1, b_2 \rangle$ and $\{D_1, D_2\}$ over $\langle b_2, b_1 \rangle$. (They are distinct because $b_1$ occurs in two relations in the $D_i$ and one in the $C_i$.)

## 4.4. $G_{\{I\}}$: elimination of imaginaries fails

**Context** We showed that $\mathrm{dcl}^*(I) = \emptyset$ and so $\mathrm{sdcl}^*(I) = \emptyset$, provided that $\mu$ triples: $\mu(C/B) \geqslant 3$ for $\delta(B) = 2$ with $|C| > 1$. So $\hat{T}_\mu$ does not admit elimination of imaginaries. Now we are going to show that the symmetric $\mathrm{sdcl}^*(I)$ is empty for any $\mu$ satisfying Hrushovski's original conditions and so elimination of imaginaries fails. That is, we now omit the adequacy hypothesis that governed Section 4.1. There may now be definable truly binary functions but elimination of imaginaries still fails. The innovation is to consider the action of $G_{\{I\}}$ rather than $G_I$, sdcl rather than dcl.

Recall that in Example 4.2.1 $d(G_I(a_1)) = 1$, since $G_I(a_1) = \{a_1\}$. The situation differs when we consider $G_{\{I\}}$. In this case, working in a $G_{\{I\}}$-normal set, $G_{\{I\}}(a_1) = \{a_1, a_2\}$, so, $d(G_{\{I\}}(a_1)) = 2$. Similarly, while $\alpha_2$ is in $\mathrm{dcl}(I)$, $\mathrm{sdcl}(I) = \emptyset$. In general, the $G_I$-invariant set generated by a set $U$ is contained in the $G_{\{I\}}$-invariant set $U$ generates.

While in the proof of Theorem 4.1.2 we showed $\dim_m$ for $m \geq 1$, here we shall prove $\mathrm{sdim}_m$ for $m \geq 0$. Allowing $m = 0$ has a crucial role for application of Claim 4.4.13 in the proof of Theorem 4.4.1, showing any $G_{\{I\}}$-invariant subset of $\mathfrak{A}^2$ is safe. In Example 4.2.1 one can see the difference between a flower and a bouquet and how the notion of bouquet works for the proof of Theorem 4.4.1 (Remark 4.2.2(3)–(5)).

**Theorem 4.4.1.** *If $\hat{T}_\mu$ is as in Definition 0.1, then there is no symmetric $\emptyset$-definable truly n-ary function for $v \geq 2$, i.e., $\mathrm{sdcl}^*(I) = \emptyset$ for any $v$-element independent set $I$. That is, there is no $\emptyset$-definable truly n-ary function whose value does not depend on the order of the arguments. Thus, $\hat{T}_\mu$ does not admit elimination of imaginaries. (See Theorem 2.13.)*

*As a corollary, we obtain that $\mathrm{sdcl}(J) = \bigcup_{a \in J} \mathrm{sdcl}(a)$ for any independent set $J$.*

In contrast to Section 4.1, we work now with a *global induction* on the height $m_0$ of $G_{\{I\}}$-decompositions of finite, $G_{\{I\}}$-invariant subsets $\mathfrak{A}$ of $\mathrm{acl}(I)$ with $I \subseteq \mathfrak{A} \leqslant M$. We show for each $m_0$, for all such decompositions of height $m_0$, for all $m \leqslant m_0$, $\mathrm{sdim}_m$ holds. While we analyze a specific $G_{\{I\}}$-normal $\mathfrak{A}$ containing $I$ and a $G_{\{I\}}$-decomposition of $\mathfrak{A}$ into strata $\mathfrak{A}^n$ as in Section 3, the contradiction will result in most involved case a second normal subset of $\mathcal{M}$. The analysis takes into account that the resulting $\mathfrak{A}^n$ are now $G_{\{I\}}$-invariant. For this we need to introduce the induction hypothesis in Lemma 4.4.3 on the dimension of $G_{\{I\}}$-invariant sets.

**Definition 4.4.2** (Safe)**.** *Let $X$ be contained in a finite $G_{\{I\}}$-invariant set $\mathfrak{A}$. We say $X$ is* safe *if $d(E) \geq 2$ for any $G_{\{I\}}$-invariant set $E \subseteq X$ that is not a subset of $\mathrm{acl}(\emptyset)$.*

*The $G_{\{I\}}$-decomposition $\mathfrak{A}^m$ of $\mathfrak{A}$ satisfies $\mathrm{sdim}_m$ if every $G_{\{I\}}$-invariant subset of $\mathfrak{A}^m$ is safe.*

In addition to changing the group, the requirement $E \not\subseteq \mathfrak{A}^0$ has been replaced by $E \not\subseteq \mathrm{acl}(\emptyset)$. So, the main differences between Theorem 4.1.2 and Theorem 4.4.1 are the following:

- There may be cases where $A_{f,k}^{m+1}$ is $G_{\{I\}}$-invariant, because there is no longer the restriction that $\mu(C/B) \geq 3$ (In Example 3.13, Figure 4, $A_{1,1}^2$ is $G_{\{I\}}$-invariant);
- Different $A_{j,i}^{m+1}$ may be shown safe for different reasons. (Lemma 4.4.10)
- For any $e \in \mathfrak{A} - \mathrm{acl}(\emptyset)$, $d(G_I(e))$ may be 1, but we show $d(G_{\{I\}}(e)) \geq 2$.

Note that $X$ is $G_{\{I\}}$-invariant implies $X$ is $G_I$-invariant. Analogously to Notation 3.16 we write $s\ell_j^{m+1}$ for the number of images under $G_{\{I\}}$ of $A_j^{m+1}$ that do not intersect $\mathfrak{A}^m$. Since $G_{\{I\}} \supseteq G_I$, $s\ell_j^{m+1} \geqslant \ell_j^{m+1}$. Results from Section 4.1 for $G_I$ decompositions do not automatically extend. We will now prove $\mathrm{sdim}_{m_0}$ holds not by a dual induction but by distinct arguments depending on whether $\mathrm{move}_m$ holds at a given stage, which requires an even more global induction on all $G$-normal decompositions rather that the length of a fixed decomposition. The main lemma becomes:

**Lemma 4.4.3.** *Let $\hat{T}_\mu$ be as in Theorem 4.4.1. Then for every finite $G$-normal $\mathfrak{A} \subseteq \mathrm{acl}(I)$ and every $G_{\{I\}}$-decomposition $\langle \mathfrak{A}^i : i < m_0^{\mathfrak{A}} \rangle$ of $\mathfrak{A}$:*

$$\text{for every } m \leqslant m_0 \ \mathrm{sdim}_m \text{ holds of } \mathfrak{A}.$$

Theorem 4.4.1 follows from Lemma 4.4.3, because if there were a $u \in \mathrm{sdcl}(I)$, then $G_{\{I\}}(u) = \{u\}$ and so $d(G_{\{I\}}(u)) \leq \delta(G_{\{I\}}(u)) = 1$.

We cannot prove that every $A_{j,k}^{m+1}$ is moved by $G_{\{I\}}$ (Remark 4.2.2). Rather, we show that if $A_{j,k}^{m+1}$ is $G_{\{I\}}$-invariant then each $s \in A_{j,k}^{m+1}$ satisfies $\dim(G_{\{I\}}(s)) \geq 2$. If $\mu_j^{m+1} = 2$ (recall that $\mu_j^{m+1} = \mu(A_{j,k}^{m+1}/B_j^{m+1})$), the argument turns out to be a short argument (Lemma 4.4.8). So we assume below that $\mu_j^{m+1} \geqslant 3$. The global induction for Theorem 4.4.1 obtains from a failure of $\mathrm{sdim}_{m+1}$ another $G_{\{I\}}$-invariant set $\tilde{\mathfrak{A}}^* = \mathfrak{A}^{m-1} \cup \tilde{A}_{1,j}^m$. The height of $\tilde{\mathfrak{A}}^*$ is $m$, but $\tilde{A}_{j,1}^m$ contains an element $e'$ such that $d(G_{\{I\}}(e)) \leq 1$, which violates the inductive hypothesis, $\mathrm{sdim}_m$, for $\tilde{\mathfrak{A}}^*$. However, $\tilde{\mathfrak{A}}^*$ need not be contained in $\mathfrak{A}$.

The proof is a lengthy induction. We start with the following claim which is blatantly false for $G_I$.

**Claim 4.4.4.** *The statement* $\mathrm{sdim}_0$ *holds: every* $G_{\{I\}}$-*invariant subset of* $\mathfrak{A}^0$ *has dimension at least 2 provided that this set is not a subset of* $\mathrm{acl}(\emptyset)$.

*Proof.* Without loss, since $e \in \mathrm{acl}(\emptyset)$ implies $d(G_{\{I\}}(e)) = 0$, let $e \in \mathfrak{A}^0 \setminus \mathrm{acl}(\emptyset)$. Then, since $d(e) \leqslant 1$, $e \in \mathrm{acl}(a_1) \cup \mathrm{acl}(a_2) \cup \cdots \cup \mathrm{acl}(a_v)$, say, $e \in \mathrm{acl}(a_1)$. Since $e \in \mathrm{acl}(a_1) \setminus \mathrm{acl}(\emptyset)$, we obtain $a_1 \in \mathrm{acl}(e)$.

Let $g_i \in G_{\{I\}}$ be such that $g_i(a_1) = a_i$. Such a $g_i$ exists because the $a_i$ are independent and strong minimality implies there is a unique non-algebraic type over the empty set. Then $a_i \in \mathrm{acl}(g_i(e))$. Thus $\{a_1, a_2, \ldots, a_v\} \subseteq \mathrm{acl}(\{g_i(e) : i = 1, \ldots, v\}) \subseteq \mathrm{acl}(G_{\{I\}}(e))$. So, $d(G_{\{I\}}(e)) = v \geq 2$. ∎

By Lemma 4.1.4 (the inductive step, '$moves_m$ implies $dim_m$, from Section 4.1) applied to a $G_{\{I\}}$-decomposition, when $A_{j,i}^{m+1}$ moves, we have:

**Claim 4.4.5.** *If* $m \geqslant 1$ *and* $A_{j,i}^{m+1}$ *is not* $G_{\{I\}}$-*invariant,* $\mathrm{sdim}_m$ *implies* $G_{\{I\}}(A_{j,i}^{m+1})$ *is safe.*

This is the first divide; it tells us two things. 1) We can prove $\mathrm{sdim}_{m+1}$ by showing individual $G_{\{I\}}$-invariant petals are safe (Definition 4.4.2) and 2) $\mathrm{sdim}_1$ is true as in Corollary 4.1.5.

*For the remainder of Section 4.4 we assume* $\mathrm{sdim}_m$ *holds for each* $G_{\{I\}}$-*normal* $\mathfrak{A}$. *We show that for any such* $\mathfrak{A}$, *any* $G_{\{I\}}$-*invariant* $A_{j,i}^{m+1}$ *is safe.*

We now establish some tools used below as well as show in Claim 4.4.8 that for each $m \geq 1$ and for $G_{\{I\}}$-invariant $A_{j,1}^{m+1}$, $\mathrm{sdim}_m$ and $\mu_j^{m+1} = 2$ imply $A_{j,1}^{m+1}$ is safe. In order to explain the main idea of the rest of the proof we review Example 3.13, Figure 4, where $A_{1,1}^2$ is $G_{\{I\}}$-invariant. Clearly, there is an isomorphism $\rho_0$ of $A_{1,1}^2$ to $C_{1,1}^2 = \{a_1, a_2\} = I$ over $B_1^2 = \{b_1, b_2\}$. Since we put $I \leq M$, we have $B_1^2 \leqslant M$, so $\rho_0$ can be extended to an automorphism $\rho$ of $M$. Thus, we have found an automorphism which takes the $G_{\{I\}}$-invariant petal $A_{1,1}^2$ into $\mathfrak{A}^1$. Moreover, $\rho(A_{1,1}^2) = C_{1,1}^2 = I$ is obviously $G_{\{I\}}$-invariant. Thus, $A_{1,1}^2$ has a $G_{\{I\}}$-invariant copy inside $\mathfrak{A}^1$ and by the inductive hypothesis, $\mathrm{sdim}_1$, this copy is safe. Now the key points are Observation 4.4.7 and Lemma 4.4.8, which allow the transfer of safeness of $\rho(A_{1,1}^2)$ to $A_{1,1}^2$. In general, given a $G_{\{I\}}$-invariant petal $A_{j,1}^{m+1}$ we find an automorphism $\rho$ and prove that $\rho$ takes $A_{j,1}^{m+1}$ into $\mathfrak{A}^m$ or possibly into another $G$-normal set $\tilde{\mathfrak{A}}^m$ of height $m$. We show that $\rho(A_{j,1}^{m+1})$ is $G_{\{I\}}$-invariant and then by the induction hypothesis is safe. Finally, we apply Lemma 4.4.8 to show that $A_{j,1}^{m+1}$ is safe. Finding $\rho$ is easy, showing that $\rho(A_{j,1}^{m+1})$ is in a $G$-normal set of height $m$ is quite simple in Lemmas 4.4.12 and 4.4.20, but is more difficult in Lemma 4.4.24. Much of the argument, including Subsection 4.3, is aimed at proving that $\rho(A_{j,1}^{m+1})$ is $G_{\{I\}}$-invariant.

**Notation 4.4.6.** *Extending Notation 3.20 we write* $A$ *and* $B$ *for* $A_{j,i}^{m+1}, B_j^{m+1}$. $C$ *represents a* $C_{j,q}^{m+1}$ *for arbitrary* $q$, *where* $C_{j,q}^{m+1}$ *for* $q = 1, \ldots, \nu = \mu_j^{m+1} - 1$ *list the isomorphic over* $B$ *copies of* $A_{j,1}^{m+1}$ *in* $\mathfrak{A}$ *that are subsets of* $\mathfrak{A}^m$. *We may write* $C^1, \ldots, C^\nu$, *when the stratum* $m$ *and* $j$ *are fixed. Recall that* $\mu_j^x$ *abbreviates* $\mu(A_{j,1}^x/B_j^x)$.

We state the next observation for $G$ as $G_I$ or $G_{\{I\}}$ to emphasize it holds for either group. Our application will be to $G_{\{I\}}$.

**Observation 4.4.7.** *Consider the action of $G$ on $M$. Suppose $A$ and $C$ are $G$-invariant subsets of $\mathfrak{A} \leqslant M$ with $A \cap \mathfrak{A}^m = \emptyset$ and $C \subseteq \mathfrak{A}^m$, and $\rho$ is an automorphism of $M$ that takes $A$ onto $C$. If for an arbitrary $\alpha \in G$,*

$$(*) \quad \hat{\alpha} \restriction (\mathfrak{A}^m \cup A) = \alpha \restriction \mathfrak{A}^m \cup (\rho^{-1} \restriction C \circ \alpha \restriction C \circ \rho \restriction A).$$

*extends to an element (also denoted $\hat{\alpha}$) of $G$ then $\rho^{-1}$ injects each orbit of $G$ on $C$ into an orbit of $G$ on $A$ as follows. For any $e \in A$ and $\alpha$; if $\alpha(\rho(e)) = e' \in C$ then $\hat{\alpha}(e) = \rho^{-1}(\hat{\alpha}(\rho(e))) = \rho^{-1}(e')$ defines an injection from the $G$-orbit of $\rho(e)$ to that of $e$. Consequently, the image $\rho(X)$ of any $G$-invariant subset $X$ of $A$ is a union of $G$-orbits and hence $G$-invariant.*

Part 1) of Lemma 4.4.8 tells us that the special case where $\delta(B) = 2$ we can extend an *isomorphism* $\rho$ from $A$ to $C$ to an automorphism of $M$ and then deduce the safety of $A$ from the safety of $C$. Part 2) asserts that the deduction of safety is fine provided $\rho$ extends to an automorphism. A major task in this section will be establishing that $\rho$ has such an extension.

