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BEYOND FIRST ORDER LOGIC: FROM NUMBER OF STRUCTURES TO STRUCTURE OF NUMBERS PART I

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Abstract. The paper studies the history and recent developments in non-elementary model theory focusing in the framework of *abstract elementary classes*. We discuss the role of syntax and semantics and the motivation to generalize first order model theory to non-elementary frameworks and illuminate the study with concrete examples of classes of models.

This first part introduces the main conceps and philosophies and discusses two research questions, namely categoricity transfer and the stability classification.

1. Introduction

Model theory studies classes of structures. These classes are usually a collection of structures that satisfy an (often complete) set of sentences of first order logic. Such sentences are created by closing a family of basic relations under finite conjunction, negation and quantification over individuals. *Non-elementary logic* enlarges the collection of sentences by allowing longer conjunctions and some additional kinds of quantification. In this paper we first describe for the general mathematician the history, key questions, and motivations for the study of non-elementary logics and distinguish it from first order model theory. We give more detailed examples accessible to model theorists of all sorts. We conclude with

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questions about countable models which require only a basic background in logic.

For the last 50 years most research in model theory has focused on first order logic. Motivated both by intrinsic interest and the ability to better describe certain key mathematical structures (e.g. the complex numbers with exponentiation), there has recently been a revival of 'non-elementary model theory'. We develop contrasts between first order and non-elementary logic in a more detailed way than just noting 'failure of compactness'. We explain the sense in which we use the words syntax and semantics in Section 2. Many of the results and concepts in this paper will reflect a tension between these two viewpoints. In part II, as we move from the study of classes that are defined syntactically to those that are defined semantically, we will be searching for a replacement for the fundamental notion of first order model theory, i.e. the notion of a complete theory. Section 2 also defines the basic notions of non-elementary model theory. Section 3 describes some of the research streams in more detail and illuminates some of the distinctions between elementary and non-elementary model theory. Subsection 3.1 describes the founding result of modern first order model theory, Morley's categoricity theorem, and sketches Shelah's generalization of it to $L_{\omega_1,\omega}$. In part II we study several generalizations of the result to in Abstract Elementary Classes (AEC). The remainder of Section 3 studies the so called stability classification and provides specific mathematical examples that illustrate some key model theoretic notions. We describe concrete examples explaining the concepts and problems in non-elementary model theory and a few showing connections with other parts of mathematics. Two of these illustrate the phrase 'to structure of numbers' in the title. Example 3.2.4, initiated by Zilber, uses infinitary methods to study complex exponentiation and covers of Abelian varieties. Example in section 2.3 of Part II studies models of Peano Arithmetic and the notion of elementary end-extension. This is the first study of models of Peano arithmetic as an AEC. Furthermore, part II contains new results and explores the proper analogy to complete theory for AECs; it answers a question asked by David Kueker and includes Kossak's example of a class of models of PA interesting from the standpoint of AEC.

Neither of the standard approaches, $L_{\kappa,\omega}$-definable class or AECs, has been successful in studying the countable models of an infinitary sentence. The first approach is too specific. It rapidly reduces to a complete infinitary sentence which has only one countable model. Results so far in studying general AECs give little information about countable models. We seek to find additional conditions on an AEC that lead to a fruitful study of the class of countable models. In particular we would like to find tools for dealing with one famous and one not so famous problem of model theory. The famous problem is Vaught's conjecture. Can a sentence of $L_{\omega_1,\omega}$ have strictly between $\aleph_0$ and $2^{\aleph_0}$ countable models? The second problem is more specific. What if we add the condition that the class is $\aleph_1$-categorical; can we provide sufficient conditions for having less than $2^{\aleph_0}$ countable models; for actually counting the number of countable models? In Part II we describe two sets of concepts for addressing this issue; unfortunately so far not very successfully. The first is the notion of a simple finitary AEC and the second is an attempt to define a notion of a 'complete AEC', which like a complete first order theory imposes enough uniformity to allow analysis of the models but without trivializing the problem to one model.

One thesis of this paper is that the importance of non-elementary model theory lies not only in widening the scope of applications of model theory but also in shedding light on the essence of the tools, concepts, methods and conventions developed and found useful in elementary model theory.

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2. Non-elementary model theory

In this section we study the history of non-elementary model theory during the second half of the twentieth century and compare that to the development of more 'mainstream' first order model theory. We identify two different trends in the development. In both the 'elementary' and non-elementary cases the focus of research has moved from 'syntactic' consideration towards 'semantic' ones - we will explain what we mean by this. We see some of the cyclic nature of science. Non-elementary classes bloom in the 60's and 70's; the bloom fades for some decades, overshadowed by the success and applications arising from the 'elementary' field. But around the turn of the 21st century, innovative examples and further internal developments lead to a rebirth.

We will focus on some 'motivating questions' that have driven both the elementary and non-elementary approaches, such as the categoricity transfer problem. While counting models seems a rather mundane problem, new innovations and machinery developed for the solution have led to the recognition of systems of invariants that are new to mathematics and in the first order case to significant mathematical advances in e.g. number theory [13]. It is hoped that the deeper developments of infinitary logic will have similar interactions with core mathematics. Boris Zilber's webpage contains many beginnings.

