Fine Classsification of Strongly minimal sets Logic Colloquium 2021 Poznan

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Strongly Minimal Theories

STRONGLY MINIMAL

Definition

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e.g. acf, vector spaces, successor

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a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Theorem

If T is strongly minimal algebraic closure defines matroid/combinatorial geometry.

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. 'bi-interpretable' with an algebraically closed field (non-locally modular)

Hrushovski's example showed there are non-locally modular examples which are far from being fields; the examples don't even admit a group structure.

The diversity of flat strongly minimal sets

The ‘Hrushovski construction’ actually has 5 parameters:

Describing Hrushovski constructions

1. σ: vocabulary
2. $\boldsymbol{L}_0$: A $\forall\exists$ collection of finite σ-structures
3. ϵ: A submodular (hence flat) function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$.
4. $\boldsymbol{L}_0$: $\boldsymbol{L}_0^*$ defined using ϵ.
5. μ: a function bounding the number of 0-primitive extensions of an $A \in \boldsymbol{L}_0$ are in L_μ.

To organize the classification of the theories each choice of a class **U** of μ yields a collection of T_μ with similar properties.

Hrushovki's basic construction

Example

1. σ has a single ternary relation R;
2. $\boldsymbol{L}_0^*$: All finite σ-structures
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.

Group Action and Definable Closure

Fix I as two independent points in the generic model M of T_μ.

2 groups

Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

Definition

1. $\mathrm{dcl}^*(I)$ consists of those elements that are fixed by G_I but not by G_X for any $X \subsetneq I$.
2. The *symmetric definable closure* of I, $\mathrm{sdcl}^*(\mathrm{I})$, consists of those elements that are fixed by $G_{\{I\}}$ but not by $G_{\{X\}}$ for any $X \subsetneq I$.

$\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ implies T does not admit elimination of imaginaries.

The main result: Classifying dcl [BV21]

Theorem

Let T_μ be a strongly minimal theory as in Hrushovski's original paper. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B\})$. Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

- G_I If T_μ triples then $\mathrm{dcl}^*(I) = \emptyset$
  $\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$
  and every definable function is essentially unary (Definition 15).
- $G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$
  $\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$
  and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly v-ary function.

Consequently, in both cases T_μ does not admit elimination of imaginaries. Nevertheless the algebraic closure geometry is not disintegrated.

The General Construction

Amalgamation and Generic model

We study classes $\boldsymbol{K}_0$ of finite structures A
with $\delta(A') \geq 0$, for every $A' \subset A$.
$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

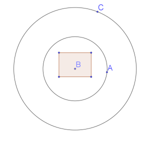
$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

α is the isomorphism type of $(\{a, b\}, \{c\})$,

Overview of construction

Realization of good pairs

1. A good pair C/B *well-placed* by $\mathcal{A}$ in a model M, if $B \subseteq A \leq M$ and C is 0-primitive over X.
2. For any good pair (C/B), $\chi_M(C/B)$ is the maximal number of disjoint copies of C over B appearing in M.
3. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(C/B) \leq \mu(C/B)$ for every good pair (C/B).

If C/B is well-placed by $\mathcal{A} \leq M$, $\chi_M(C/B) = \mu(C/B)$

The structure of $\mathrm{acl}(X)$

Finite Coding

Definition

A finite set $F = \{\overline{a}_1, \ldots, \overline{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples F is coded by some set S over $\emptyset$.

If $\mathrm{dcl}^*(I) = \emptyset$, T does not have the finite set property.

dcl^* and elimination of imaginaries

Fact: Elimination of imaginaries

A theory T admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no

Fact

If T admits weak elimination of imaginaries then T satisfies the finite set property if and only T admits elimination of imaginaries.

Since every strongly minimal theory weak elimination of imaginaries.

If a strongly minimal T has only essentially unary definable binary functions it does not admit elimination of imaginaries.

$\mathrm{dcl}^*(I) = \emptyset$ implies no elimination of imaginaries:

Lemma

Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and M is a generic model of a strongly minimal theory.

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then I is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter free definable binary function.*

‘Non-trivial definable functions’

Definition

Let T be a strongly minimal theory. function $f(x_0 \ldots x_{n-1})$ is called *essentially unary* if there is an $\emptyset$-definable function $g(u)$ such that for some i, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.

G-decomposable sets

Definition

$\mathcal{A} \subseteq M$ is G-decomposable if

1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is G-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

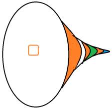
Fact

There are G-decomposable sets.
Namely for any finite U with $d(U/I) = 0$,

$$\mathcal{A} = \mathrm{icl}(\mathrm{I} \cup \mathrm{G}(\mathrm{U}))$$

Constructing a *G*-decomposition

Linear Decomposition



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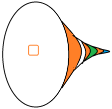
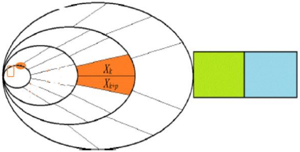


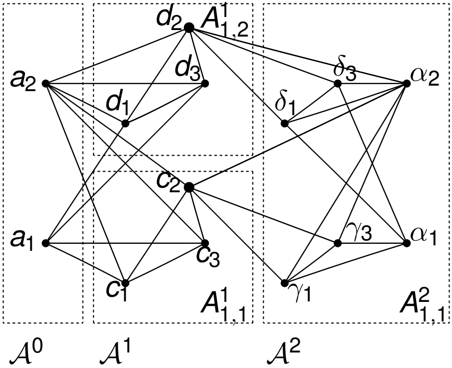
Tree Decomposition



Prove by induction on levels that $\mathrm{dcl}^*(I) = \emptyset$. $(\mathrm{sdcl}^*(\mathrm{I}) = \emptyset)$

A non-trivial definable binary function

In the diagrams, we represent a triple satisfying R by a triangle.



Conclusion

Strongly minimal theories with non-locally modular algebraic closure

1. Diversity
    1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
    2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
    3. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
    4. 2-ample but not 3-ample sm sets (not flat) [MT19]
    5. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)

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2. Classifying
   1. discrete
   2. non-trivial but no binary function
   3. non-trivial but no commutative binary function
   4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]

Combinatorial connections

Unlike many construction in infinite combinatorics these methods give a family of infinite structures with similar properties. Among the properties investigated are:

1. cycle graphs in 3-Steiner systems [CW12] generalized to paths in Steiner k-system; Omitting or demanding finite cycles.
2. preventing or demanding 2-transitivity
3. controlling the lengths of chains.
4. sparse Steiner systems: forbidding specific configurations [CGGW10]

References I

John T. Baldwin.
Some projective planes of Lenz Barlotti class I.
Proceedings of the A.M.S., 123:251–256, 1995.

John T. Baldwin and V. Verbovskiy.
Towards a finer classification of strongly minimal sets.
preprint: Math Arxiv:2106.15567, 2021.

K. M. Chicot, M. J. Grannell, T. S. Griggs, and B. S. Webb.
On sparse countably infinite Steiner triple systems.
J. Combin. Des., 18(2):115–122, 2010.

P. J. Cameron and B. S. Webb.
Perfect countably infinite Steiner triple systems.
Australas. J. Combin., 54:273–278, 2012.

References II

I. Muller and K. Tent.
Building-like geometries of finite morley rank.
J. Eur. Math. Soc., 21:3739–3757, 2019.
DOI: 10.4171/JEMS/912.