**Lemma 4.4.8.** *Let $A = A_{j,1}^{m+1}$ be $G_{\{I\}}$-invariant and*

   *(1) $\rho$ an isomorphism fixing $B = B_j^{m+1}$ pointwise and taking $A_{j,1}^{m+1}$ to $C \subseteq \mathfrak{A}^m$.*
      *(a) If $C$ is $G_{\{I\}}$-invariant and there is an automorphism $\alpha' \in G_{\{I\}}$ moving $\rho(e)$ to $e'$ for some $e \in A_{j,1}^{m+1}$, $e' \in C$, then there exists $\hat{\alpha} \in G_{\{I\}}$ moving $e$ to $\rho^{-1}(e')$. So, if $C$ is safe so is $A_{j,1}^{m+1}$.*
      *(b) In particular, if $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant, $\mathrm{sdim}_m$ holds, and $\mu_j^{m+1} = 2$, then $d(G_{\{I\}}(s)) = 2$ for each $s \in A_{j,1}^{m+1}$.*
   *(2) More generally, let $A = A_{j,1}^{m+1}$ be $G_{\{I\}}$-invariant and $\rho$ is an automorphism of $M$ moving $B = B_j^{m+1}$ inside $\mathfrak{A}^m$ and taking $A_{j,1}^{m+1}$ to a $G_{\{I\}}$-invariant $D = \rho(A_{j,1}^{m+1})$ with $D \subseteq \mathfrak{A}^m$. Then if $D$ is safe so is $A_{j,1}^{m+1}$.*

*Proof.* 1a) We must show that the $\hat{\alpha}$ on $\mathfrak{A}^m \cup A$ defined at (*) in Observation 4.4.7 extends to an element of $G_{\{I\}}$. Note $\hat{\alpha}$ is well-defined on $\mathfrak{A}^m \cup A$, since $\rho$ fixes $B$ pointwise and $C$ is fixed setwise by $\alpha$. Since $R(A, \mathfrak{A}^m) = R(A, B)$ (Definition 1.2.1) and $\rho$ is a $B$-isomorphism from $A$ to $C$, $\hat{\alpha} \restriction \mathfrak{A}^m \cup A$ is an automorphism of $\mathfrak{A}^m \cup A$. And, since $\mathfrak{A}^m \cup A \leqslant M$, $\hat{\alpha}$ extends to the required map in $G_{\{I\}}$.

1b) Since $B \subseteq \mathrm{acl}(I)$ but $B \not\subseteq \mathrm{acl}(\emptyset)$, $\mathrm{sdim}_m$, and the conditions on $\mu$ (in Definition 1.1.7) imply $2 \leqslant d(B) \leqslant \delta(B) \leqslant 2$. Thus, $B \leqslant M$ and $\rho$ extends to an automorphism of $M$. Note that all petals over $B$ are disjoint, because $B \leqslant M$, so the total number of petals that are isomorphic to $A$ over $B$ is equal to $\mu_j^{m+1} = 2$, namely, they are $A$ and $C$. Since $A = A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant, $B_j^{m+1}$ is $G_{\{I\}}$-invariant. Now we prove that $C$ is $G_{\{I\}}$-invariant. By Lemma 4.3.5 the bouquet on $A_{j,1}^{m+1}/B_j^{m+1}$ is equal to the flower on $A_{j,1}^{m+1}/B_j^{m+1}$ and has only two elements. The global $G_{\{I\}}$-isomorphism $\rho$ guarantees the same holds for $C$. Consequently, $G_{\{I\}}$ fixes each of $A$, $B$, $C$ setwise. By 1a) $\rho^{-1} \restriction C$ induces a $G_{\{I\}}$-isomorphism from $G_{\{I\}}(e)$ into $G_{\{I\}}(\rho(e))$. The induction hypothesis gives $d(G_{\{I\}}(c)) = 2$ for any $c \in C$, e.g., $\rho(e)$; so $d(G_{\{I\}}(e)) = 2$.

2) Let $\alpha \in G_{\{I\}}$ fix $D$ setwise. Now consider

$$\hat{\alpha} \restriction (\mathfrak{A}^m \cup A) = \alpha \restriction \mathfrak{A}^m \cup (\rho^{-1} \restriction D \circ \alpha \restriction D \circ \rho \restriction A).$$

$\hat{\alpha}$ is well-defined and fixes $I$ as in case 1a). Since $R(A, \mathfrak{A}^m) = R(A, B)$ and $\rho$ is an isomorphism of $BA$ to $\rho(B)D$, $\rho(A)$ is good over $\rho(B)$. But since $\mathfrak{A}^m \leqslant M$,

this implies $R(\rho(A), \mathfrak{A}^m) = R(\rho(A), \rho(B))$. So, $\hat{\alpha}{\restriction}\mathfrak{A}^m \cup A$ is an automorphism of $\mathfrak{A}^m \cup A$. And, since $\mathfrak{A}^m \cup A \leqslant M$, $\hat{\alpha}$ extends to the required map in $G_{\{I\}}$. By Observation 4.4.7 and since $\rho$ is an automorphism, if $D$ is safe, so is $A$.  ∎

The following notation will be used to study the relationship between a $G_{\{I\}}$-invariant set and the set it determines (Definition 3.18). If $\mu(A/B) \geqslant 3$ we will have the following situation.

**Notation 4.4.9.** *We extend Notation 4.4.6 to consider two levels. We will let $\mathfrak{D}$ range over subsets of the $G$-decomposable $\mathfrak{A}$; in applications they will usually be initial segments of the decomposition. Let $A, B$ denote a good pair well-placed by $\mathfrak{D} \leqslant M$ such that $A$ is $G_{\{I\}}$-invariant. $C$ denotes an arbitrary petal of the flower of $A/B$. We write $\hat{A}, \hat{B}, \hat{\mathfrak{D}}$ for a similar triple determined (Definition 3.18) by the first.*

Here is the way in which this situation arises. Suppose a $G_{\{I\}}$-invariant $A_{j,1}^{m+1}$ with base $B_j^{m+1}$ determines $A_{i,1}^m$. Then each of the $C_{j,k}^{m+1}$ intersects $A_{i,1}^m$. When $A_{i,1}^m$ is also $G_{\{I\}}$-invariant, then we get a new iteration. We call the first level $A, B, C$ and the second $\hat{A}, \hat{B}, \hat{C}$. Similarly $\mathfrak{D}$ and $\hat{\mathfrak{D}}$ refer to (are instantiated as) $\mathfrak{A}^m$ and $\mathfrak{A}^{m-1}$. We introduce this notation to avoid the distraction of the multiple super/sub scripts and focus on certain relationships which will appear several times in the sequel. In the crucial case where $\mu(\hat{A}/\hat{B}) = 2$, we will be able to extend the partial isomorphism $\rho$ over $\hat{B}$ taking $\hat{A}$ to its unique copy $\rho(\hat{A}) = \hat{C} \subseteq \hat{\mathfrak{D}}$ to an automorphism of $\mathcal{M}$ also called $\rho$.

At this stage we must invoke our induction hypothesis.

**Lemma 4.4.10.** *Suppose $\mathfrak{A}$ satisfies* $\mathrm{sdim}_m$.
   (1) $A_{j,1}^{m+1}$ *is safe if either*
       (a) $A_{j,1}^{m+1}$ *is not $G_{\{I\}}$-invariant or*
       (b) $A_{j,1}^{m+1}$ *is $G_{\{I\}}$-invariant and $\mu_j^{m+1} = 2$.*
   (2) *If $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant, $|A_j^{m+1}| > 1$ and $\mu_j^{m+1} \geq 3$ then $A_{j,1}^{m+1}$ determines $A_{i,1}^m$ for some $i$. Moreover, $B_j^{m+1} \cap A_{i,1}^m \neq \emptyset$ and $B_+ = B_j^{m+1} - A_{i,1}^m \subseteq$ $\mathrm{acl}(\emptyset)$.*

*Proof.* Case 1) follows from Lemmas 4.4.5 and 4.4.8.1a. For Case 2), since $|A_{j,1}^{m+1}| > 1$ the hypotheses of Lemma 3.22 hold, we may apply Lemma 3.21 and then Lemma 3.23. Thus, $\delta(B_+) \leq 1$. Since $\mathfrak{A}^m$ witnesses $\mathfrak{A}$ satisfies $\mathrm{sdim}_m$, and $B_+ - \mathrm{acl}(\emptyset)$ is $G_{\{I\}}$-invariant, if $B_+ - \mathrm{acl}(\emptyset)$ were nonempty it would have dimension 2. Thus, $B_+ \subseteq \mathrm{acl}(\emptyset)$.  ∎

Claim 4.4.12.1 shows a stronger form of case 2) ($B_+ = \emptyset$) when $|A_{j,1}^{m+1}| = 1$.

**Lemma 4.4.11.** *In the situation of Notation 4.4.9, suppose $\mu(\hat{A}/\hat{B}) = 2$. Let $\rho$ be a partial isomorphism over $\hat{B}$ from $\hat{A}$ to its unique copy $\rho(\hat{A}) = \hat{C} \subseteq \hat{\mathfrak{D}}$. Then $\rho(B)$ is $G_{\{I\}}$-invariant.*

*Proof.* By Lemma 4.4.8.1b, $\rho$ extends to an automorphism of $M$. Suppose $\pi \in G_{\{I\}}$ fixes $\hat{B}$ setwise. By Lemma 4.4.10.2, $B \subseteq \hat{A} \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0))$ and so

$$\rho(B) \subseteq \rho(A \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0)) = \hat{C} \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0).$$

Obviously, $\rho(B \cap \mathrm{acl}(\emptyset)) = B \cap \mathrm{acl}(\emptyset)$ is $G_{\{I\}}$-invariant, because $B$ is $G_{\{I\}}$-invariant. By Lemma 4.4.10.2, $B - \mathrm{acl}(\emptyset) = B \cap A_{i,1}^m$ is $G_{\{I\}}$-invariant. We know that both $\hat{A}$ and $\hat{C}$ are $G_{\{I\}}$-invariant, and that $\rho(B \cap \hat{A}) \subseteq \hat{C}$. Assume for contradiction that $\rho(B \cap \hat{A})$ is not $G_{\{I\}}$-invariant, witnessed by $\pi \in G_{\{I\}}$ such that

$$\pi(\rho(B \cap \hat{A})) \neq \rho(B \cap \hat{A}).$$

Then we put

$$\tau = \pi {\restriction} \mathfrak{A}^{m-1} \cup (\rho^{-1} \circ \pi \circ \rho) {\restriction} \hat{A}$$

Obviously, $\tau$ can be extended to an automorphism of $M$ and $\tau(B \cap \hat{A}) \neq B \cap \hat{A}$, contradicting $G_{\{I\}}$-invariance of $B$. ∎

We continue to rely on our induction hypothesis, $\mathrm{sdim}_m$; we show a $G_{\{I\}}$-invariant $A_{j,1}^{m+1}$ with only one element determines an $A_{i,1}^m$ with at least two elements and $\mu_i^m \geqslant 3$. Parts 2) and 3) foreshadow the main argument below.

**Claim 4.4.12.** *Assume* $\mathrm{sdim}_m$. *If* $|A_{j,1}^{m+1}| = 1$ *and is* $G_{\{I\}}$*-invariant then*

  (1) $B = B_j^{m+1} \leqslant M$, *and each* $C^k = C^{m+1,k}$ *is contained in* $A_{i,1}^m \cup B_i^m$ *and* $A_{j,1}^{m+1}$ *determines* $A_{i,1}^m$ *for some* $i$.
  (2) *Moreover,* $\mu_j^{m+1} \geqslant 3$;
  (3) *and* $\mu_i^m \geqslant 3$.

*Proof.* 1) Lemma 3.19.1 asserts $A_{j,1}^{m+1} = \{e\}$ determines some $A_{i,1}^m$ for some $i$ and by $\mathrm{sdim}_m$, Lemma 3.19.2 yields $B \subseteq A_{i,1}^m$. By $\mathrm{sdim}_m$ again, $d(B) = \delta(B) = 2$; so $B \leqslant M$. Since $B \subseteq A_{i,1}^m$ has relations in $\mathfrak{A}^m$ only with elements of $A_{i,1}^m$ and its base $B_i^m$ and each $C^k$ is a singleton, each $C^k \subseteq A_{i,1}^m \cup B_i^m$.

2) Assume to the contrary that $\mu_j^{m+1} = 2$. By Lemma 4.3.5.2, $A_{j,1}^{m+1}$, $B_j^{m+1}$, and $C_{j,1}^{m+1}$ are $G_{\{I\}}$-invariant. Since $C_{j,1}^{m+1} \subseteq \mathfrak{A}^m$ and is $G_{\{I\}}$-invariant, $C_{j,1}^{m+1}$ is safe by induction. But $|C_{j,1}^{m+1}| = 1$, so $C_{j,1}^{m+1} = \{c\}$. Then $d(G_{\{I\}}(c)) = d(\{c\}) \leq \delta(\{c\}) = 1$, for a contradiction.

3) Assume to the contrary that $\mu_i^m = 2$. Using the notation and result of Lemma 4.4.11, we are given a partial isomorphism $\rho$ taking $A_{i,1}^m$ to $C_{i,1}^m \subseteq \mathfrak{A}^{m-1}$. Moreover, $A_{i,1}^m, B_i^m, C_{i,1}^m, \rho(A_{j,1}^{m+1})$ are all $G_{\{I\}}$-invariant. By $\mathrm{sdim}_m$, $d(B_i^m) \geq 2$ and so $2 \leq \delta(B_i^m) \leq \mu_i^m = 2$ and $A_{i,1}^m$ is a 0-primitive extension of $B_i^m$; thus, $B_i^m \cup A_{i,1}^m \leq M$. So $\rho$ can be extended to an automorphism $\hat{\rho}$ of $M$. The automorphism $\hat{\rho}$ is not in $G_{\{I\}}$ as it doesn't respect strata. Indeed, it may not fix $\mathfrak{A}$ setwise.

Clearly, $\hat{\rho}(B)$, $\hat{\rho}(C^1)$, $\ldots$, $\hat{\rho}(C^\nu) \subseteq \hat{\rho}(A_{i,1}^m \cup B_i^m) = C_{i,1}^m \cup B_i^m \subseteq \mathfrak{A}^{m-1}$. Since $B \leqslant M$, $\hat{\rho}(B) \leqslant M$ so by Lemma 3.7 $\mu(\hat{\rho}(B), \hat{\rho}(C^1)) = \mu(B, A_{j,1}^{m+1}) = \nu + 1$, so $\hat{\rho}(A_{j,1}^{m+1}) = \{e'\}$ is a $(\nu + 1)$th copy of $\hat{\rho}(C^1)$ over $\hat{\rho}(B)$. Note that $\hat{\rho}(A_{j,1}^{m+1})$ is the unique such copy which is not in $C_{i,1}^m$.

As $C_1^m$ is $G_{\{I\}}$-invariant, by the 'consequently' of Observation 4.4.7, $\rho(B)$ is a $G_{\{I\}}$-invariant set and so is $\{\hat{\rho}(C^1), \ldots, \hat{\rho}(C^\nu)\}$. By Lemma 4.3.5 the bouquet of $A_{j,1}^{m+1}/B_j^{m+1}$ consists of one flower. Again by Observation 4.4.7, the bouquet of $\hat{\rho}(A_{j,1}^{m+1})/\hat{\rho}(B_j^{m+1})$ consists of one flower $\{\hat{\rho}(C^1), \ldots, \hat{\rho}(C^\nu), \hat{\rho}(A_{j,1}^{m+1})\}$. Since $\hat{\rho}(C^1), \ldots, \hat{\rho}(C^\nu)$ are in the $G_{\{I\}}$-invariant set $C_{i,1}^m$, $\hat{\rho}(A_{j,1}^{m+1})$ is a $G_{\{I\}}$-invariant set, because it is disjoint from $C_{i,1}^m$. Thus, $e'$ as a unique element of $\hat{\rho}(A_{j,1}^{m+1})$, is fixed by $G_{\{I\}}$.