2.1. **Syntax and semantics.** The distinction between *syntax* and *semantics* has been present throughout the history of modern logic starting from the late 19th century: *completeness theorems* build a bridge between the two by asserting that a sentence is provable if and only if it is true in all models. By syntax we refer to the formalism of logic, objects of language as strings of symbols and deductions as manipulations of these strings according to certain rules. Semantics, however, has to do with interpretations, 'meaning' and 'sense' of the language. By the *semantics* for a language we mean a 'truth definition' for the sentences of the language, a description of the conditions when a structure is considered to be a model for that sentence. 'Semantic properties' have to do with properties of such models.

In fact these two notions can also be seen as methodologies or attitudes toward logic. The extreme (formalist) view of the syntactic method avoids reference to any 'actual' mathematical objects or meaning for the statements of the language, considering these to be 'metaphysical objects'. The semantic attitude is that logic arises from the tradition of mathematics. The method invokes a trace of Platonism, a search for the 'truth' of statements with less regard for formal language. The semantic method would endorse 'proof in metamathematics or set theory' while the syntactic method seeks a 'proof in some formal system'. Traditionally model theory is seen as the intersection of these two approaches. Chang and Keisler[17] write: *universal algebra + logic = model theory.* Juliette Kennedy[33] discusses ideas of 'formalism freeness', found in the work of Kurt Gödel. Motivated by issues of incompleteness and faithfulness and hence the 'failure' of first order logic to capture truth and reasoning, Gödel asked if there is some (absolute) concept of proof (or definability) 'by all means imaginable'. One interpretation of this absolute notion (almost certainly not Gödel's) is as the kind of semantic argument described above. We will spell out this contrast in many places below.

Model theory by definition works with the semantic aspect of logic, but the dialectics between the syntactic and semantic attitudes is central. This becomes even clearer when discussing questions arising from *non-elementary model theory*. Non-elementary model theory studies formal languages other than 'elementary' or first order logic; most of them extend first order. We began by declaring that model theory studies classes of models. Traditionally, each class is the collection of models that satisfy some (set of) sentence(s) in a particular logic. Abstract Elementary Classes provide new ways of determining classes: a class of structures in a fixed vocabulary is characterized by semantic properties. The notion of AEC does not designate the models of a collection of sentences in some formal language, although many examples arise from such syntactic descriptions. In first order logic, the most fruitful topic is classes of models of complete theories. A *theory* T is a set of sentences in a given language. We say that T is *complete*, if for every sentence ϕ in the language, either T implies ϕ, or T implies $\neg\phi$. In Part II we seek an analogue to completeness for AEC.

Model-Theoretic Logics, edited by Barwise and Feferman [8], summarizes the early study of non-elementary model theory. In this book, 'abstract model theory' is a study comparing different logics with regard to such properties as interpolation, expansions, relativizations and projections, notions of compactness, Hanf and Löwenheim-Skolem numbers.

A vocabulary[1] L consists of constant symbols, relation symbols and function symbols, which have a prescribed number of arguments (arity). An L-structure consists of a universe, which is a set, and interpretations for the symbols in L. When L' is a subset of a vocabulary L, and M is an L-structure, we can talk about the reduct of M to L', written $M \restriction L'$. Then M is the expansion of $M \restriction L'$ to L. If M and N are two L-structures, we say that M is an L-*substructure* of N if the domain of M is contained in the domain of N and the interpretations of all the symbols in L in M agree with the restriction of N to M.

A *formal language* or logic in the vocabulary L is a collection of formulas that are built by certain rules from the symbols of the vocabulary

[1] Another convention specifies the vocabulary by a small Greek letter and the L with decorations describes the particular logic. What we call a vocabulary is sometimes called a language. We have written language or logic for the collections of sentences; more precisely this might be called the language and the logic would include proof rules and even semantics.

and from some 'logical symbols'. In this paper we focus on countable vocabularies but don't needlessly restrict definitions to this case.

L-terms are formed recursively from variables and the constant and function symbols of the vocabulary by composing in the natural manner. With a given interpretation for the constants and assignment of values for the variables in a structure, each term designates an element in the structure.

An *atomic formula* is an expression $R(t_1, \ldots, t_n)$ where R is an n-ary relation symbol (including equality) of the vocabulary and each t_i is a term.

Definition 2.1.1 (The language $L_{\lambda\kappa}$). *Assume that L is a vocabulary. The language $L_{\lambda\kappa}$ consists of formulas $\phi(\bar{x})$, where the free variables of the formula are contained in the finite sequence $\bar{x}$ and where the formulas are built with the following operations:*