If $\hat{\rho}(A_{j,1}^{m+1}) \subseteq \mathfrak{A}$, by the definition of decomposition it is contained in $\mathfrak{A}^m \leqslant M$. But this means $e' \in \mathrm{sdcl}^*(I)$ contradicting the induction assumption that $\mathfrak{A}$ satisfies $\mathrm{sdim}_m$, since $G_{\{I\}}(e') = \{e'\}$, implies that

$$d(G_{\{I\}}(e')) = d(\{e'\}) \leq \delta(\{e'\}) = 1.$$

The final possibility is that $\hat{\rho}(A_{j,1}^{m+1}) \subseteq M - \mathfrak{A}$. Now we use our 'global induction'. Let $\tilde{\mathfrak{A}} = \mathfrak{A}^{m-1} \cup \{e'\}$. Then $\tilde{\mathfrak{A}} \leqslant M$ (since $\delta(\{e'\}/\rho(B)) = 0$) and $\tilde{\mathfrak{A}}$ is $G_{\{I\}}$-invariant. Thus $\tilde{\mathfrak{A}}$ admits a decomposition of height $m$ and $\{e'\} = \tilde{A}_{1,1}^m$. But $G_{\{I\}}(e') = \{e'\}$ contradicting the inductive hypothesis that $\mathrm{sdim}_m$ hold for all decompositions. ∎

The argument for Lemma 4.4.12.3 shows the main idea of the proof of (Lemmas 4.4.20, 4.4.24) for $m \geqslant 2$. We cut an intermediate strata out, preserving the top, in order to obtain a counterexample with smaller height. There are three possibilities for $A_{j,i}^{m+1}$: item 1) of Lemma 4.4.10 details that we have finished the proof for two of them and item 2) specifies the conditions for further analysis.

So we need only study case 2 of Lemma 4.4.10. We describe the case $m + 1 = 2$ to illuminate a major issue in the remainder of the proof.

**Claim 4.4.13.** *Assume that $A_{j,1}^2$ is $G_{\{I\}}$-invariant, then $\mu_j^2 = 2$ and $A_{j,1}^2$ is safe.*

*Proof.* If some $A_{j,1}^2$ is $G_{\{I\}}$-invariant and $\mu_j^2 \geq 3$, Claim 4.4.10.2 gives that $A_{i,1}^1$ is $G_{\{I\}}$-invariant for some $i$; this contradicts Lemma 3.11 (moves$_1$). So, $\mu_j^2 \leq 2$. Since $\delta(B) \geq d(B) \geq 2$ by $\mathrm{sdim}_1$, we obtain that $\mu_j^2 = 2$ and $\mathrm{sdim}_2$ follows from Lemma 4.4.8. ∎

The difficulty is that this argument depended on every $A_{j,i}^1$ being moved; not merely being safe. In order to deal with this, we introduce a new system of indexing which is expounded more fully in Notation 4.4.21. Note that a $G_{\{I\}}$-invariant $\overleftarrow{A}^0$ engenders by Lemma 3.23 a decreasing sequence of $G_{\{I\}}$-invariant petals $\overleftarrow{A}^i$ such that $\overleftarrow{A}^i$ determines $\overleftarrow{A}^{i+1}$ which continues as long as $\overleftarrow{\mu}^i = \mu(\overleftarrow{A}^i/\overleftarrow{N}^i) \geqslant 3$. However, we know that no petal $A_{u,v}^1$ is $G_{\{I\}}$-invariant. So this sequence must terminate with an $s \leqslant m - 1$ such that $\overleftarrow{\mu}^s = 2$. We begin the study of such sequences with the case $\overleftarrow{\mu}^1 = 2$, where the chain has only two levels: *The next five Lemmas, 4.4.14 through 4.4.20, complete the proof when $\mu_i^m = 2$.* We apply the technical Lemma 4.4.14 in the proof of Lemmas 4.4.15 and 4.4.16.

**Lemma 4.4.14.** *Let $D, E \subseteq_\omega M$ satisfy $\delta(D) = 0$; then $\delta(D/E) \leqslant 0$. Thus, if $E \leqslant M$ then $\delta(D/E) = 0$.*

*Proof.* Monotonicity of $\delta$ implies the first inequality

$$\delta(D/E) \leq \delta(D/E \cap D) = \delta(D) - \delta(E \cap D) = -\delta(E \cap D) \leq 0$$

and the second equality holds since any subset of $M$ has non-negative dimension. ∎

Lemma 4.4.20, Claim 4.4.17, Lemma 4.4.23 and Lemma 4.4.24 rely indirectly on the following ostensibly technical claim about the location of $A$, with $A$ in Claim 4.4.15 getting different interpretations. It is the crucial point that allows us to anchor (Definition 4.4.21.4) our inductive analysis in $\mathrm{acl}(\emptyset) \cup B$ where $B$ is the base of good pair rather than the $\mathfrak{A}_r$, where the sequence in Definition 4.4.21 stops.

**Claim 4.4.15.** *Let $A$ and $B$ be disjoint finite subsets of $M$, with $A$ good over $B$ and $\delta(B) = d(B) \geq 1$. Then $A \cap \mathrm{acl}(\emptyset) = \emptyset$.*

*Proof.* Let $d \in A \cap \mathrm{acl}(\emptyset)$ and $D = \mathrm{icl}(d)$. Then $\delta(D) = 0$. Suppose for contradiction that $D \subseteq A$. Since $B \leqslant M$, Lemma 4.4.14 implies $0 \leq \delta(D/B) \leq \delta(D/\emptyset) = 0$; this contradicts the definition of a good pair, as $D$ must equal $A$ and then, since $\delta(A) - \delta(D) = 0$, $A$ is 0-primitive over $\emptyset \subsetneq B$.

Let $D_0 = D \cap B$ and $D_1 = D \cap (B \cup A)$. Since, $D, B$ and $B \cup A$ are all strong in $M$, so are $D_0$ and $D_1$. So, $\delta(D) = 0$ implies $\delta(D_0) = \delta(D_1) = 0$. By Lemma 4.4.14 $0 = \delta(D_1/B)$. Repeating the reasoning of the first paragraph with $D_1$ playing the role of $D$, we obtain the same contradiction; so, $A$ is 0-primitive over $\emptyset$. ∎

**Lemma 4.4.16.** *Suppose $\mu_j^{m+1} \geq 3$, $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant and determines $A_{i,1}^m$. Then $C^{m+1,q} \subseteq A_{i,1}^m \cup \mathrm{icl}(B_i^m)$ for each $q$ with $1 \leqslant q \leqslant \nu^{m+1}$, i.e. $C_+^{m+1,q} \subseteq \mathrm{icl}(B_i^m)$.*

*Proof.* Let $B$ denote $B_j^{m+1}$, $\hat{B}$ denote $B_i^m$, $\hat{A}$ denote $A_{i,1}^m$. We write $C$, for a fixed but arbitrary $q$, $C = C^{m+1,q}$, and $C_- = C_-^{m+1,q} = (C^{m+1,q} - \mathfrak{A}^{m-1})$. First we show $C_- = C^{m+1,q} \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ is contained in $\hat{A}$.

Assume that $C_-$ intersects some other petal $A'$ on strata $m$. By monotonicity and since $C$ is 0-primitive over $B$, taking into account $B \cup C \subseteq \mathfrak{A}^m$,

$$(*) \quad \delta((C \cap A')/\mathfrak{A}^m - A') \leqslant \delta((C \cap A')/(B \cup (C - A'))) < 0.$$

But, also

$$(**) \quad \delta((C \cap A')/\mathfrak{A}^m - A') = \delta((C \cap A')/\mathfrak{A}^{m-1}) \geq 0.$$

(**) holds because $R(A', \mathfrak{A}^m - A') = R(A', \mathfrak{A}^{m-1})$ as all petals in $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ are fully independent over $\mathfrak{A}^{m-1}$ and because $\mathfrak{A}^{m-1} \leqslant M$. But (*) and (**) are contradictory, so $C_- \subseteq A_{i,1}^m = \hat{A}$.

Our goal is to show $C_+^{m+1,q} \subseteq \mathrm{icl}(\hat{B})$, where $\hat{B} = B_i^m$. So, suppose for contradiction that for some $q$ with $1 \leqslant q \leqslant \nu^{m+1}$ where $\nu^{m+1} = \mu^{m+1} - 1$, and with $C_+ = C_+^{m+1,q} = C^{m+1,q} \cap \mathfrak{A}^{m-1}$, we have $C_+ - \mathrm{icl}(\hat{B}) \neq \emptyset$. Since $C/B$ is a good pair and $C = (C_+ - \mathrm{icl}(\hat{B})) \cup (C_+ \cap \mathrm{icl}(\hat{B})) \cup C_-$:

$$(9) \qquad 0 > \delta((C_+ - \mathrm{icl}(\hat{B}))/B \cup C_- \cup (C_+ \cap \mathrm{icl}(\hat{B}))).$$

**Claim 4.4.17.** *Let $\overline{B}_+ = \mathrm{icl}(B_+)$. Inequality (9) simplifies to:*

$$(10) \quad \delta((C_+ - \mathrm{icl}(\hat{B}))/B \cup C_- \cup (C_+ \cap \mathrm{icl}(\hat{B}))) = \delta(C_+ - \mathrm{icl}(\hat{B})/B_+ \cup (C_+ \cap \mathrm{icl}(\hat{B})))$$

*and*

$$(11) \qquad 0 > \delta(C_+ - \mathrm{icl}(\hat{B})/\overline{B}_+ \cup (C_+ \cap \mathrm{icl}(\hat{B})))$$

*Proof.* Since $B_- \cup C_- \subseteq A_{i,1}^m$, each relation between $B_- \cup C_-$ and $\mathfrak{A}^{m-1}$ is a relation on $B_- \cup C_-$ as a subset of $\hat{A}$ and the base, $\hat{B}$, of $\hat{A}$. So we can delete $B_- \cup C_-$ from the base of Equation (9) and obtain Equation (10). By Lemma 3.23 $\delta(B_+) \leq 1$. Then Lemma 4.4.10.2 implies that $B_+ \subseteq \mathrm{acl}(\emptyset)$, so $\delta(\overline{B}_+) = \delta(\mathrm{icl}(B_+)) = 0$. By Claim 4.4.15, $C \cap \mathrm{acl}(\emptyset) = \emptyset$, then $C_+ \cap \overline{B}_+ = 0$. Monotonicity of $\delta$ implies that

$$\delta(C_+ - \mathrm{icl}(\hat{B})/B_+ \cup (C_+ \cap \mathrm{icl}(\hat{B}))) \geq \delta(C_+ - \mathrm{icl}(\hat{B})/\overline{B}_+ \cup (C_+ \cap \mathrm{icl}(\hat{B})))$$

The last inequality and inequality (9) yield inequality (11). ∎

**Remark 4.4.18.** *In the special case that $B_+ = \emptyset$, the Lemma 4.4.16 is easy. By monotonicity of $\delta$ and by* $\mathrm{icl}(\hat{B}) \leq M$ *we obtain*

$$\delta(C_+ - \mathrm{icl}(\hat{B})/C_+ \cap \mathrm{icl}(\hat{B})) \geq \delta(C_+ - \mathrm{icl}(\hat{B})/\mathrm{icl}(\hat{B})) \geq 0$$

*The last contradicts* (9).

Verification of Equation (9) in the general situation of Claim 4.4.17 requires a further technical calculation.

More generally, we only know from the proof of Claim 4.4.17 that $\delta(\overline{B}_+) = 0$ so we must consider more carefully the connections of $C_+$ and $\overline{B}_+$.

We apply the identity $(\delta(X/Y \cup Z) = \delta(X \cup Y/Z) - \delta(Y/Z))$ by putting $X = C_+ - \mathrm{icl}(\hat{B})$, $Y = (\overline{B}_+ - \mathrm{icl}(\hat{B}))$, and $Z = (\overline{B}_+ \cap \mathrm{icl}(\hat{B})) \cup (C_+ \cap \mathrm{icl}(\hat{B})) = (\overline{B}_+ \cup C_+) \cap \mathrm{icl}(\hat{B})$. Thinking of $\overline{B}_+$ as $(\overline{B}_+ - \mathrm{icl}(\hat{B})) \cup (\overline{B}_+ \cap \mathrm{icl}(\hat{B}))$, we rewrite the right hand side of (10) as follows:

$$(12) \qquad \qquad \delta(C_+ - \mathrm{icl}(\hat{B})/\overline{B}_+ \cup (C_+ \cap \mathrm{icl}(\hat{B}))) =$$

$$(13) \qquad = \delta((C_+ - \mathrm{icl}(\hat{B})) \cup (\overline{B}_+ - \mathrm{icl}(\hat{B}))/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})) -$$

$$(14) \qquad \qquad -\delta(\overline{B}_+ - \mathrm{icl}(\hat{B})/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})).$$

Now we show the subtracted term, $\delta(Y/Z)$ (Line (14)) is 0. We apply monotonicity of $\delta$ in Line (15) and Lemma 4.4.14 in Line (16).

$$(15)\ \delta(\overline{B}_+ - \mathrm{icl}(\hat{B})/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})) \quad \leq \quad \delta(\overline{B}_+ - \mathrm{icl}(\hat{B})/\overline{B}_+ \cap \mathrm{icl}(\hat{B})) =$$

$$(16) \qquad \qquad \qquad \qquad \qquad = \quad \delta(\overline{B}_+/\overline{B}_+ \cap \mathrm{icl}(\hat{B})) \leq 0$$

On the other hand, applying monotonicity of $\delta$ and $\mathrm{icl}(\hat{B}) \leq M$ we obtain

$$\delta(\overline{B}_+ - \mathrm{icl}(\hat{B})/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})) \geq \delta(\overline{B}_+ - \mathrm{icl}(\hat{B})/\mathrm{icl}(\hat{B})) \geq 0$$

So, Line (14) is 0. By Line (10) and Claim 4.4.17, Line (12) is negative, so Line (13) is negative, too. Below we sequentially apply the distributive law, monotonicity of $\delta$ and $\mathrm{icl}(\hat{B}) \leq M$ to Line (13).

$$\delta((C_+ - \mathrm{icl}(\hat{B})) \cup (\overline{B}_+ - \mathrm{icl}(\hat{B}))/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})) =$$

$$= \delta((C_+ \cup \overline{B}_+) - \mathrm{icl}(\hat{B})/(C_+ \cup \overline{B}_+) \cap \mathrm{icl}(\hat{B})) \geq$$

$$\geq \delta((C_+ \cup \overline{B}_+) - \mathrm{icl}(\hat{B})/\mathrm{icl}(\hat{B})) \geq 0$$

The contradiction obtained with Equation (9) completes the proof of Lemma 4.4.16.
∎

We avoid the subscripts and isolate in Lemma 4.4.19 the connections imposed by determination, which drive the proof, and to emphasize that these results do not require any inductive hypotheses.

In combination with Lemma 4.4.15 (which is used in Lemma 4.4.20), Claim 4.4.17, Lemma 4.4.23 and Lemma 4.4.24, we now extend Lemma 4.4.11 from petals to flowers. While this larger set being $G$-invariant is *a priori* weaker, we recover the result for petals by a substantial induction.

**Lemma 4.4.19.** *Let $A$ be well-placed over $\mathfrak{D}$ by $B$ and $\hat{A}$ be well-placed over $\hat{\mathfrak{D}} \leqslant \mathfrak{D}$ by $\hat{B}$. Further suppose that $A$ is $G_{\{I\}}$-invariant, $A$ determines $\hat{A}$, and $\mu(\hat{A}/\hat{B}) = 2$. Further, let $A, C^1, \ldots C^\nu$ list the (by Lemma 4.3.5) flower associated with $A/B$. Let $\rho$ be a partial isomorphism from $\hat{A}$ over $\hat{B}$ to its unique copy $\hat{C} \subseteq \hat{\mathfrak{D}}$. Then*

(1) $\rho$ extends to an automorphism of $M$.
(2) there is a unique $G_{\{I\}}$-invariant flower over $\rho(C)/\rho(B)$ and $\rho(A)$ is $G_{\{I\}}$-invariant.

*Proof.* i) In fact, we will make the extension of $\rho$ fix $\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0$. Let $\mathcal{W}_1 = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \hat{B} \cup \hat{A})$ and $\widetilde{\mathcal{W}} = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \hat{B} \cup \hat{C}) \subseteq \mathfrak{D}$. Then, by Claim 4.4.15, with $\hat{C} = \rho(\hat{A})$ playing the role of $A$, $\hat{C} \cap \mathrm{acl}(\emptyset) = \emptyset$. Now, since $\mu(\hat{A}/\hat{B}) = 2$, $\hat{A}$ and $\hat{C}$ are isomorphic by $\rho$ not only over $\hat{B}$ but over $\mathcal{W}_2 = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \hat{B}$. (We know $R(\hat{A}, \mathfrak{D}) = R(\hat{A}, \hat{B})$. So if the isomorphism is not over $(\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \hat{B}$, there is a relation between $\hat{C}$ and $(\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) - \hat{B}$. But then $\delta((\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))/\hat{C} \cup \hat{B}) < 0$, contradicting $\hat{C} \cup \hat{B} \leqslant M$.) Note that $\mathcal{W}_2 \leq M$ because

$$2 = d(\hat{B}) \leq d(\hat{B} \cup (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))) \leq \delta(\hat{B} \cup (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)))$$
$$\leq \delta(\hat{B}) - \delta(\hat{B} \cap \mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) + \delta(\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) = 2 - 0 + 0 = 2$$

We obtain the first zero because $\hat{B} \cap \mathfrak{A}^0 \cap \mathrm{acl}(\emptyset) \leq M$.