- *$L_{\lambda\kappa}$ contains all atomic formulas in the vocabulary L.*
- *If $\phi(\bar{x})$, $\psi(\bar{x})$ are in $L_{\lambda\kappa}$, then the negation $\neg\phi(\bar{x})$ and implication $(\phi(\bar{x}) \to \psi(\bar{x}))$ are in $L_{\lambda\kappa}$.*
- *If $\phi_i(\bar{x})$ is in $L_{\lambda\kappa}$ for every i in the index set I, and $|I| < \lambda$ the conjunction $\bigwedge_{i\in I} \phi_i(\bar{x})$ and disjunction $\bigvee_{i\in I} \phi_i(\bar{x})$ are in $L_{\lambda\kappa}$.*
- *If $\phi(y_i, \bar{x})$ is in $L_{\lambda\kappa}$ for each i in the well-ordered index set I, and $|I| < \kappa$, then the quantified formula $(Q_i y_i)_{i\in I}\phi(\bar{x})$ is in $L_{\lambda\kappa}$, where each quantifier Q_i is either $\forall$ ('for all y_i') or $\exists$ ('there exists y_i').*

First order logic is the language $L_{\omega\omega}$, i.e. only finite operations are allowed. We define that $L_\infty\kappa$ is the union of all $L_{\lambda\kappa}$ for all cardinal numbers λ.

The languages $L_{\lambda\omega}$ allowing only finite strings of quantifiers are much better behaved. We will later introduce *abstract elementary classes* generalizing, among other things, classes of structures definable with a sentence in $L_{\lambda\omega}$. The definition of the *truth* of a formula in a structure is crucial. For a formula $\phi(\bar{x})$, with the sequence $\bar{x}$ containing all the free variables of ϕ, we define what it means that the formula $\phi(\bar{x})$ is true in an L-structure M with the variables $\bar{x}$ interpreted in a particular way as elements $\bar{a}$, written $M \models \phi(\bar{a})$. The definition is done by induction on the complexity of the formula, following the inductive definition of the formula in Definition 2.1.1.

Definition 2.1.2 (The language $L(Q)$). *The language $L(Q)$ is formed as the first order logic $L_{\omega\omega}$, but we allow also formulas of the form $Qy\phi(y, \bar{x})$*

with the following truth definition: $M \models Qy\phi(y, \bar{a})$ *if there are uncountable many* $b \in M$ *such that* $M \models \phi(b, \bar{a})$*.*

Definition 2.1.3 (Elementary substructure with respect to a fragment)**.** *A subset* $\mathcal{F} \subseteq L$ *is a* fragment *of some formal language* L *if it contains all atomic formulas and is closed under subformulas, substitution of variables with* L*-terms, finite conjunction and disjunction, negation and the quantifiers* $\forall$ *and* $\exists$*, applied finitely many times. For two* L*-structures* M *and* N*, we say that* M *is an* $\mathcal{F}$*-elementary substructure of* N*, written* $M \preccurlyeq_{\mathcal{F}} N$*, if* M *is an* L*-substructure of* N *and for all formulas* $\phi(\bar{x})$ *of* $\mathcal{F}$ *and sequences* $\bar{a}$ *of elements in* M*,* $M \models \phi(\bar{a})$ *if and only if* $N \models \phi(\bar{a})$*.*

Definition 2.1.4 (Elementary class and PC-class)**.** *An* elementary class $\mathbb{K}$ *of* L*-structures is the class of all models of a given theory in first order logic. A* pseudoelementary *(PC) class* $\mathbb{K}$ *is the class of reducts* $M \restriction L$ *of some elementary class in a larger vocabulary* $L' \supseteq L$*.*

We say that a formal language (logic) L is compact if whenever a set of sentences is *inconsistent*, that is, has no model, then there is some finite subset which already is inconsistent. This is a crucial property that, along with the upwards Löwenheim-Skolem property, fails in most non-elementary logics.

The Löwenheim-Skolem number and the Hanf number are defined for a formal logic L (i.e. 'the Löwenheim-Skolem or Hanf number of L'). In the following definitions $\mathbb{K}$ is a class definable with a sentence of L, $\preccurlyeq_{\mathbb{K}}$ is given as the $\mathcal{F}$-elementary substructure relation in some given fragment $\mathcal{F}$ of L, usually the smallest fragment containing the sentence defining $\mathbb{K}$, and the collection $\mathcal{C}$ is the collection of all classes definable with a sentence L.

Definition 2.1.5 (Löwenheim-Skolem number)**.** *The Löwenheim-Skolem number* $\mathrm{LS}(\mathbb{K})$ *for a class of structures* $\mathbb{K}$ *and a relation* $\preccurlyeq_{\mathbb{K}}$ *between the structures is the smallest cardinal number* λ *with the following property: For any* $M \in \mathbb{K}$ *and a subset* $A \subseteq M$ *there is a structure* $N \in \mathbb{K}$ *containing* A *such that* $N \preccurlyeq_{\mathbb{K}} M$ *and* $|N| \leq \max\{\lambda, |A|\}$*.*

Definition 2.1.6 (Hanf Number)**.** *The Hanf number* H *for a collection* $\mathcal{C}$ *of classes of structures is the smallest cardinal number with the property: for any* $\mathbb{K} \in \mathcal{C}$*, if there is* $M \in \mathbb{K}$ *of size at least* H*, then* $\mathbb{K}$ *contains arbitrarily large structures.*

Modern model theory began in the 1950's. Major achievements in the mid 60's and early 70's included Morley's categoricity transfer theorem in

1965 [43] and Shelah's development of stability theory [49]. These works give results on counting the number of isomorphism types of structures in a given cardinality and establishing invariants in order to classify the isomorphism types. Such invariants arise naturally in many concrete classes: the dimension of a vector space or the transcendence degree of an algebraically closed field are prototypical examples. A crucial innovation of model theory is to see how to describe structures by *families* of dimensions. The general theory of dimension appears in e.g. ([49, 45]); it is further developed and applied to valued fields in [23].

Non-elementary model theory thrived in the mid 60's and early 70's. Results such as Lindström theorem in 1969, Barwise's compactness theorem for *admissible fragments* of $L_{\omega_1\omega}$ published in 1969, Mostowski's work on generalized quantifiers in 1957 [44] and Keisler's beautiful axiomatization of $L(Q)$ in [31] gave the impression of a treasury of new formal languages with amenable properties, a possibility to extend the scope of definability and maybe get closer to the study of provability with 'all means imaginable'. However, the general study turned out to be very difficult. For example, the study of the languages $L_{\lambda\kappa}$ got entangled with the set-theoretical properties of the cardinals λ and κ. Since the real numbers are definable as the unique model of a sentence in $L_{2^\omega,\omega}$, the continuum hypothesis would play a major role. But perhaps the study was focused too much on the syntax and trying to study the model theory of *languages*? Why not study the properties of classes of structures, defined semantically. One might replace compactness with, say, closure under unions of chains?

One can argue that a major achievement of non-elementary model theory has been to *isolate* properties that are crucial for classifying structures, properties that might not be visible to a mathematician working with only a specific application or even restricted to the first order case. Excellence (see below) is a crucial example. Some examples of applications of non-elementary model theory to 'general mathematics' are presented in the chapter 'Applications to algebra' by Eklof in [8]. In many of these applications we can see that some class of structures is definable in $L_{\omega_1\omega}$ or in $L_{\infty\omega}$ and then the use the model theory of these languages to, for example, count the number of certain kind of structures or classify them in some other way. Barwise writes in *Model-theoretic logics* [8]:

> Most important in the long run, it seems, is where logic contributes to mathematics by leading to the formation

> of concepts that allow the right questions to be asked and answered. A simple example of this sort stems from 'back-and-forth arguments' and leads to the concept of partially isomorphic structures, which plays such an important role in extended model theory. For example, there is a classical theorem by Erdos, Gillman and Henriksen; two real-closed fields of order type η_1 and cardinality $\aleph_1$ are isomorphic. However, this way of stating the theorem makes it vacuous unless the continuum hypothesis is true, since without this hypothesis there are no fields which satisfy both hypotheses. But if one looks at the proof, there is obviously something going on that is quite independent of the size of the continuum, something that needs a new concept to express. This concept has emerged in the study of logic, first in the work of Ehrenfeucht and Fraïssé in first-order logic, and then coming into its own with the study of infinitary logic. And so in his chapter (in the book [8]), Dickmann shows that the theorem can be reformulated using partial isomorphisms as: Any two real-closed fields of order type η_1, of any cardinality whatsoever, are strongly partially isomorphic. There are similar results on the theory of abelian torsion groups which place Ulm's theorem in its natural setting. ... Extended model theory provides a framework within which to understand existing mathematics and push it forward with new concepts and tools.

One of the foundational discoveries of abstract model theory was Per Lindström's theorem that first order logic is the strongest logic which has both the compactness property and a countable Löwenheim-Skolem number. In order to study such concepts as 'the strongest logic', one has to define the notion of an 'abstract logic'. The book [8] presents the syntax as a crucial part: an abstract logic is a class of sentences with a satisfaction relation between the sentences and the structures, where this relation satisfies certain properties. However, Barwise comments on Lindström's formulation of his theorem [38]:

> To get around the difficulties of saying just what a logic is, they dealt entirely with classes of structures and closure conditions on these classes, thinking of the classes

> definable in some logic. That is, they avoided the problem of formulating a notion of a logic in terms of syntax, semantics, and satisfaction, and dealt purely with their semantic side.

Lindström defined a logic to be a non-empty set of objects called sentences, but the role of these is only to name a class of structures as 'structures modeling one sentence'. Then it is possible to define for example compactness as the property that if a countable intersection of such classes is empty, then already some finite intersection must be empty.

Saharon Shelah built on these insights and introduced Abstract Elementary Classes in [51]. Semantic properties of a class of structures $\mathbb{K}$ and a relation $\preccurlyeq_{\mathbb{K}}$ are prescribed, which are sufficient to isolate interesting classes of structures. But more than just the class is described; the relation $\preccurlyeq$ between the structures in $\mathbb{K}$ provides additional information that, as examples in Subsection 3.2 illustrate, may be crucial.