As $\mathcal{W}_2 \leq M$, $\rho$ extends to an automorphism of $M$ fixing $\mathcal{W}_2$, also denoted $\rho$.

ii) By Lemma 4.4.11, $\rho(B)$ is $G_{\{I\}}$-invariant. Suppose $\pi \in G_{\{I\}}$, and so fixes $B$ setwise. By Lemma 3.23.A, $B \subseteq \hat{A} \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0))$ and so

$$\rho(B) \subseteq \rho(\hat{A} \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0)) = \hat{C} \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0).$$

Put

$$\check{\pi} = \pi{\restriction}\hat{\mathfrak{D}} \cup ((\rho^{-1}{\restriction}\hat{C}) \circ (\pi{\restriction}\hat{C}) \circ (\rho{\restriction}\hat{A})).$$

Since $\hat{\mathfrak{D}} \cup \hat{A} \leq M$, we can extend $\check{\pi}$ to $\pi' \in G_{\{I\}}$. By Lemma 4.3.5, there is a unique flower $\mathcal{F}$ over $A/B$. $\pi'$ maps it to a flower $\rho(\mathcal{F})$ over $\rho(C)/\rho(B)$, which must also be unique. As, $\pi'^{-1}$ of a second flower over $\rho(C)/\rho(B)$ would contradict the uniqueness of $\mathcal{F}$.

Since $A, C^1, \ldots C^\nu$ enumerate the $G_{\{I\}}$-invariant flower $\mathcal{F}$, its $G_{\{I\}}$-invariant-image (by Lemma 4.3.5.2) under $\rho$ is $\{\rho(A), \rho(C^1), \ldots \rho(C^\nu)\}$. Since $B_i^m \leqslant M$, Lemma 4.4.16 implies all the $C^q \subseteq \hat{C} \cup \hat{B}$. But then, since $\pi$ fixes both $\hat{B} = B_i^m$ and $\hat{C}$ (since $\mu(\hat{A}/\hat{B}) = 2$), each $\rho(C^q)$ is contained in $\hat{C} \cup \hat{B}$ which is $G_{\{I\}}$-invariant and $\rho(A) \cap (\hat{C} \cup \hat{B}) = \emptyset$. So $\rho(A)$ is $G_{\{I\}}$-invariant as the complement within the $G_{\{I\}}$-invariant flower $\rho(\mathcal{F})$ of the set of the $\rho(C^q)$ that are contained in $\hat{C} \cup \hat{B}$. $\blacksquare$

We continue the inductive proof of $\mathrm{sdim}_m$ for all $m < m_0$. The idea is similar to Claim 4.1.6.2, where with $|A_{j,1}^{m+1}| = 1$ we have constructed $\tilde{\mathfrak{A}}$, a counterexample with smaller decomposition height, but the situation is more complicated. We have reduced to the case where $\mu_j^{m+1} \geq 3$ and some for $m' < m$, $\mu_i^{m'} = 2$. Building on Lemma 4.4.11, we first consider the special case when $\mu_i^m = 2$. Note that by Lemma 4.4.12.3 the hypothesis $|A_{j,1}^{m+1}| > 1$ is essential.

**Claim 4.4.20.** *Suppose $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant, determines $A_{i,1}^m$, and in addition that $|A_{j,1}^{m+1}| > 1$ and $\mu_i^m = 2$. Then, $A_{j,1}^{m+1}$ is safe.*

*Proof.* Recall that we are doing a global induction to show $\mathfrak{A}$ satisfies sdim. The next constructions will allow us to show $A_{j,1}^{m+1}$ is safe by finding an isomorphic copy of it with lower height.

Lemma 3.23 and Lemma 4.4.10 imply that since $A_{j,1}^{m+1}$ is not a singleton and is $G_{\{I\}}$-invariant, $\mu_j^{m+1} \geq 3$ implies $B_j^{m+1} \subseteq A_{i,1}^m \cup (\mathrm{acl}(\emptyset) \cap \mathfrak{A}^0)$. By Lemma 4.4.11

the bouquet of $A_{i,1}^m$ over $B_i^m$ is $G_{\{I\}}$-invariant. Since both $A_{j,1}^{m+1}$ and $A_{i,1}^m$ are $G_{\{I\}}$-invariant, Lemma 4.3.5 implies the bouquets of $A_{j,1}^{m+1}$ over $B_j^{m+1}$ and of $A_{i,1}^m$ over $B_j^m$ are each $G_{\{I\}}$-invariant. The second of these consists of two petals: $A_{i,1}^m$ and $C^{m,1}$. Since the bouquet and $A_{i,1}^m$ are $G_{\{I\}}$-invariant so is $C^{m,1}$.

We now apply Lemma 4.4.19, taking $A$ as $A_{j,1}^{m+1}$, $\hat{A}$ as $A_{i,1}^m$, $B$ as $B_j^{m+1}$, $\hat{B}$ as $B_i^m$, $C^i$ as $C_{j,i}^{m+1}$, $\hat{C}$ as $C^{m,1}$, $\mathfrak{D}$ as $\mathfrak{A}^{m-1}$, and $\rho$ as constructed in Lemma 4.4.19 to conclude: $\rho(B_j^{m+1})$ is $G_{\{I\}}$-invariant, the flower over $\rho(C)/\hat{B}$ is $G_{\{I\}}$-invariant, and $\rho(A)$ is $G_{\{I\}}$ invariant.

Using the notation of Lemma 4.4.19 and Claim 4.4.20, we illustrate the location of $\rho(B_j^{m+1}) = \rho(B)$ and $\rho(C_{j,x}^{m+1}) = \rho(C^x)$, for $x = 1, 2$, in Diagram 8. So, $\rho(B) = B_+ \cup \rho(B_-)$. We denote $\rho(B_-)$ by $\widetilde{B}_-$ in Figure 8. As in Figure 5 we draw only two petals $A_{i,1}^m$ and $A_{i,2}^m$ from $\mathfrak{A}^m$ and $A_{j,1}^{m+1}$ from $\mathfrak{A}^{m+1}$. While, for simplicity, $B_i^m \subseteq \mathfrak{A}^{m-1}$ is omitted from the diagram, we clarify its properties. Since $B_i^m$ is $G_{\{I\}}$-invariant and is safe by $\mathrm{sdim}_m$, general properties of the construction yield

$$2 \leq d(B_i^m) \leq \delta(B_i^m) \leq \mu(A_{i,1}^m/B_i^m) = 2.$$

This implies $B_i^m \leq M$ and $B_i^m = \mathrm{icl}(B_i^m)$. Lemma 4.4.16 implies, writing $C_+^x$ for $C_{j,x}^{m+1} \cap \mathfrak{A}^{m-1}$, that $C^x \subseteq A_{i,1}^m \cup \mathrm{icl}(B_i^m)$; more precisely, $C_+^x \subseteq \mathrm{icl}(B_i^m) = B_i^m$ and $C_-^x \subseteq A_{i,1}^m$. Since $\rho$ fixes $B_i^m$ pointwise, $\rho(C^x) = \rho(C_+^x \cup C_-^x) = C_+^x \cup \rho(C_-^x)$. We denote $\rho(C_-^x)$ by $\widetilde{C}_-^x$. By construction, $\rho$ moves $A_{i,1}^m$ to $C^{m,1}$. So, any subset of $A_{i,1}^m$ is moved to $C^{m,1}$, in particular $C_-^x$.

Obviously, $\rho(B)$, $\rho(C^1)$, and $\rho(C^2)$ are subsets of $\mathfrak{A}^{m-1}$, so the third petal over $\rho(B)$ will be on at most $m$-th strata of some $G_{\{I\}}$-normal set, as we show it below.



FIGURE 8. Illustrating Claim 4.4.20

In Figure 8 we abbreviate as follows: $A = A_{j,1}^{m+1}$, $B = B_j^{m+1}$, $C^x = C^{m+1,x}$, $\widetilde{C}_-^x = \rho(C^{m+1,x})$ and $\widetilde{B}_- = \rho(B_-)$.

Clearly $\chi_M(\rho(A_{j,1}^{m+1})/\rho(B_j^{m+1})) \leqslant \mu_j^{m+1} = \mu(A_{j,1}^{m+1}/B_j^{m+1})$. The $\rho(C^{m+1,q})$ gives us $\mu_j^{m+1} - 1$ witnesses. Since $\rho(B^{m+1,q}) \subseteq \mathfrak{A}^{m-1} \leqslant M$, $\rho(A_{j,1}^{m+1})$ cannot split over $\mathfrak{A}^{m-1}$ (Definition 3.6). Similarly $\mathfrak{A} \leqslant M$ implies $\rho(A_{j,1}^{m+1})$ cannot split over $\mathfrak{A}^{m-1}$. We now have three cases depending on the exact location of $\rho(A_{j,1}^{m+1})$.

Case 1. $\rho(A_{j,1}^{m+1}) \subseteq \mathfrak{A}^{m-1}$. Immediately, the induction hypothesis $\text{sdim}_m$ (in fact, $\text{sdim}_{m-1}$) implies $\rho(A_{j,1}^{m+1})$ is safe.

To complete the proof, we show an extension of Lemma 4.4.8.

Case 2: $\rho(A_{j,1}^{m+1}) \subset \mathfrak{A}$ and $\rho(A_{j,1}^{m+1}) \cap \mathfrak{A}^{m-1} = \emptyset$: Since $\mathfrak{A}^{m-1} \leqslant M$, we must have $\delta(\rho(A_{j,1}^{m+1})/\mathfrak{A}^{m-1}) = 0$. But $(\rho(A_{j,1}^{m+1})/\rho(B_j^{m+1}))$ is a good pair. We know $(B_j^{m+1})_- \subseteq A_{1,1}^m$ so $(\rho(B_j^{m+1}))_- \subseteq C^{m,1} \subseteq \mathfrak{A}^{m-1}$. And $(\rho(B_j^{m+1}))_+ \subseteq \mathfrak{A}^0$. So $\rho(B_j^{m+1}) \subseteq \mathfrak{A}^{m-1}$. Since $\mathfrak{A}$ is $G_{\{I\}}$-normal and $\rho(A_{j,1}^{m+1})$ is well-placed by $\rho(B)$ over $\mathfrak{A}^{m-1}$ the construction places $\rho(A_{j,1}^{m+1})$ in $\mathfrak{A}^m$. So by $\text{sdim}_m$, $\rho(A_{j,1}^{m+1})$ is safe.

Case 3. $\rho(A_{j,1}^{m+1}) \subset M - \mathfrak{A}$: Then we put $\tilde{\mathfrak{A}}^m = \mathfrak{A}^{m-1} \cup \rho(A_{j,1}^{m+1})$. Note that $\tilde{\mathfrak{A}}^m$ is a $G_{\{I\}}$-normal with height $m$. Applying the global induction hypothesis $\text{sdim}_m$ to $\tilde{\mathfrak{A}}^m$, we see $\rho(A_{j,1}^{m+1})$ is safe.

Thus, in each case $\rho(A_{j,1}^{m+1})$ is safe. So, by Lemma 4.4.8.3, $A_{j,1}^{m+1}$ is safe. $\blacksquare$

We have finished if the descending sequence described before Lemma 4.4.14 stops immediately; we now consider the alternative. Because '$A_{j,1}^{m+1}$ determines $A_{i,1}^m$' produces a decreasing chain of complicated sub/superscripts, we introduce a notation for a descending sequence, which is relative to a given $G_{\{I\}}$-invariant petal $A_{j,1}^{m+1}$, and describes the 'root' below $A_{j,1}^{m+1}$ that controls its intersection with $\text{sdcl}^*(I)$. Recall that capital Roman letters $(A, B)$ denote petals, while script letters $\mathfrak{A}$ denote initial segments of a tree-decomposition.

**Definition 4.4.21.** *[Determined Sequences] We write $\Upsilon(A)$ for the petal determined (Definition 3.18) by $A$. Then $\Upsilon^k(A)$ denotes the kth iteration of this operation.*

(1) *As usual, $\mathfrak{A}^m = \bigcup_{i \leqslant m} \mathfrak{A}^i$.*

(2) *Fix $\overleftarrow{A}^0, \overleftarrow{A}^1$ such that $\overleftarrow{A}^0 = A_{j,1}^{m+1}$ determines $\Upsilon(\overleftarrow{A}^0) = \overleftarrow{A}^1 = A_{i,1}^m$ (Definition 3.18). For fixed $q$, $\overleftarrow{C}^{0,q} = C^{m+1,q}$, $\overleftarrow{B}^0 = B_j^{m+1}$.*

*The crucial inductive definition is*

$$\overleftarrow{A}^{k+1} = \begin{cases} \Upsilon(\overleftarrow{A}^k), & \text{if } \mu(\overleftarrow{A}^k, \overleftarrow{B}^k) \geqslant 3 \\ stop, & \text{if } \mu(\overleftarrow{A}^k, \overleftarrow{B}^k) = 2 \end{cases}$$

*So, $\overleftarrow{A}^k = A_{t_k,1}^{m+1-k}$ for some $t_k$ for each $k \leq s$. Increment indices for $B, C, \mu$ in the same way. E.g. $\overleftarrow{C}^{k,q}$ is $C_{j,q}^{m+1-k}$ in the notation for decompositions.*

(3) *The order of $\overleftarrow{A}^0 = A_{j,1}^{m+1}$ is the least index $s$ such that $\mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$.*

(4) *Suppose the order of $\overleftarrow{A}^0$ is $s$. We work from the bottom to define the root that supports $\overleftarrow{A}^0$.*

$$\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{B}^s.$$

*For $k \leqslant s$, we define $\mathcal{W}_k$ by downward induction.*

$$\mathcal{W}_k = \mathcal{W}_{s+1} \cup \overleftarrow{A}^s \cup \cdots \cup \overleftarrow{A}^k = \mathcal{W}_{k+1} \cup \overleftarrow{A}^k \subseteq \overleftarrow{\mathfrak{A}}^k.$$

Recall that increasing the superscript of an $\overleftarrow{\mathfrak{A}}$ moves to lower strata. Since we are analyzing $A_{j,1}^{m+1}$, for any $k < m+1$, $\overleftarrow{\mathfrak{A}}^0 = \mathfrak{A}^{m+1}$, $\overleftarrow{\mathfrak{A}}^{k+1}$ is the initial segment

preceding $\overleftarrow{\mathfrak{A}}^k$ in the original decomposition. In particular,

$$\overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m+1-(s+1)} = \mathfrak{A}^{m-s} = \bigcup_{k \leqslant m-s} \mathfrak{A}^k.$$

Also, the $\overleftarrow{C}^{k,q} = C_{t_k}^{m+1-k,q} \subseteq \overleftarrow{\mathfrak{A}}^{k+1}$ are isomorphic over $\overleftarrow{B}^k$ copies of $\overleftarrow{A}^k$.

Since $\mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$ and $\overleftarrow{B}^s$ is safe, $2 \leq d(\overleftarrow{B}^s) \leq \delta(\overleftarrow{B}^s) \leq \mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$. So the next lemma is easy.

**Lemma 4.4.22.** *Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leqslant k \leqslant s \rangle$ stops with $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$, then $\delta(\mathcal{W}_k) = 2$ for each $k \leq s+1$.*

*Proof.* Since $\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \overleftarrow{B}^s \leq M$ and $\delta(\overleftarrow{B}^s) = 2$, $\delta(\mathcal{W}_{s+1}) = 2$. Then we can finish by induction, because at each step we consider a 0-primitive extension. $\blacksquare$

**Lemma 4.4.23.** *Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leqslant k \leqslant s \rangle$ stops with $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$ then*

(1) *each $\mathcal{W}_k \leqslant M$ and is $G$-invariant;*
(2) *$\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$ for every $q \in \{1, \ldots, \mu(\overleftarrow{A}^0/\overleftarrow{B}^0) - 1\}$.*

*Proof.* 1) To start the induction, note $\overleftarrow{B}^s \leqslant M$ since $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$. $G_{\{I\}}$-invariance follows from the definition of determined, noting that $\overleftarrow{A}^k$ is $G_{\{I\}}$-invariant by Lemma 3.23 as $\mu(\overleftarrow{A}^{k-1}/\mu(\overleftarrow{B}^{k-1}) \geqslant 3$. But each $\mathcal{W}_{k+1} \leqslant \mathcal{W}_k$ since all have dimension 2.