Definition 2.1.7. *For any vocabulary τ, a class of τ-structures* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* an abstract elementary class *(AEC) if*

1. *Both $\mathbb{K}$ and the binary relation $\preccurlyeq_{\mathbb{K}}$ are closed under isomorphism.*
2. *If $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$, then $\mathcal{A}$ is a substructure of $\mathcal{B}$.*
3. *$\preccurlyeq_{\mathbb{K}}$ is a partial order on $\mathbb{K}$.*
4. *If $\langle \mathcal{A}_i : i < \delta \rangle$ is an $\preccurlyeq_{\mathbb{K}}$-increasing chain:*
   (a) $\bigcup_{i<\delta} \mathcal{A}_i \in \mathbb{K}$;
   (b) *for each $j < \delta$, $\mathcal{A}_j \preccurlyeq_{\mathbb{K}} \bigcup_{i<\delta} \mathcal{A}_i$*
   (c) *if each $\mathcal{A}_i \preccurlyeq_{\mathbb{K}} \mathcal{M} \in \mathbb{K}$, then $\bigcup_{i<\delta} \mathcal{A}_i \preccurlyeq_{\mathbb{K}} \mathcal{M}$.*
5. *If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{C}$, $\mathcal{B} \preccurlyeq_{\mathbb{K}} \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$.*
6. *There is a Löwenheim-Skolem number* $\mathrm{LS}(\mathbb{K})$ *such that if $\mathcal{A} \in \mathbb{K}$ and $B \subset \mathcal{A}$ a subset, there is $\mathcal{A}' \in \mathbb{K}$ such that $B \subset \mathcal{A}' \preccurlyeq_{\mathbb{K}} \mathcal{A}$ and $|\mathcal{A}'| = |B| + \mathrm{LS}(\mathbb{K})$.*

When $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$, we say that $\mathcal{B}$ is an $\mathbb{K}$-extension of $\mathcal{A}$ and $\mathcal{A}$ is an $\mathbb{K}$-submodel of $\mathcal{B}$. If $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $f : \mathcal{A} \to \mathcal{B}$ an embedding such that $f(\mathcal{A}) \preccurlyeq_{\mathbb{K}} \mathcal{B}$, we say that f is a $\mathbb{K}$-*embedding*. Category-theoretic versions of the axioms are studied by Kirby [34], Liebermann [37] and Beke and Rosick [11].

A basic example of an AEC is the class of models defined by some sentence $\phi \in L_{\infty\omega}$, where $\preccurlyeq_{\mathbb{K}}$ is taken as the elementary substructure relation in the smallest fragment of $L_{\infty\omega}$ containing ϕ. Then the Löwenheim-Skolem number is the size of the fragment. An even simpler example is that of an elementary class, where ϕ is a complete theory in first order logic.

A class defined with a sentence in $L_{\omega_1\omega}(Q)$ with the quantifier $Qx\phi(x)$ standing for 'there exists uncountably many x such that $\phi(x)$ holds' can be an AEC. The natural syntactic notion of elementary submodel is inadequate but substitutes are available. Arbitrary psuedoelementary classes are often not AEC. E.g. If $\mathbb{K}$ is the class of all structures A in a language L with a single unary predicate such that $|A| \leq 2^{|U(A)|}$ then $\mathbb{K}$ fails to be an AEC with respect to L-elementary submodel as it is not closed under unions of chains. (See chapter 5 and 4.29 of [4].)

In contemporary first order model theory, the most fundamental concept is the class of models of a complete theory in first order logic. This can be seen as a form of *focusing*; instead of studying different vocabularies, expansions and projections, one fixes one class: the class of differentially closed fields of fixed characteristic (see [41]) or the class of models of 'true' arithmetic. This focus on classes and of properties determining 'similar' classes has become a crucial tool in applications to algebra. The difference from the 'Lindström-style' study of classes of structures is significant: we do not study many classes of structures each corresponding to the 'models of one sentence', but focus on a fixed class, 'models of a theory'. *Abstract elementary classes*, which will be one of the main notions studied in this paper, takes the 'semantic view' to the extreme by eliminating the syntactic definition.

3. Several research lines in non-elementary logic

3.1. Categoricity transfer in $L_{\omega\omega}$ and $L_{\omega_1\omega}$.

Definition 3.1.1 (Categoricity). *Let κ be a cardinal. We say that a class of structures $\mathbb{K}$ is κ-categorical if there is exactly one model of size κ in $\mathbb{K}$, up to isomorphism. A theory T is κ-categorical, if $Mod(T)$, the class of models of T, is κ-categorical.*

The transition to the focus on classes of models begins with Morley's theorem:

Theorem 3.1.2 (Morley's categoricity transfer theorem). *Assume that T is a complete theory in $L_{\omega\omega}$, where L is countable. If there exists an uncountable cardinal κ such that T is κ-categorical, then T is λ-categorical for all uncountable cardinals λ.*

Categoricity transfer will be our first example of a motivating question in the history of model theory. Its proof gave many new tools and concepts that are nowadays contained in every basic course in model theory. Furthermore, both the tools and the theorem itself have been generalized to different frameworks. A categoricity transfer theorem for elementary classes in an uncountable vocabulary was proved by Shelah in [47] (announced in 1970): if the language has cardinality κ and a theory is categorical in some uncountable cardinal greater than κ then it is categorical in all cardinalities greater than κ. This widening of scope led to many tools, such as weakly minimal sets and a greater focus on the properties of individual formulas, that proved fruitful for countable vocabularies. We will look more closely at some of the many extensions of categoricity results to non-elementary classes.

We consider a syntactical *type* in some logic $\mathcal{L}$ as a collection of $\mathcal{L}$-formulas in some finite sequence of variables $\bar{x}$ with *parameters* from a given subset A of a structure M such that an element $\bar{b}$ in an $\mathcal{L}$-elementary extension N of M realizes (simultaneously satisfies) p. If no such sequence exists in a model N, we say that the type is *omitted* in N. In elementary classes, the compactness theorem implies all *finitely consistent* such collections p of formulas really are *realized*. If there is a structure N and a finite sequence $\bar{b} \in N$ such that $M \preccurlyeq N$ and

$$p = \{\phi(\bar{x}, \bar{a}) : \bar{a} \in A \subseteq M, N \models \phi(\bar{b}, \bar{a})\}.$$

then p is called a *complete type* over A for two reasons. Semantically: it gives a complete description of the relation of $\bar{b}$ and A. Syntactically; every formula $\phi(\bar{x}, \bar{a})$ over A or its negation is in p.

An essential concept in Morley's argument is a *saturated structure* M: M is *saturated* if all *consistent* types over parameter sets of size strictly less than $|M|$ are realized in M. Two saturated models of T of size κ are always isomorphic. Morley shows that if T is categorical in some uncountable power, saturated models exist in each infinite cardinality. Then he concludes that if T is *not* categorical in some uncountable power λ, there is a model of power λ which is *not* saturated or even $\aleph_1$-saturated; some type over a countable subset is omitted. But then he shows that if some model of uncountable power λ omits a type over a countable set, then in any other uncountable power κ some model omits the type. Hence, T cannot be categorical in κ either. This method, *saturation transfer*, generalizes to many other frameworks. While proving saturation transfer for elementary classes he introduced many new concepts such as a *totally transcendental theory* ($\aleph_0$-*stable* theory), *prime models over sets* and *Morley sequences.*

Keisler generalized many of the ideas from Morley's proof to the logic $L_{\omega_1\omega}$; see [32]. He studies a class of structures $(\mathbb{K}, \preccurlyeq_{\mathcal{F}})$, where $\mathbb{K}$ is definable with a sentence in $L_{\omega_1\omega}$ and $\mathcal{F}$ is some countable fragment of $L_{\omega_1\omega}$ containing the sentence. He uses a concept of *homogeneity*, which is closely related to saturation.

Definition 3.1.3. *For L-structures M and N and a fragment $\mathcal{F}$ of $L_{\omega_1\omega}$, $A \subset M$ a subset and $f : A \to N$ a function, write $(M, A) \equiv_{\mathcal{F}} (N, f(A))$ if for every formula $\phi(\bar{x}) \in \mathcal{F}$ and every $\bar{a} \in A$,*

$$M \models \phi(\bar{a}) \text{ if and only if } N \models \phi(f(\bar{a})).$$

A model is $(\kappa, \mathcal{F})$-homogeneous, if for every set $A \subseteq M$ of cardinality strictly less than κ and every $f : A \to M$, if $(M, A) \equiv_{\mathcal{F}} (M, f(A))$, then for all $b \in M$ there exists $c \in M$ such that

$$(M, A \cup \{b\}) \equiv_{\mathcal{F}} (M, f(A) \cup \{c\}).$$

Keisler proved the following theorem (Theorem 35 of [32]):

Theorem 3.1.4. *(Keisler 1971, announced in 1969) Let $\mathcal{F}$ be a countable fragment of $L_{\omega_1\omega}$, $T \subseteq \mathcal{F}$ a set of sentences and $\kappa, \lambda > \omega$. Assume that:*

1. *T is κ-categorical.*
2. *For every countable model M of T, there are models N of T of arbitrarily large power such that $M \preccurlyeq_{\mathcal{F}} N$.*
3. *Every model M of power κ is $(\omega_1, \mathcal{F})$-homogeneous.*

Then T is λ-categorical. Moreover, every model of T of power λ is $(\lambda, \mathcal{F})$-homogeneous.

One stage in the transition from strictly syntactic to semantic means of defining classes is Shelah's version of Theorem 3.1.4. To understand it, we need the following fact, which stems from Chang, Scott and Lopez-Escobar (see for example [16] from 1968); the current formulation is Theorem 6.1.8 in the book [4].

Theorem 3.1.5. *(Chang-Scott-Lopez-Escobar) Let ϕ be a sentence in $L_{\omega_1\omega}$ in a countable vocabulary L. Then there is a countable vocabulary L' extending L, a first-order L' theory T and a countable collection Σ of L'-types such that reduct is a 1-1 map from the models of T which omit Σ onto the models of ϕ.*

A crucial point is that the infinitary aspects are translated to a first order context, at the cost of expanding the vocabulary. If ϕ is a complete sentence, the pair (T, Σ) can be chosen so that the associated class of models is the class of atomic models of T (every tuple realizes a principal type). Saharon Shelah generalized this idea to develop a more general context, *finite diagrams* [46]. A *finite diagram* D is a set of types over the empty set and the class of structures consists of the models which only realize types from D. Shelah defined a structure M to be (D, λ)-homogeneous if it realizes only types from D and is $(|M|, L_{\omega\omega})$-homogenous (in the sense of Definition 3.1.3). He (independently) generalized Theorem 3.1.4 to finite diagrams. His argument, like Keisler's, required the assumption of homogeneity. Thus, [46] is the founding paper of *homogeneous model theory*, which was further developed in for example [21], [15], [30], [29]. The compact case ('Kind II' in [48]) was transformed into the study of continuous logics and abstract metric spaces [12] and finally generalized to metric abstract elementary classes [24]. These last developments have deep connections with Banach space theory.