2) By Lemma 4.4.16, for each $k \leqslant s$, $q \leqslant \mu(\overleftarrow{A}^k/\overleftarrow{B}^k)$, $\overleftarrow{C}^{k,q} \subseteq \overleftarrow{A}^{k+1} \cup \mathrm{icl}(\overleftarrow{B}^{k+1})$. Since each $\overleftarrow{B}^{k+1} \subseteq \mathcal{W}_k \leqslant M$, this implies $\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$. $\blacksquare$

Having dealt with the case $\mu_i^m = 2$, we consider the general case of Lemma 4.4.20. The key difficulty is that we cannot deduce $\rho(B)$ is $G_{\{I\}}$-invariant in one step as in Lemma 4.4.11. We have a sequence that stops with an $\overleftarrow{A}^s$ such that $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$ so that there is an automorphism $\rho$ mapping $\overleftarrow{A}^s$ into $\overleftarrow{C}^s$. With this $\rho$ fixed we argue inductively that each $\overleftarrow{A}^k$ for $s \geqslant k \geqslant 0$ is safe.

But, we must perform a dual induction with the proof that $\rho(A)$ is $G_{\{I\}}$-invariant.

**Lemma 4.4.24.** *Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leqslant k \leqslant s \rangle$ stops with $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$. Then $\overleftarrow{A}^k$ is safe for each $k \leqslant s$. In particular, when $k = 0$, we see $A_{j,1}^{m+1}$ is safe.*

*Proof.* We use Definition 4.4.21 of $\mathcal{W}_k$. By Lemma 4.4.15, as used in Lemma 4.4.20, fix an automorphism $\rho$ of $M$, that sends $\overleftarrow{A}^s$ to its unique copy $\overleftarrow{C}^s$ and which fixes $\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \overleftarrow{B}^s$ pointwise. Recall $\mathcal{W}_s = \mathcal{W}_{s+1} \cup \overleftarrow{A}^s$ and that $\overleftarrow{C}^s \subseteq \overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m-s}$. Let $\widetilde{\mathcal{W}}_s = \mathcal{W}_s \cup \overleftarrow{C}^s = (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \overleftarrow{B}^s \cup \overleftarrow{C}^s$. Then $\rho(\mathcal{W}_s) = \widetilde{\mathcal{W}}_s$. For $k \leqslant s+1$, we build on Definition 4.4.21.4 of $\mathcal{W}_k$. We define

$$\widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_s \cup \rho(\overleftarrow{A}^{s-1}) \cup \cdots \cup \rho(\overleftarrow{A}^k).$$

Note that $\widetilde{\mathcal{W}}_k$ need not be contained in $\mathfrak{A}$. In particular, $\mathcal{W}_s \subseteq \overleftarrow{\mathfrak{A}}^s$ while $\widetilde{\mathcal{W}}_s - \overleftarrow{\mathfrak{A}}^{s+1} = \rho(\overleftarrow{A}^{s-1})$.

Finally, to obtain a $G_{\{I\}}$-normal structure with a well-defined height, we define:

$$\widetilde{\mathcal{R}}_k = \overleftarrow{\mathfrak{A}}^{s+1} \cup \rho(\overleftarrow{A}^s) \cup \cdots \cup \rho(\overleftarrow{A}^k).$$

Note that the height of $\widetilde{\mathcal{R}}_s = \overleftarrow{\mathfrak{A}}^{s+1} \cup \rho(\overleftarrow{A}^s)$ is $m-s$ because $\rho(\overleftarrow{A}^s) \subseteq \overleftarrow{\mathfrak{A}}^{s+1}$ and $\overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m-s}$. Moving from $\widetilde{\mathcal{R}}_{k+1}$ to $\widetilde{\mathcal{R}}_k$ increases the height at most by 1; that is why the height of $\widetilde{\mathcal{R}}_0$ is at most $m$. Since $\overleftarrow{A}^k$ determines $\overleftarrow{A}^{k+1}$, Lemma 4.4.10.2 implies that $\overleftarrow{B}^k \subseteq (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)) \cup \overleftarrow{A}^{k+1}$: obviously, then,

$$(*) \quad \rho(\overleftarrow{B}^k) \subseteq \widetilde{\mathcal{W}}_{k+1} = \widetilde{\mathcal{W}}_s \cup \rho(\overleftarrow{A}^s) \cup \cdots \cup \rho(\overleftarrow{A}^{k+1}) \subseteq \widetilde{\mathcal{R}}_k.$$

By Claim 4.4.23 $\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$. Thus, $\rho(\overleftarrow{C}^{k,q}) \subseteq \widetilde{\mathcal{W}}_{k+1}$. Obviously, $\rho(\overleftarrow{A}^k) \cap \widetilde{\mathcal{W}}_{k+1} = \emptyset$ because $\overleftarrow{A}^k \cap \mathcal{W}_{k+1} = \emptyset$. By Lemma 4.3.5 the bouquet of $\rho(\overleftarrow{A}^k)$ over $\rho(\overleftarrow{B}^k)$ consists just of one flower.

We conclude Lemma 4.4.24 from the following, which we show below for each $k \leqslant s$:

(1) $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$-invariant;
(2) $\overleftarrow{A}^{s+1} \cup \widetilde{\mathcal{W}}_k$, $\widetilde{\mathcal{W}}_k$, and $\rho(\overleftarrow{A}^k)$ are $G_{\{I\}}$-invariant.

We prove these two assertions by simultaneous induction on $k$. The induction is downward from $s$ and the base step is the third paragraph of the proof of Lemma 4.4.20. So, we assume that (1)–(2) hold for $k+1$ and show that they hold for $k$.

(1) Recall that by Lemma 3.23.1 and Lemma 4.4.10.2, $\overleftarrow{B}^k \subseteq \overleftarrow{A}^{k+1} \cup (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))$. So,

$$\rho(\overleftarrow{B}^k) \subseteq \rho(\overleftarrow{A}^{k+1} \cup (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))) = \rho(\overleftarrow{A}^{k+1}) \cup (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)).$$

We consider $\rho(\overleftarrow{B}^k) \cap (\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))$ and $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})$ separately. Since $\rho$ fixes $\mathfrak{A}^0 \cap \mathrm{acl}(\emptyset)$ pointwise and $\overleftarrow{B}^k$ is $G_{\{I\}}$-invariant, $\rho(\overleftarrow{B}^k \cap \mathfrak{A}^0 \cap \mathrm{acl}(\emptyset))$ is $G_{\{I\}}$-invariant.

We show an arbitrary $\pi \in G_{\{I\}}$ fixes $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})$ setwise. By the induction hypothesis, $\rho(\overleftarrow{A}^{k+1})$ is $G_{\{I\}}$-invariant, so $\pi(\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})) \subseteq \rho(\overleftarrow{A}^{k+1})$. Now we put

$$\tau = (\pi \upharpoonright \overleftarrow{\mathfrak{A}}^{k+2}) \cup (\rho^{-1} \circ \pi \circ \rho) \upharpoonright \overleftarrow{A}^{k+1}.$$

Obviously, this isomorphism extends to an automorphism from $G_{\{I\}}$. Since $\overleftarrow{B}^k$ is $G_{\{I\}}$-invariant, $\tau(\overleftarrow{B}^k) = \rho^{-1} \circ \pi \circ \rho(\overleftarrow{B}^k) = \overleftarrow{B}^k$; so $\pi(\rho(\overleftarrow{B}^k)) = \rho \circ \pi(\overleftarrow{B}^k) = \rho(\overleftarrow{B}^k)$.

(2) By the induction hypotheses $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, $\widetilde{\mathcal{W}}_{k+1}$, and $\rho(\overleftarrow{A}^{k+1})$ are $G_{\{I\}}$-invariant. Whence, by (1) $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$-invariant. Since $\widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_{k+1} \cup \rho(\overleftarrow{A}^k)$, it is sufficient to prove that $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$-invariant to deduce that $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_k$ and $\widetilde{\mathcal{W}}_k$ are $G_{\{I\}}$-invariant. So, we consider $\rho(\overleftarrow{A}^k)$. In fact, we repeat some reasoning from Lemma 4.4.20. We put $\overleftarrow{\nu}^k = \mu(\overleftarrow{A}^k / \overleftarrow{B}^k) - 1$.

Case 2a) $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$: Since $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$-invariant, so is

$$(**) \quad \bigcup_{q=1}^{\overleftarrow{\nu}^k} \rho(\overleftarrow{C}^{k,q}) \cup \rho(\overleftarrow{A}^k)$$

because it is the flower of $\rho(\overleftarrow{A}^k)$ over $\rho_0(\overleftarrow{B}^k)$. Clearly, each $\overleftarrow{C}^{k,q}$ intersects $\overleftarrow{A}^{k+1}$; *otherwise*, there must be $c_1, c_2 \in \overleftarrow{C}^{k,q} \subseteq \overleftarrow{\mathfrak{A}}^s \cup \mathcal{W}_{k+2}$ and $b \in \overleftarrow{B}^k \cap \overleftarrow{A}^{k+1}$ with $R(b, c_1, c_2)$. The last implies that $|\overleftarrow{A}^{k+1}| = 1$, for a contradiction. Thus, each $\rho(\overleftarrow{C}^{k,q})$ intersects $\rho(\overleftarrow{A}^{k+1})$, which is $G_{\{I\}}$-invariant by the induction hypothesis.

Let $\tau \in G_{\{I\}}$ be arbitrary. Since $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$-invariant $\tau(\rho(\overleftarrow{B}^k)) = \rho(\overleftarrow{B}^k)$ and so
$$\tau(\rho(\overleftarrow{A}^k)) \in \{\rho(\overleftarrow{C}^{k,q}) : q = 1, \ldots, \overleftarrow{\nu}^k\} \cup \{\rho(\overleftarrow{A}^k)\}.$$
By construction, $\overleftarrow{A}^k \cap \overleftarrow{A}^{k+1} = \emptyset$; so $\rho(\overleftarrow{A}^k) \cap \rho(\overleftarrow{A}^{k+1}) = \emptyset$ and $\tau(\rho(\overleftarrow{A}^k))$ does not intersect $\tau(\rho(\overleftarrow{A}^{k+1})) = \rho(\overleftarrow{A}^{k+1})$. But, we showed in the last paragraph $\rho(\overleftarrow{C}^{k,q}) \cap \rho(\overleftarrow{A}^{k+1}) \neq \emptyset$, so $\tau(\rho(\overleftarrow{A}^k))$ cannot be equal to any of the $\rho(\overleftarrow{C}^{k,q})$. Hence, using (**), $\tau(\rho(\overleftarrow{A}^k)) = \rho(\overleftarrow{A}^k)$ and $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$-invariant.

Since $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, by the global induction hypothesis $\rho(\overleftarrow{A}^k)$ is safe.

Case 2b) $\rho(\overleftarrow{A}^k) \not\subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$: As $\rho(\overleftarrow{A}^k)$ is a 0-primitive extension of $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, $\rho(\overleftarrow{A}^k) \cap (\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}) = \emptyset$. By (*), $\rho(\overleftarrow{B}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$; moreover $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1}) \neq \emptyset$.

**Claim 4.4.25.** *In case 2b, for each $k < s$, $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1} - \overleftarrow{\mathfrak{A}}^{k+2}$ or $\rho(\overleftarrow{A}^k) \cap \mathfrak{A} = \emptyset$. In either case $\overleftarrow{\mathfrak{A}}^{k+1} \cup \widetilde{\mathcal{W}}_{k+2}$ is $G_{\{I\}}$-normal.*

*Proof.* We have $\rho(\overleftarrow{B}^k) \subseteq \rho(\overleftarrow{A}^{k+1}) = \overleftarrow{C}^{k+1} \subseteq \overleftarrow{\mathfrak{A}}^{k+2} \cup \widetilde{\mathcal{W}}_{k-1}$, $\rho(\overleftarrow{A}^k)$ is 0-primitive over $\rho(\overleftarrow{B}^k)$, and $\overleftarrow{\mathfrak{A}}^{k+1} \leqslant M$. Thus, $\rho(\overleftarrow{A}^k)$ is 0-primitive over $\overleftarrow{\mathfrak{A}}^{k+1}$ and based on $\rho(\overleftarrow{B}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1}$. If $\rho(\overleftarrow{A}^k) \subseteq \mathfrak{A}$ by, construction, $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1} - \overleftarrow{\mathfrak{A}}^{k+2}$. If not, since $\rho(\overleftarrow{A}^k)$ cannot split (Definition 3.6) over $\mathfrak{A}$, $\mathfrak{A} \cap \rho(\overleftarrow{A}^k) = \emptyset$ and so $\overleftarrow{\mathfrak{A}}^{k+1} \cup \{\widetilde{\mathcal{W}}_k\}$ is $G_{\{I\}}$-normal. ∎

Since $\overleftarrow{\nu}^k$ copies of $\rho(\overleftarrow{A}^k)$ over its base are inside $\widetilde{\mathcal{W}}_{k+1}$, $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$-invariant. This completes the proof of Case 2b.

Since $\rho(A_{j,1}^{m+1}) = \rho(\overleftarrow{A}^0) \subseteq \widetilde{\mathcal{W}}_0 \subseteq \widetilde{\mathcal{R}}_0$ and the height of $\widetilde{\mathcal{R}}_0$ is at most $m$, by the global induction $\rho(A_{j,1}^{m+1})$ is safe; by Lemma 4.4.8.2, we conclude $A_{j,1}^{m+1}$ is safe. We finish Lemma 4.4.24. ∎

This completes the proof of Lemma 4.4.3 showing $\mathrm{sdim}_m$ for $m \leqslant m_0$; thus we have the main conclusion, Theorem 4.4.1.

## 5. Steiner Systems

In this section we study the strongly minimal $k$-Steiner systems discovered in [BP21]. A $k$-Steiner system is a collections of points and lines so that two points determine a line and all lines have the same finite length $k$. A quasigroup (binary operation with unique solutions of $ax = b$ and $xa = b$) such that every 2-generated sub-quasigroup has $k$ elements determines a $k$-Steiner system where the lines are the two generated subalgebras. Our interest in the existence of definable truly binary functions arose from the discovery that while a Steiner system with line length three admits a quasigroup operation definable in the vocabulary of the ternary collinarity predicate and Steiner systems with prime power length admit the imposition of quasigroups that preserve lines (e.g. [GW80]), it seemed very unlikely in the second case that those quasigroups were *definable from R* ([Bal23]).

There are two examples of strongly minimal 3-Steiner systems in [BP21] and [Hru93, Section 5]. By explicitly adding multiplication to the vocabulary, [Bal23] constructs strongly minimal quasigroups which determine $k$-Steiner systems for each prime power $k$. We show below that this separate operation is essential. The following problem/example inspired this research and is solved here.

**Problem 5.1.** We can impose a quasigroup structure on any 4-Steiner system. There are two obvious ways: one commutative, one not [Bal23]. In fact, [GW75] a quasi-group can be imposed in any Steiner $k$-system when $k = p^n$ for a prime $p$. 1) Prove the operations of these quasi-groups are not $R$-definable in a strongly minimal 4-Steiner system $(M, R)$. 2) More generally, is there an $\emptyset$-definable truly binary function?

We now use $\boldsymbol{K}$ rather than $\boldsymbol{L}$ to emphasize the distinctions from Section 4. Having said that, $\boldsymbol{K}^* = \boldsymbol{L}^*$ and $\boldsymbol{K}_0^* = \boldsymbol{L}_0^*$ while $\boldsymbol{K}_0 \neq \boldsymbol{L}_0$ . We work in a vocabulary $\tau$ with one ternary relation $R$, and assume always that $R$ can hold only of three distinct elements and then in any order (a 3-hypergraph). The basic definitions are in Section 1.2. In the language of $*$-petals, $\mu$ *triples* if for every *non-linear* (Definition 5.6) $*$-petal $(C/B)$ with $\delta(B) = 2$ and $|C| > 1$, $\mu(C/B) \geqslant 3$.