Baldwin and Lachlan in 1971[7] give another method for first order categoricity transfer. They develop some *geometric* tools to study structures of a theory categorical in some uncountable cardinal: any model of such a theory is prime over a *strongly minimal* set and the isomorphism type is determined by a certain *dimension* of the strongly minimal set. This gives a new proof for the Morley theorem for elementary classes but also the *Baldwin-Lachlan Theorem*: if an elementary class is categorical in some uncountable cardinal, it has either just one or $\aleph_0$-many countable models. The geometric analysis of uncountably categorical elementary classes was developed even further by Zilber (see [57], earlier Russian version [56]), giving rise to *geometric stability theory*. We discuss the number of countable models of an $\aleph_1$-categorical *non-elementary* class in Part II.

A further semantic notion closely tied to categoricity is Shelah's 'excellence'. Excellence is a kind of generalized amalgamation; (details in [4]). The rough idea is to posit a type of unique *prime models* over certain *independent diagrams* of models. 'Excellence' was discovered independently by Boris Zilber while studying the model theory of an algebraically closed field with *pseudoexponentiation*, (a homomorphism from $(F, +)$ to $(F^*, \cdot)$. He defines the notion of a quasiminimal excellent (qme) class by 'semantic conditions'; Kirby [35] proved they can be axiomatized in $L_{\omega_1\omega}(Q)$. Zilber showed any qme class is categorical in all uncountable powers and finds such a class of pseudo-exponential fields. Natural algebraic characterizations of excellence have been found in context of algebraic groups by Zilber and Bays [60], [10], [9]. Excellence implies that the class of structures has models in all cardinalities, has the *amalgamation property* (see Part II), and admits full categoricity transfer. Zilber's notion of 'excellence' specializes Shelah's notion of excellence for sentences in $L_{\omega_1\omega}$, invented while proving the following general theorem for transferring categoricity for sentences in $L_{\omega_1\omega}$[50][2]. The theorem uses a minor assumption on cardinal arithmetic.

Theorem 3.1.6. *(Shelah 1983) Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. Let $\phi \in L_{\omega_1\omega}$ be a sentence which has an uncountable model, but strictly less than the maximal number of models in each cardinality $\aleph_n$ for $0 < n < \omega$. Then the sentence is* excellent.

(ZFC:) Assume that a sentence ϕ in $L_{\omega_1\omega}$ is excellent and categorical in some uncountable cardinality. Then ϕ is categorical in every uncountable cardinality.

The excellence property is defined only for complete sentences in $L_{\omega_1\omega}$, more precisely for the associated classes of *atomic models* (each model omits all non-isolated types) of a first order theory T in an extended vocabulary. Excellent classes have been further studied in [36], [27] [20]. Theorem 3.1.6, expounded in [4], extends easily to incomplete sentences:

Corollary 3.1.7. *Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. Let $\phi \in L_{\omega_1\omega}$ be a sentence which is categorical in $\aleph_n$ for each $n < \omega$. Then ϕ is categorical in every cardinality.*

Shelah and Hart [22], made more precise in [6], show the necessity of considering categoricity up to $\aleph_\omega$; there are examples of $L_{\omega_1\omega}$-sentences ϕ_n which are categorical in each $\aleph_k$ for $k \leq n$ but have the maximal number of models in $\aleph_{n+1}$. However, it is not known whether the assumption on cardinal arithmetic can be removed from the theorem.

In the discussion above we isolated properties such as *homogeneity* and *excellence*, which enable one to prove categoricity transfer theorems. More important, they support the required tools for classifying and analyzing structures with model-theoretic methods; both generated subfields: *homogeneous model theory* and *model theory for excellent classes.* These properties have applications to 'general mathematics': $L_{\infty\omega}$-free algebras [42] for homogeneous model theory or Zilber's pseudoexponentiation and the work on covers of Abelian varieties [59] for excellence.

[2] The important first order notion of the OTOP discussed in Subsection 3.2 was derived from the earlier concept of excellence for $L_{\omega_1,\omega}$.

We argue that finding such *fundamental properties* for organizing mathematics is one of the crucial tasks of model theory.

The investigation of $L_{\omega_1,\omega}$ surveyed in this section makes no assumption that the class studied has large models; the existence of large models is deduced from sufficient categoricity in small cardinals. Shelah pursues a quite different line in [52]. He abandons the syntactic hypothesis of definability in a specific logic. In attempting to prove eventual categoricity, he chooses smaller AEC's in successive cardinalities. Thus he attempts to construct a smaller class which is categorical in all powers. Crucially, this work does not assume the existence of arbitarily large models.