**Theorem 5.2.** *Let $M \models T_\mu^S$ be a strongly minimal Steiner system described in Notation 0.1. Then*

(1) *The naturally imposed quasigroups ([GW75]) on $M$ are not $\emptyset$-definable in $M$.*

(2) *If $\mu(\boldsymbol{\alpha}) \geqslant 2$ and $\mu$ triples, then there is no $\emptyset$-definable truly binary function in $T_\mu^S$.*

(3) *There is no symmetric $\emptyset$-definable truly $v$-ary function for $v \geq 2$, i.e., $\mathrm{sdcl}^*(I) = \emptyset$ for any $v$-element independent set $I$.*

After a short introduction establishing 1), we prove 2) and 3). The major obstacles to adapting the earlier proofs of these results in the Hrushovski case are a) the need to modify the notion of base (Lemma 5.4) and b) the analysis of distinct occurrences of $R$ (e.g. Lemma 3.21).

The following example (Figure 9) shows that as in the Hrushovski case, we must pass to $G_{\{I\}}$ and strengthen the hypothesis to get $\mathrm{sdcl}^*(I) = \emptyset$. Definable truly binary functions may appear when $\mu(A/B) = 2$ and $d(B) = 2$ is allowed. We put the following lines: $\{a_1, d_2, d_1\}$, $\{a_1, d_4, d_5\}$, $\{a_2, d_5, d_3, d_1\}$, and $\{d_2, d_3, d_4\}$. The elements $c_i$ is the isomorphic copy of $d_i$ over $\{a_1, a_2\}$, for each $i$. In order to construct $A_{1,1}^2$ we make $\alpha_i$ a copy of $a_i$, $\delta_i$ a copy of $d_i$ and each $\gamma_i$ a copy of $c_i$ for each appropriate $i$, where the isomorphism under consideration is over $\{d_3, c_3\}$. Then $\alpha_1 \in \mathrm{dcl}^*(a_1, a_2)$.

We defined linear spaces and the appropriate $\delta$ for studying them in Definition 1.2.2. In [BP21, Lemma 3.7] we showed that this $\delta$ is flat, submodular, and computes exactly on free products defined as in Definition 5.7. Thus, the notion of decomposition and the arguments for the basic properties of the standard Hrushovski construction in earlier sections go through below with minor changes. However, Lemma 5.11 shows some significant differences in the resulting decomposition. This finer analysis of the decomposition, which is the chief novelty of this section, powers the understanding of definable closure in these Steiner systems.

Recall from Conclusion 1.1.10: for each $3 \leqslant k < \omega$, there are continuum-many strongly minimal infinite linear spaces in the vocabulary $\tau$ that are Steiner $k$-systems. A crucial invariant for these systems is 'line length'. The length of each line in a model of the Steiner system is $\mu(\boldsymbol{\alpha}) + 2$ where $\boldsymbol{\alpha}$ is from Notation 1.2.6. However, there may be maximal cliques in a substructure $A$ with smaller cardinality. We refer to such configurations as *partial lines*; a line of length $\mu(\boldsymbol{\alpha}) + 2$ may be
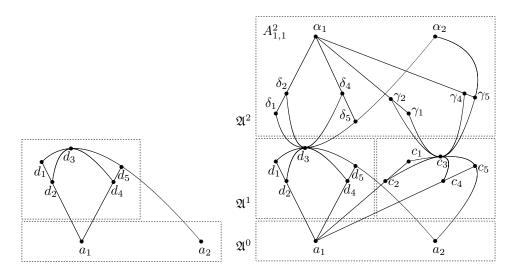
FIGURE 9. Example of $\mathrm{dcl}^*(\{a_1, a_2\}) \neq \emptyset$

called *full* for emphasis. Following a convention established in [BP21], we think of two independent points as a *trivial (and therefore partial) line*.

The Hrushovski restraint in defining the $\mu$-function: an integer $\mu(\beta) = \mu(A/B) \geqslant \delta(B)$ was relaxed in [BP21] to hold only when $|A - B| \geqslant 2$. To allow lines of length three, we required only $\mu(\boldsymbol{\beta}) \geqslant 1$, if $\boldsymbol{\beta} = \boldsymbol{\alpha}$. Thus for the case when $\mu(\boldsymbol{\alpha}) = 1$ we got a strongly minimal Steiner system with lines of length three. Obviously, there is a definable symmetric truly binary function $H$ on pairs of distinct elements; $H(x, y)$ is the third point on the line determined by $x$ and $y$ and $H(x, x) = x$. So we restrict here to lines of length at least four.

**Assumption 5.3.** $\mu(\boldsymbol{\alpha}) \geqslant 2$.

With longer line length $k$ one can always introduce a $k$-ary *partial function* saying its value on $k - 1$ distinct elements is the remaining point on the line. But, now there is no clear way to give a uniform definition of a $k$-ary *function* on sequences with repetition. With the following variant on the results in Section 4, we show there is no such truly binary function in the vocabulary: $\{R\}$.

As noted in Remark 1.1.6 the original Hrushovski construction supports a happy coincidence. The minimal subset $B$ of $D$ (the base: Definition 1.2.5) such that $A$ is 0-primitive over $B$ is also the maximal subset such that every element of $B$ is $R$ related to some element of $A$. But for linear spaces, the two notions diverge. Allowing for and exploiting this difference is one of the two major changes from the proof for the Hrushovski construction in Section 4. We recast [BP21, Lemma 4.8] (Lemma 1.2.5) in case 2) of the next lemma.

**Lemma 5.4.** *Let $D \leqslant D \cup A \in \boldsymbol{K}_0$ be a 0-primitive extension with $D \cap A = \emptyset$. Then there are two cases:*

   *(1) If $A = \{a\}$ there is a unique line $\ell$ with $\ell \cap D \geqslant 2$. In that case, any $B \subseteq (\ell \cap D)$ with $|B'| = 2$ yields a good pair $(B, a)$. Furthermore, $d \in D$ is in the relation $R$ with the element $a$ if and only if $d$ is on $\ell$.*

(2) If $|A| \geqslant 2$ then there is a unique maximal subset $B$ of $D$ with every point $b$ in $B$ incident with a line $\ell_b$ with $|\ell_b \cap A| \geqslant 2$ containing $b$.

On the basis of Lemma 5.4 we add the new notion of *extended base*.

**Definition 5.5.** Let $A$ be a 0-*primitive extension of* $D$ *(in* $M$*), where we assume that* $D \cap A = \emptyset$. *If* $A = \{a\}$, *then the* extended base *for* $A$ *is the maximal set* $\check{B} = \ell \cap D$ *where* $\ell$ *is the line through* $b_1, b_2$ *for any elements* $b_1, b_2 \in D$ *such that* $R(b_1, b_2, a)$. *Note that if* $A = A_{j,i}^{m+1}$ *and* $D = \mathfrak{A}^m$ *the extended base for* $A$ *is* $B_j^{m+1} = \{d \in \mathfrak{A}^m - \mathfrak{A}^{m-1} : R(b_1, b_2, d)\}$ *for any* $b_1, b_2 \in \mathfrak{A}^m$ *with* $R(b_1, b_2, a)$.

If $A_{j,i}^{m+1} = \{a\}$ is 0-primitive over $\mathfrak{A}$ with extended base $B = B_j^{m+1}$, any two element subset $B_0$ of $B$ can act as a base. If $\ell_{m+j} < \mu(\boldsymbol{\alpha})$, the $C^{m+1,q}$ must be mapped into $B - B_0$.

Definition 5.6 will be clarified by Lemma 5.11 showing that all types of $*$-petals have been described.

**Definition 5.6.** Let $G \in \{G_I, G_{\{I\}}\}$, and let $\mathfrak{A}$ be a $G$-normal set. Fix a decomposition of $\mathfrak{A}$ into strata $\mathfrak{A}^m$ constructed inductively as in Construction 3.8.

(1) We say $A = A_{j,1}^{m+1} = \{a\} \in \mathfrak{A} - \mathfrak{A}^m$ is an $\boldsymbol{\alpha}$-point if there exist $b_1, b_2 \in \mathfrak{A}^m$ with $R(b_1, b_2, a)$.

(2) A set $A$ is a linear cluster if $A = \{a \in \mathfrak{A}^{m+1} - \mathfrak{A}^m : R(b_1, b_2, a)\}$ for some $b_1, b_2 \in \mathfrak{A}^m$. We denote the linear cluster with extended base $\check{B} = B_f^m \subseteq \mathfrak{A}^m$ as $A_f^{m+1} = \bigcup A_{f,i}^{m+1}$ where the $A_{f,i}^{m+1}$ are the $\alpha$-petals over $\check{B}$.

(3) A $*$-petal is either an $A_{f,i}^{m+1}$ with cardinality greater than 1 (called a non-linear petal) or a linear cluster.

(4) We write Steiner-moves$_m$ if every non-linear petal $A_{f,k}^m$ is moved by some $g \in G_I$.

(5) Recall that we say $X$ is safe if $d(E) \geq 2$ for any $G_{\{I\}}$-invariant set $E \subseteq X$ which is not a subset of $\mathrm{acl}(\emptyset)$. The $G_{\{I\}}$-decomposition $\mathfrak{A}^m$ of $\mathfrak{A}$ satisfies Steiner-sdim$_m$ if every $G_{\{I\}}$-invariant subset of $\mathfrak{A}^m$ is safe.

Now any $\gamma \in G$ that fixes $A$ setwise fixes an extended base set-wise but it does not need to fix a base of an $\alpha$-point even setwise.

**Definition 5.7.** [BP21, Lemma 3.14] Let $A \cap C = B$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_B C$ as follows:

(1) the domain of $D$ is $A \cup C$;

(2) a pair of points $a \in A - B$ and $b \in C - B$ are on a non-trivial line $\ell'$ in $D$ if and only if there is line $\ell$ based in $B$ such that $a \in \ell$ (in $A$) and $b \in \ell$ (in $C$). Thus $\ell' = \ell$ (in $D$).

**Lemma 5.8.**   (1) If $D \supseteq A \cup B \cup C$ where $A$ and $C$ are 0-primitive over $B$, $B \leqslant D$, and there is a relation among elements $a_1 \in A - B$ and $a_2 \in C - B$ then both $|A - B|$ and $|C - B|$ are 1.

(2) Each $\mathfrak{A}^m$ is partitioned into $*$-petals and there is no non-trivial line (even through $\mathfrak{A}^{m-1}$) connecting distinct $*$-petals. That is, the $*$-petals are fully independently joined

*Proof.* 1) If $R(a_1, a_2, b)$ then $\delta(A/BC) < \delta(A/C)$, unless there is a line $\ell \subseteq D$ with $|\ell \cap B| \geqslant 2$ that contains both $a_i$. But each $a_i$ is then the only element of a

0-primitive extension of $B$. 2) Thus the collection of $*$-petals (i.e. non-linear petals and the linear clusters $A_f^{m+1}$ of $\alpha$-points) are fully freely joined as a partition of $\mathfrak{A}^{m+1} - \mathfrak{A}^m$.                                                                ∎

**Lemma 5.9.** *Fix a $G$-normal $\mathfrak{A}$ and a decomposition of height at least 3, where $G \in \{G_I, G_{\{I\}}\}$. Every non-trivial partial line $\ell$ in $\mathfrak{A}$ is either contained (except for at most one point) in a single petal of the topmost strata $\mathfrak{A}^{m_0}$ or extends to a full line that intersects at most three strata.*

*Proof.* Let $m$ be least so that $\ell$ is based in $\mathfrak{A}^m$. If $|\ell| < \mu(\boldsymbol{\alpha}) + 2$, adding a new point in $\mathfrak{A}^{m+1}$, that is related only to $\ell \cap \mathfrak{A}^m$ is a 0-primitive extension giving an $\boldsymbol{\alpha}$-petal $A_{f,i}^{m+1}$. By Corollary 3.7, $|\ell \cap \mathfrak{A}^{m+1}| = \mu(\boldsymbol{\alpha}) + 2$. It is possible that one point of $\ell$, but, by choice of $m$, not two, is in $\mathfrak{A}^{m-1}$. That is, it may be $|\ell \cap (\mathfrak{A}^{m+1} - \mathfrak{A}^{m-1})| = \mu(A_{j,1}^{m+1}/B_j^{m+1}) + 1$. This is the possibility that intersects three strata. If $m = m_0$, the line may remain partial but includes at most one point of $\mathfrak{A}^{m-1}$.                                                                ∎

**Definition 5.10.** *We say a petal $A_{j,1}^{m+1}$ Steiner-determines a $*$-petal, if there is a non-linear petal $A_{i,f}^m$ or a linear cluster $A_i^m$ which is the unique $*$-petal based in $\mathfrak{A}^{m-1}$ that intersects $B_j^{m+1} - \mathfrak{A}^{m-1}$. (More precisely, $\langle A_{j,1}^{m+1}, B_j^{m+1}, \mathfrak{A}^m \rangle$ determines $\langle A_{i,f}^m, B_i^m, \mathfrak{A}^{m-1} \rangle$.)*

**Lemma 5.11.** *Fix a decomposition of a $G$-normal set $\mathfrak{A}$, where $G \in \{G_I, G_{\{I\}}\}$. Suppose $A = \{a\}$ is an $\alpha$-point of $\mathfrak{A}^{m+1}$ based on $B = \{b_1, b_2\} \subseteq \mathfrak{A}^m$ and a subset of the linear cluster $A_j^{m+1}$. Let $\check{B}$ be the extended base of $a$ in $\mathfrak{A}^m$. Then,*

- *(1) If $m = 0$, $\{a\}$ is in a linear cluster $A_j^1$ with $|A_j^1| = \mu(\boldsymbol{\alpha}) - |I|$. Since $I$ is independent, this is possible only if $|I| = 2$.*
- *(2) Let $m > 0$. If a linear cluster satisfies $|A_j^{m+1}| = 1$ then $\check{B} - \mathfrak{A}^{m-1}$ is a subset of one $*$-petal, say, $A_{f,i}^m$, which is not a linear cluster. So, $A_j^{m+1}$ determines $A_{f,i}^m$ in this case.*
- *(3) Let $G = G_I$. Then Steiner-moves$_m$ implies each $\alpha$-point $\{a\}$ over $\mathfrak{A}^m$ is moved by $G_I$.*
  - *(a) $G_B$ acts as the symmetric group $S_{|A_f^m|}$ on a linear petal $A_f^m$ based on $B = \{b_1, b_2\}$. Thus, $G_I$ moves such $\alpha$-points.*
  - *(b) $A$ is a line based on $B \subsetneq A_{f,i}^m$ for some $f, i$. By Steiner-moves$_m$, $A_{f,i}^m$ is moved and a fortiori so is $A$.*
- *(4) Let $A_j^{m+1}$ be a linear cluster which contains at least two elements (that is, at least two $\alpha$-points) and which is $G$-invariant. If $d(\check{B}) \geq 2$ then $d(A_j^{m+1}) = 2$.*

*Proof.* 1) Suppose $m = 0$. We have $R(b_1, b_2, a)$; $b_1, b_2$ are algebraically independent; else $a \in \mathfrak{A}^0$. Moreover the definition of $\mathfrak{A}^0$ decrees $\neg R(b_1, b_2, b_3)$ for any 3 distinct $b_i \in \mathfrak{A}^0$. By Corollary 3.7, $\chi_M(\{a\}/\{b_1, b_2\}) = \mu(\boldsymbol{\alpha})$ yielding a linear cluster of cardinality $\mu(\boldsymbol{\alpha}) - 2$.

2) By Lemma 3.7 the line $\ell$ passing through $a, b_1, b_2$ is full and is equal to $\check{B} \cup A_j^{m+1}$. Then $|\check{B}| = \mu(\boldsymbol{\alpha}) + 2 - |A_j^{m+1}| \geq 3$, because $\mu(\boldsymbol{\alpha}) \geq 2$ and $|A_j^{m+1}| = 1$. By Lemma 5.9 $|\check{B} \cap \mathfrak{A}^{m-1}| \leq 1$, so at least two elements of $\check{B}$ are in $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. If these two elements belong to different $*$-petals, then these $*$-petals are not free over $\mathfrak{A}^{m-1}$, for a contradiction. (If there is a point on the line and in $\mathfrak{A}^{m-1}$ or

if there are three points in different petals, the petals are dependent over $\mathfrak{A}^{m-1}$.) Note that $\ell \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ is not a linear cluster because $|\check{B} \cap \mathfrak{A}^{m-1}| \leq 1$, while a base for a linear cluster contains at least 2 elements.

3) Any $\alpha$-point $e$ is either on a linear petal with size $\geqslant 2$ or $\mathrm{icl}(G_I(e))$ intersects two distinct $*$-petals:

3a) $|A_j^{m+1}| > 1$ *and is a linear cluster:* Then for $k \leqslant |A_j^{m+1}|$, all $k$-sequences from $A_j^{m+1}$ realize the same quantifier-free type over $\check{B}$ (and so over $\mathfrak{A}^m$ since $R(A_j^{m+1}, \mathfrak{A}^m) = R(A_j^{m+1}, \check{B})$, so they are automorphic over $\mathfrak{A}^m$ in $\mathfrak{A}$ since $\mathfrak{A}^m A_j^{m+1} \leqslant \mathfrak{A}$.

3b) $|A_j^{m+1}| = 1$: Then $B_j^{m+1} \subseteq A_{f,i}^m$ which is a non-linear petal and so Steiner-moves$_m$ implies $A = A_{j,1}^{m+1}$ is moved by $G_I$.

4) Since $\check{B}$ is a partial line, $\delta(\check{B}) = 2$. So, $\check{B} \leq M$ because by the hypothesis $d(\check{B}) \geq 2$. Then $\mathrm{icl}(A_j^{m+1}) \subseteq A_j^{m+1} \cup \check{B}$, because $A_j^{m+1}$ is a 0-primitive extension of $\check{B}$ and $\check{B} \leq M$. Since $|\mathrm{icl}(A_j^{m+1})| \geq |A_j^{m+1}| \geq 2$ and $\mathrm{icl}(A_j^{m+1})$ is contained in the line $A_j^{m+1} \cup \check{B}$, $d(A_j^{m+1}) = \delta(\mathrm{icl}(A_j^{m+1})) = 2$. ∎

**Remark 5.12.** Note that there are $R$-relations within a linear cluster; it lies on one line. And at least one linear cluster is $G_I$-invariant, the line through $I = \{a, b\}$; others are easy to find. But Lemma 5.11 shows no $\boldsymbol{\alpha}$-point is in $\mathrm{dcl}^*(I)$. There are partial lines of various lengths in the $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ that are not linear clusters. But each is within a single non-linear petal (Lemma 5.11). (This depends essentially on the decomposition of the ambient $G$-normal $\mathfrak{A}$; every pair of points is contained in a nontrivial line in $M$, but perhaps not in $\mathfrak{A}$.)

Lemma 5.11.3a yields immediately the answer to the motivating Problem 5.1.1. Recall a quasigroup satisfies for all $x$ and $y$, there exist unique $l$ and $r$ such that $lx = y$ and $xr = y$ (the multiplication table is a Latin square). [GW75] show that if Steiner system has line-length $k$, where $k$ is a prime-power, then it is possible to impose a binary function $*$ on the universe such that:

(#) $a, b$, $a * b$ is on the line through $a, b$ and $*$ is a quasigroup such that the restriction of $*$ to each line is generated by any two elements of the line.

However, this function cannot be definable (without parameters) in a strongly minimal structure $(M, R)$ studied here. It suffices to find one line on which the function is not defined. This is straight forward since any finite configuration is strongly embedded in $M$. In detail,

**Theorem 5.13.** *No quasigroup $*$ restricted to each line and satisfying (#) is definable in a strongly minimal Steiner system from* [BP21] *with line length at least four.*

*Proof.* Take any independent pair $I = \{a_1, a_2\}$ contained in some $\mathfrak{A}^m$ and suppose they generate the line $A = \{a_1, a_2, \dots a_k\}$. Then $A - I \subseteq \mathfrak{A}^{m+1} - \mathfrak{A}^m$ is a linear cluster and by Lemma 5.11.3a, $G_I$ induces the symmetric group on $A - I$.

Suppose $a_1 * a_2 = a_i$ and $a_2 * a_1 = a_j$. Choose an element $a_k$ of $A$ distinct from $a_i$. There is a $g \in G_I$ with $g(a_i) = a_k$. The definition of a quasigroup is contradicted unless $a_i = a_j = a_k$; in that event replace $a_k$ with an $a_{k'}$ distinct from all $a$'s previously considered; this is easy since $|A| \geqslant 4$. ∎

While this solution to Motivating Problem 5.1.1 invokes the decomposition, a more direct argument yields that result in [Bal23]. However, we now show the much

stronger consequence of the decomposition asked for in Problem 5.1.2, no truly $n$-ary function. For smoother reading, we mention results from Sections 3 and 4 that go through without any changes and pay attention to those results which requires some adaptations.

Lemmas 3.11 and 3.17 work for Steiner systems. Lemma 5.11.(2) and (3).(b) yield a stronger version Lemma 3.19.(1): If $|A_{i,j}^{m+1}| = 1$ is $G_I$-invariant then $A_{i,k}^{m+1}$ determines a $G_I$-invariant non-linear petal $A_{f,i}^m$. Multiple realizations of $\boldsymbol{\alpha}$ in $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ represent distinct petals but only one $*$-petal (linear cluster). We incorporate the role of Lemma 3.19.(2) in proving Lemma 3.22 into the proof of Lemma 5.14.

Comparing the argument for Lemma 3.19 with Figure 10 explains the main differences between Lemma 3.19 for Hrushovski's examples and Lemma 5.14 for Steiner systems. In Hrushovski's examples we obtain that $b_2$ is in two relations $R(b_2, c_1^0, c_3^0)$ and $R(b_2, c_1^1, c_3^1)$ with $\mathfrak{A}^{m-1}$, which contradicts $\mathfrak{A}^{m-1} \leq M$. But, in Steiner systems we have just one line $\ell$, which contains points from different copies $C^0$ and $C^1$ of $A_{j,1}^{m+1}$.
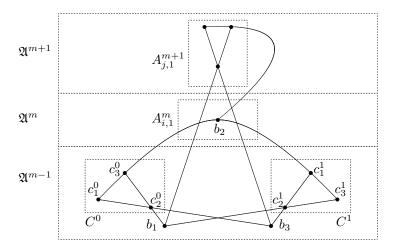


FIGURE 10. Example with one line and two $C^d$'s

**Lemma 5.14.** *Fix a decomposition of a $G$-normal set $\mathfrak{A}$, where $G \in \{G_I, G_{\{I\}}\}$. Suppose $B = B_j^{m+1}$ is the base of a non-linear petal $A_{j,1}^{m+1}$ which is $G$-invariant and $\ell_j^{m+1} + 1 < \mu_j^{m+1}$.*

*(1) Let $G = G_I$ and Steiner-dim$_m$ hold; or*
*(2) let $G = G_{\{I\}}$ and Steiner-sdim$_m$ hold.*

*It is impossible that $B$ has a non-empty intersection with a linear cluster $A_f^m$.*

*Proof.* Suppose for contradiction that $(B \cap \ell) - \mathfrak{A}^{m-1} \neq \emptyset$, witnessed by $b$ for some $\ell$ such that $A_f^m \cap \ell \neq \emptyset$ and $\ell \cap \mathfrak{A}^{m-1} = B_f^m = B'$, the extended base of $b/\mathfrak{A}^{m-1}$.

Step 1: We show $B$ contains a single point $b$ from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. By Lemma 5.4.2, there exist $x_1, x_2 \in A_{j,1}^{m+1}$ with $R(x_1, x_2, b)$. Since $\ell_j^{m+1} + 1 < \mu_j^{m+1}$, there are (Figure 10) at least two disjoint embeddings $C^0$ and $C^1$ over $B$ of $A_{j,i}^{m+1}$ into $\mathfrak{A}^m$; the image $C^i$ must contain copies $c_1^i$ and $c_2^i$ of $x_1$ and $x_2$, which satisfy $R(c_1^i, c_2^i, b)$

for $i < 2$ and are disjoint from $B$. Without relying on the inductive hypotheses, the proof of Lemma 5.11.3.a shows that if the $G$-invariant $B$ intersects a linear cluster $A_f^m$, $B \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ contains $A_f^m$. Since the $*$-petals are freely joined, all the $c_j^i$ are in $\mathfrak{A}^{m-1}$. So they must be in $\ell$ since any element in $\mathfrak{A}^{m-1}$ related to $b$ is in $\ell$. And $b$ is on a line with at least five elements[26] based in $\mathfrak{A}^{m-1}$.

In fact, $A_f^m = (\ell \cap \mathfrak{A}^m) - \mathfrak{A}^{m-1} \subseteq B$ must be a singleton. As, if $b'$ is a second point in $A_f^m$, $(\{b, b'\}, c_1^i)$ realizes $\boldsymbol{\alpha}$ with the base contained in $B$. But this is a contradiction, because $A_{j,1}^{m+1}$ is a non-linear petal based on $B$ and $C^i$ is isomorphic to $A_{j,1}^{m+1}$ over $B$.

Step 2: Having shown $B$ contains a single point $b$ from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ there are two cases. In the first case suppose this $b$ and so its extended base $B'$ are $G$-invariant. Since $\{b\}$ is $G$-invariant but not safe, this contradicts Steiner-dim$_m$ or Steiner-sdim$_m$ depending on $G = G_i$ or $G = G_{\{I\}}$.

We are left with the case that $A_f^m$ is a singleton but not $G$-invariant, i.e. there exists $g \in G$ such that $g(A_f^m) \neq A_f^m$, but $|A_f^m| = 1$. Let $b = b_0, b_1, \dots b_{k-1}$ enumerate the orbit of $b$ under $G$. Then, for $u < k$ there is a $g_u \in G$ satisfying $R(g_u(x_1), g_u(x_2), g_u(b))$ and $g_u(x_1), g_u(x_2) \in A_{j,1}^{m+1}$, $g_u(b) \in B$ because both $A_{j,1}^{m+1}$ and $B$ are $G$-invariant. Let $\langle C^i : i < \nu = \mu_j^{m+1} - 1 \rangle$ enumerate the copies of $A_{j,1}^{m+1}$ in $\mathfrak{A}^m$. Now, as in step 1, for each $b_u, C^i$, there are elements $d_1^{i,u}, d_2^{i,u} \in C^i \cap \mathfrak{A}^{m-1}$ satisfying $R(d_1^{i,u}, d_2^{i,u}, b_u)$. Again as in step 1, all the $C^i \subseteq \mathfrak{A}^{m-1}$ and for each $u$ all the $d_w^{i,v}$ for $w < 2, i < \nu$ are on the same line. Now we consider the substructure $\overline{C} = B \cup \bigcup_{v < \nu} C^v$. If $C^1$ and $C^2$ are freely joined over $B$, $\delta(C^1 \cup C^2 / B) = 0$. For each fixed $b_u$ we have one new line $\ell_u$ with at least five points on it and the nullity of $\ell \cap (C^1 \cup C^2)$ is $4 - 2 = 2$. As no points are added this reduces $\delta(C^1 \cup C^2 / B)$ by 1. (One line of length 3 in $BC^2$ in the computation of $\delta(C^1 C^2 / B)$ has been replaced by two points added to $\ell_u$.) Each additional $C^i$ decrements another 1 so with respect to $b_u \in B \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ the line $\ell_u$ reduces $\delta(\overline{C}/B)$ by $(\nu - 1)$. But there are $k$ such $b_u$ and $\nu = \mu_{j,1}^{m+1} - 1$, so $\delta(\overline{C}/B) \leq k(1 - \nu) = k(1 - (\mu_{j,1}^{m+1} - 1)) = k(2 - \mu_j^{m+1})$. Hence,

$$\delta(\overline{C} \cup B) - \delta(B) = \delta(\overline{C}/B) \leq 2k - 2\mu_j^{m+1} \leq 2k - k\delta(B).$$

Consider the first and last terms and move $\delta(B)$ and $\delta(\overline{C} \cup B)$ to the opposite sides of the inequality; then divide by $k - 1$ to get

$$\delta(B) \leq \frac{2k - \delta(\overline{C} \cup B)}{k - 1} \leq \frac{2k - 2}{k - 1} = 2.$$

Recall, that $B$ is safe, so $2 \leq d(B) \leq \delta(B) \leq \delta(\overline{C} \cup B)$. This justifies the second inequality. Thus, $d(B) = \delta(B) = 2$ and $B \leq M$ and all 0-primitive extensions of $B$ must be independent; this contradicts the existence of the lines $\ell_u$.  ∎

Lemma 3.21 1) and 3) concern only non-linear petals and so goes through without changes. However, a small new argument is needed for part 2).

**Lemma 5.15.** *Assume that $A_{j,1}^{m+1}$ is $G$-invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$. Then, for any $d$ with $1 \leqslant d \leqslant \nu = \nu_j^{m+1}$:*

*A  For any $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$, $C^d \cap A_{i,f}^m \neq \emptyset$, i.e., $C_-^d \neq \emptyset$.*

---

[26]Note that this situation is impossible unless $\mu(\boldsymbol{\alpha}) \geqslant 3$.

$B$  $\delta(B_-/B_+ \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d) = \delta(B_-/B_+)$.

$C$  If $C^d \cap \mathfrak{A}^{m-1} = \emptyset$, that is $C_+^d = \emptyset$, then there is a unique petal $A_{i,f}^m$ that contains both $C^d$ and $B_-$. So, $A_{i,f}^m$ is $G$-invariant.

*Proof.* A) As in Lemma 3.21.1, for each $f, i$ for each $d$, $A_{i,f}^m \cap B \neq \emptyset$ implies $C^d \cap A_{i,f}^m \neq \emptyset$. For B) note that if $\delta(B_-/B_+ \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d) = \delta(B_-/B_+)$ fails it is because there is a line $\ell$ with $|\ell| \geqslant 3$ intersecting $B_-$ and $B_+ \cup \bigcup_{1 \leqslant d \leqslant \nu} C_+^d$ with at most one point in $B_+$. If $|\ell \cap B| = 2$, then each $C^d$ is a linear petal. Since $A_{j,1}^{m+1} \cong_B C^d$, $A_{j,1}^{m+1}$ is also linear; contradiction. Then $|\ell \cap B| = |\ell \cap B_-| = 1$ and $\ell$ is based in $\mathfrak{A}^{m-1}$. Let $\{b\} = \ell \cap B_-$. Then $b \in A_{i,f}^m$ for some $i$ and $f$, and $\{b\}$ is a linear petal over $\mathfrak{A}^{m-1}$, contradicting the hypothesis that $|A_{i,f}^m| > 1$ for each $i$, $f$ such that $A_{i,f}^m \cap B \neq \emptyset$. C) follows as in Lemma 3.21.3.  ∎

From Lemma 5.14, we know that if $B_j^{m+1}$ is the base of a $G$-invariant $A_{j,1}^{m+1}$, $B_j^{m+1}$ intersects only non-linear petals. Lemma 3.23 relies on '$\mu$-triples' but involves only calculations justified by the axiomatic properties of $\delta$, so we can apply it here to conclude:

**Corollary 5.16.** *Fix a decomposition of a $G$-normal set $\mathfrak{A}$, where $G \in \{G_I, G_{\{I\}}\}$. Suppose a non-linear petal $A_{j,1}^{m+1}$ is $G$-invariant. Assume $\mu$ triples, (so $\mu_j^{m+1}(A/B) \geqslant 3$ when $B$ is not a singleton).*

*(1)  Let $G = G_I$ and Steiner-dim$_m$ hold; or*
*(2)  let $G = G_{\{I\}}$ and Steiner-sdim$_m$ hold.*

*Then, $A_{j,1}^{m+1}$ Steiner-determines a non-linear petal $A_{i,1}^m$.*

We restate and prove Theorem 5.2 using essentially the same induction as in Section 4.1; the difference is that Lemma 5.11 makes the treatment of $\alpha$-petals easier while we apply Corollary 5.16 for determinacy of non-linear petals.

**Theorem 5.17** (no definable truly $n$-ary function). *Suppose $T_\mu^S$ is a Steiner-system as in Definition 0.1. Assume $\mu$ triples. Let $I$ be a finite independent set that contains at least 2 elements. Fix a $G$-normal $\mathfrak{A} \leqslant M \models \hat{T}_\mu$ with height $m_0$.*

*Then for every $m \leqslant m_0$, $\mathfrak{A}^m \cap \mathrm{dcl}^*(I) = \emptyset$.*

*Thus, $\mathrm{dcl}^*(I) \cap \mathfrak{A} = \emptyset$; so there is no $n$-ary $\emptyset$-definable function for $n \geq 2$ and $\hat{T}_\mu$ does not admit elimination of imaginaries.*

*As a corollary, we obtain that $\mathrm{dcl}(J) = \bigcup_{a \in J} \mathrm{dcl}(a)$ for any independent set $J$.*

As in Section 4.1, the decomposition for Theorem 5.17 is with respect to $G_I$.

*Proof.* We show Steiner-moves$_m$ and Steiner-dim$_m$ jointly imply Steiner-moves$_{m+1}$. Suppose for contradiction that $A_{j,1}^{m+1}$ is $G_I$-invariant. By Lemma 5.11.3, we can assume $A_{j,1}^{m+1}$ is non-linear. Then Corollary 5.16 implies that $A_{j,1}^{m+1}$ Steiner-determines a non-linear petal $A_{i,1}^m$; but this contradicts Steiner-moves$_m$.

Fix $m$ with $1 \leqslant m \leqslant m_0$. Since Lemma 4.1.4 uses only the notions depending on abstract properties of the $\delta$-function: Steiner-moves$_{m+1}$ and Steiner-dim$_m$ imply Steiner-dim$_{m+1}$. Thus by induction as in Section 4.1, we have Steiner-moves$_m$ for all $m \leqslant m_0$ and finish.  ∎

Before attacking the symmetric function case in general, we prove the Steiner version of Claim 4.4.12, describing the implications of the existence of a $G_{\{I\}}$-invariant $\alpha$-petal.

**Lemma 5.18.** *Let $T_\mu^S$ be a strongly minimal Steiner-system as described in Definition 0.1 with $\mu(\boldsymbol{\alpha}) > 1$. Suppose $\mathrm{sdim}_m$, $|A_{j,1}^{m+1}| = 1$ and $A_{j,1}^{m+1}$ is $G_{\{I\}}$-invariant. Then*

    (1) *$A_{j,1}^{m+1}$ determines some non-linear $A_{i,1}^m$; $B = B_j^{m+1} \leqslant M$, $B$ and each $C^k = C^{m+1,k}$ is contained in $A_{i,1}^m$.*

    (2) *Moreover, $\mu_j^{m+1} \geqslant 3$;*

    (3) *and $\mu_i^m \geqslant 3$.*

*Proof.* Lemma 5.11.2 shows $A_{j,1}^{m+1}$ determines some non-linear $A_{i,1}^m$. The analog of Lemma 4.4.12.1, showing each $C^{m+1,q} \subseteq A_{i,1}^m \cup B_i^m$, has both a shorter proof and is stronger. Let $B = B_j^{m+1}$ be the extended base of $A_{j,1}^{m+1}$. By Lemma 5.9 $B \cap \mathfrak{A}^{m-1}$ contains at most one element, but since this set is $G_{\{I\}}$-invariant and a one-element set cannot be safe, Steiner-$\mathrm{sdim}_m$ implies that $B \cap \mathfrak{A}^{m-1} = \emptyset$. Also Steiner-$\mathrm{sdim}_m$ implies $B \leqslant M$. So, each $C^{m+1,q} \subseteq B \subseteq A_{i,1}^m$. 2) and 3) now follow exactly by the argument in Claim 4.4.12. ∎

As in Section 4.4, we now drop the $\mu$-triples requirement and still show there is no *symmetric* definable function.

**Theorem 5.19** (no definable symmetric function). *If $T_\mu^S$ is a Steiner-system as in Definition 0.1, then there is no symmetric $\emptyset$-definable $v$-ary function for $v \geq 2$, i.e., $\mathrm{sdcl}^*(I) = \emptyset$ for any $v$-element independent set $I$.*

*That is, there is no definable function of $v$ variables whose value does not depend on the order of the arguments. Thus, $\hat{T}_\mu$ does not admit elimination of imaginaries.*

*As a corollary, we obtain that $\mathrm{sdcl}(J) = \bigcup_{a \in J} \mathrm{sdcl}(a)$ for any independent set $J$.*

*Proof.* We break the proof from Section 4.4 into several sections and indicate changes necessary for Steiner systems.

First, note Claim 4.4.4 obviously works for Steiner systems, i.e. Steiner-$\mathrm{sdim}_0$ holds. The inductive proof of safety of $A_{j,1}^{m+1}$ from $\mathrm{sdim}_m$, Claim 4.4.5, follows for non-$G_{\{I\}}$-invariant non-linear petals (or $\alpha$-petals) from $\delta$-calculations as in Lemma 4.1.4.

We now fix on a $G_{\{I\}}$-invariant $A_{j,1}^{m+1}$ that determines $G_{\{I\}}$-invariant $A_{i,1}^m$. Our aim is to prove $A_{j,1}^{m+1}$ is safe. They respectively have $\mu_j^{m+1}$ and $\mu_i^m$ realizations in $M$. Results 4.4.7 through 4.4.11 establish the result when $\mu_i^m = 2$. This can only happen when $A_{j,1}^{m+1}$ is non-linear by Lemma 5.18, which is the analog of Lemma 4.4.12. These results are properties of automorphisms of finite structures and hold for the same reasons as in Subsection 4.4.

As in Lemma 4.4.10 we have reduced to the case that $\mu_j^{m+1} \geqslant 3$. But with Lemmas 5.16 and 5.18, while $A_{j,1}^{m+1}$ may be linear, every element of the sequence it determines is non-linear. Moreover, if $A_{j,1}^{m+1}$ is linear, $B_j^{m+1} \subseteq A_{i,1}^m$.

The analogs of Lemmas 4.4.14 through 4.4.20 complete the proof when $\mu_i^m = 2$. They go through in the Steiner case with little change. (Lemma 5.18 includes for Steiner systems the more difficult conclusion in Lemma 4.4.16.)

This leaves us with the analog of Lemmas 4.4.21 to 4.4.25, which formulate and carry out the complicated double induction. But again, one can check that the arguments go through with minor modifications. ∎

## 6. Further work

We worked throughout in this paper in a vocabulary with a single ternary relation symbol. We now explain a conjectured sufficient condition for the elimination of imaginaries in arbitrary finite and infinite vocabularies, using Hrushovski's $\delta$ and definition of $\boldsymbol{K}_0$.

In [Ver02], the second author constructed a variant of Hrushovski's example with elimination of imaginaries. The idea is that for each $n \geq 3$ we add an $n$-ary relation $R_n$ and put $\mu(\{a_1\}/\{a_2, \ldots, a_n\}) = 1$, where the tuple $(a_1, a_2, \ldots, a_n)$ satisfies $R_n$. This gives us an $(n-1)$-ary symmetric function. Thus we can construct *in an infinite vocabulary* a Hrushovski strongly minimal set which has elimination of imaginaries. The conjecture is that in some sense it is the only way to get a symmetric function in Hrushovski's examples. Recall that the constraint $\mu(A/B) \geq \delta(B)$ has a crucial role in proving the amalgamation property. However, as it was shown in [Ver06], for good pairs $(A/B)$ satisfying $r(\{a\}, A \cup B, \{b\}) > 0$ for each $a \in A$ and $b \in B$ we may put $\mu(A/B)$ equal to any positive number while preserving the amalgamation property. A slight modification should construct definable truly $n$-ary functions. So, the exact formulation of the conjecture is the following:

**Conjecture 6.1.** *We take the class $\boldsymbol{L}_0$ to be all finite $\tau$-structures that satisfy the hereditarily positive $\epsilon$ dimension defined in Axiom 1.1.1.2. Assume that there is a natural number $N$, such that $\mu(A/B) \geq \delta(B)$ for any good pair $(A/B)$ with $\delta(B) \geq N$; then $\mathrm{sdcl}^*(I) = \emptyset$ for any independent set $I$ with $|I| \geq \max\{N, 5\}$. Thus, no Hrushovski construction in a finite relational vocabulary $\tau$ (that is, where $\boldsymbol{K}_0$ contains all finite $\tau$-structures) has elimination of imaginaries.*

More generally, one might ask

**QUESTION 6.2.** *No strongly minimal set in a finite vocabulary with a strictly flat* acl-*geometry admits elimination of imaginaries.*

[Mer20] makes a step in this direction by representing each strictly flat geometry by a Hrushovski construction. However, he deals only with the $\omega$-stable case and takes $\boldsymbol{L}_0^*$ as all finite structures of a given relational vocabulary. So this question has to be made more precise. The methods here must be extended as examples in [Bal23, Bal22] show that with two ternary relations one can define global ternary functions. Moreover, the binary function can be commutative[27]. *This contradiction with the conclusion of Theorem 5.2.3 is explained because we have widened our method of construction to include amalgamation classes of finite structures, which are defined by $\forall\exists$-formulas.*

**QUESTION 6.3** (Flat Geometries). *We have provided several properties distinguishing among strongly minimal theories with flat geometries and provided some examples. Three directions of inquiry are:*

   *(1) Are there further useful distinction among the theories of flat* acl-*geometries?*
   *(2) Are there further useful syntactic distinctions among the theories themselves?*
   *(3) Are there further applications in combinatorics using the methods developed here?*

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[27]E.g.,we built built the strongly minimal set from a commutative variety of block algebras over $F_5$ ([GW80, p 7]) by the method of [Bal22, §3]. These examples fall into Remark 0.3.(2).(b).(ii).

Linear spaces and quasi-groups are only a glimpse at the structures that can obtained when we remove the restriction that we are imposing the dimension function on *all* finite structures for a given vocabulary. Moreover as exemplified in [Bal22], new phenomena are obtained by varying $\mu$. In a different direction, one can ask whether these methods might be useful higher in the complexity classification.

**QUESTION 6.4** (Model Theoretically Complex Classes)**.** [Bal23, Bal22], constructs strongly minimal quasigroups using the graph of the quasigroup operation as in the study of model complete Steiner triple system of Barbina and Casanovas [BC19]. As noted in Remark 5.27 of [BP21], their generic structure $M$ differs radically from ours: $\mathrm{acl}_M(X) = \mathrm{dcl}_M(X)$.

*Do the strongly minimal quasigroups in last paragraph satisfy elimination of imaginaries? Is it possible to develop a theory of q-block algebras for arbitrary prime powers similar to that for Steiner quasigroups in their paper? That is, to find a model completion for each of the various varieties of quasigroups discussed in* [Bal23]. *Where do the resulting theories lie in the stability classification?* [HW21] *prove the existence of generics for classes of Steiner systems omitting certain finite Steiner Systems. Are there strongly minimal theories for such classes?*

## REFERENCES

[Bai96] B. Baizhanov. O sushchestvennosti o-minimalnogo obogashcheniya. In *Proceedings of Informatics and Control Problems Institute*, pages 71–76. 1996. in Russian: (English title: On essentiality of an o-minimal expansion).

[Bal73] John T. Baldwin. The number of automorphisms of models of $\aleph_1$-categorical theories. *Fundamenta Mathematicae*, 93:1–6, 1973.

[Bal95] John T. Baldwin. Some projective planes of Lenz Barlotti class I. *Proceedings of the A.M.S.*, 123:251–256, 1995.

[Bal22] John T. Baldwin. Strongly minimal Steiner Systems III: Path Graphs and Sparse configurations. submitted, 2022.

[Bal23] John T. Baldwin. Strongly minimal Steiner Systems II: Coordinatizaton and Strongly Minimal Quasigroups. *Algebra Universalis*, 84, 2023. Math arXiv:2106.13704; `https://doi.org/10.1007/s00012-023-00812-w`.

[BC19] Silvia Barbina and Enrique Casanovas. Model theory of Steiner triple systems. *Journal of Mathematical Logic*, 20, 2019. `https://doi.org/10.1142/S0219061320500105`.

[BL71] John T. Baldwin and A.H. Lachlan. On strongly minimal sets. *Journal of Symbolic Logic*, 36:79–96, 1971.

[BP21] John T. Baldwin and G. Paolini. Strongly Minimal Steiner Systems I. *Journal of Symbolic Logic*, 86:1486–1507, 2021. published online Oct 22, 2020 `arXiv:1903.03541`.

[BS96] John T. Baldwin and Niandong Shi. Stable generic structures. *Annals of Pure and Applied Logic*, 79:1–35, 1996.

[CF04] E. Casanovas and R. Farré. Weak forms of elimination of imaginaries. *Mathematical Logic Quarterly*, 50:126–140, 2004.

[CGGW10] K. M. Chicot, M. J. Grannell, T. S. Griggs, and B. S. Webb. On sparse countably infinite Steiner triple systems. *J. Combin. Des.*, 18(2):115–122, 2010.

[CW12] P. J. Cameron and B. S. Webb. Perfect countably infinite Steiner triple systems. *Australas. J. Combin.*, 54:273–278, 2012.

[EF12] David M. Evans and Marco S. Ferreira. The geometry of Hrushovski constructions, II. the strongly minimal case. *J. Symbolic Logic*, 77(1):337–349, 2012.

[Fuj06] Yuichiro Fujiwara. Infinite classes of anti-mitre and 5-sparse Steiner triple systems. *J. Combin. Des.*, 14(3):237–250, 2006.

[Grä79] George Grätzer. *Universal Algebra*. Springer-Verlag, 1979.

[GW75] Bernhard Ganter and Heinrich Werner. Equational classes of Steiner systems. *Algebra Universalis*, 5:125–140, 1975.

[GW80]    Bernhard Ganter and Heinrich Werner. Co-ordinatizing Steiner systems. In C.C. Lind-
          ner and A. Rosa, editors, *Topics on Steiner Systems*, pages 3–24. North Holland,
          1980.
[Hol99]   Kitty Holland. Model completeness of the new strongly minimal sets. *The Journal of
          Symbolic Logic*, 64:946–962, 1999.
[Hru93]   E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–
          166, 1993.
[HW21]    D. Horsley and B. Webb. Countable homogeneous Steiner triple systems avoiding
          specified subsystems. *Journal of Combinatorial Theory, Series A*, 180, 2021. `https:`
          `//www.sciencedirect.com/science/article/pii/S0097316521000339`.
[Mer20]   M. Mermelstein. The generic flat pregeometry. `https://arxiv.org/abs/2001.00609`,
          2020.
[MT19]    I. Muller and K. Tent. Building-like geometries of finite morley rank. *J. Eur. Math.
          Soc.*, 21:3739–3757, 2019. DOI: 10.4171/JEMS/912.
[Pao20]   Gianluca Paolini. New $\omega$-stable planes. *Reports on Mathematical Logic*, 55:87–111,
          2020.
[Pil99]   Anand Pillay. Model theory of algebraically closed fields. In E. Bouscaren, editor,
          *Model Theory and Algebraic Geometry : An Introduction to E. Hrushovski's Proof
          of the Geometric Mordell-Lang Conjecture*, pages 61–834. Springer-Verlag, 1999.
[Poi83]   Bruno Poizat. Une théorie de Galois imaginaire. *Journal of Symbolic Logic*, pages
          1151–1170, 1983.
[Poi85]   Bruno Poizat. *Cours de Théories des Modèles*. Nur Al-mantiq Wal-ma'rifah, 82, Rue
          Racine 69100 Villeurbanne France, 1985.
[She78]   S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-
          Holland, 1978.
[Ste56]   Sherman K. Stein. Foundations of quasigroups. *Proc. Nat. Acad. Sci.*, 42:545–546,
          1956.
[Tsu93]   Akito Tsuboi. Algebraic types and automorphism groups. *The Journal of Symbolic
          Logic*, 58:232–239, 1993.
[Ver02]   Viktor Verbovskiy. *Properties of functions definable in structures with conditions of
          minimality of families of definable subsets*. PhD thesis, Novosibirsk State University,
          Pirogov St., 1, Academgorodok, Novosibirsk, Russia, 5 2002. in Russian.
[Ver06]   V. Verbovskiy. On the elimination of imaginaries for the strongly minimal sets of
          Hrushovski. In M.M.Erimbetov, editor, *Model Theory in Kazakhstan*, pages 41–55.
          Eco Study, Almaty, 2006.
[Yon22]   I. Yoneda. Pillay's alternative definition for weak elimination of imaginaries coincides
          with poizat's original one. In *RIMS kokyuroku (2018)*, pages 59–63. 2022.
[Yon23]   I. Yoneda. On weak elimination of imaginaries, an appendix of casanova and farre's
          paper and basic examples. In *RIMS Kokyuroku (2023)*. 2023.
[Zie13]   Martin Ziegler. An exposition of Hrushovski's new strongly minimal set. *Ann. Pure
          Appl. Logic*, 164(12):1507–1519, 2013.