We discuss more on categoricity transfer in AECs in Part II. There we will concentrate on certain type of AECs, namely Jónsson classes, where some categoricity transfer results are known and some stability theory along with a natural notion of type can be constructed. These classes are generalizations of homogeneous and excellent classes and they have arbitrarily large models and for example the *amalgamation property* by assumption.

3.2. **The stability classification: First order vs. non-elementary.** One of the major themes of contemporary model theory is the notion of classification theory. Classification is used in two senses. On the one hand models in a particular class can be classified by some assignment of structural invariants. On the other hand, the classes of models[3] are split into different groups according to common properties, which may be semantic or syntactic; many examples are given below. Shelah (e.g. [52]) has stressed the importance of certain properties of theories, those which are dividing lines: both the property and its negation has strong consequences. In the following we discuss various important classes of theories and emphasize those properties which are dividing lines.

Saharon Shelah originated *stability theory* for elementary classes [49] and produced much of the early work. Now however, the field embraces much of model theory and the tools are pervasive in modern applications of model theory. Among the many texts are: [14],[2],[45].

We can define *stability* in λ, as the property that there are no more than λ many distinct complete types over any subset of size λ. However, stability has many equivalent definitions in elementary classes. A

[3]The word class is vastly overloaded in this context. In first order logic, a complete theory is a natural unit. In studying infinitary logic, the natural unit often becomes an AEC (in the first order case this would be the class of models of the theory.)

remarkable consequence of the analysis is that counting the number of types is related to the geometry of the structures in the class. For example, if the class of structures is stable in any cardinal at all, one can define a notion of *independence* between arbitrary subsets of any model, which is a useful tool to analyze the properties of the structures in the class. The importance of such a notion of independence is well established and such *independence calculus* has been generalized to some unstable elementary classes such as classes given by *simple* [54] or *NIP* theories [1]. Stability theory has evolved to such fields as *geometric stability theory*[45], which is the major source for applications of model theory to 'general mathematics'.

Stability theory divides classes into four basic categories. This division is called the *stability hierarchy*:

(1) $\aleph_0$-stable classes;
(2) superstable classes, that is, classes stable from some cardinal onwards;
(3) stable classes, that is, stable in at least one cardinal;
(4) unstable classes.

In elementary classes $\aleph_0$-stable classes are stable in all cardinalities and hence we get a hierarchy of implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. Uncountably categorical theories are always $\aleph_0$-stable whereas non-superstable classes have the maximal number of models in each uncountable cardinal. An $\aleph_0$-stable or superstable class can also have the maximal number of models, e.g., if it has one of the properties DOP or OTOP, discussed in Examples 3.2.2 and 3.2.3.

Developing stability theory for non-elementary classes is important not only because it widens the scope of applications but also because it forces further analysis of the tools and concepts developed for elementary classes. Which of the tools are there only because first order logic 'happens' to be compact and which could be cultivated to extend to non-elementary classes? Especially, can we *distinguish* some core properties enabling the process? What are the problems met in, say, categoricity transfer or developing independence calculus? Why does the number of *types* realized in the structure seem to affect the geometric properties of structures and can we analyze the possible geometries arising from different frameworks? For example, Hrushovski [25] proved a famous theorem in geometric stability theory: under assumptions of a logical nature the geometry given by the notion of independence on the realizations of a regular type, must fall into one of three natural categories involving group actions. In the available non-elementary versions of the same theorem ([28],[26]), we cannot rule out a fourth possibility: existence of a so-called *non-classical group*, a non-abelian group admitting an ω-homogeneous pre-geometry. We can identify some quite peculiar properties of such groups. Even their existence is open.

The established notion of type for abstract elementary classes is a so called *Galois type*, which we will define more carefully in Part II. Then κ-stability is defined with respect to these types: A class of structures is stable in a cardinal κ if no structure in the class realizes more than κ many Galois types over an $\preccurlyeq_{\mathbb{K}}$-elementary substructure of size $\leq \kappa$. For the remainder of this section the reader can think of the following descriptive notion on a Galois type: Let $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$ and a, b be elements in $\mathcal{B}$. We say that a and b have the same Galois type over the structure $\mathcal{A}$ if there is $\mathcal{C}$ such that $\mathcal{B} \preccurlyeq_{\mathbb{K}} \mathcal{C}$ and an automorphism of $\mathcal{C}$ fixing $\mathcal{A}$ pointwise and mapping a to b.

We present here some examples of AECs where the choice of the relation $\preccurlyeq_{\mathbb{K}}$ affects the placement of the class in the stability hierarchy. How 'coincidental' is the division of elementary classes according to the stability hierarchy? The placement of a class of structures in the hierarchy has been shown to affect a huge number of properties that at first sight do not seem to have much to do with the number of types. Which of these connections are 'deep' or 'semantic', or especially, which extend to non-elementary frameworks? Can an appropriate hierarchy be found?

The moral of these examples is that properties of the 'same' class of structures might look different if definitions in logics with more expressive power are allowed or a different notion $\preccurlyeq_{\mathbb{K}}$ for an abstract elementary class is chosen.

Example 3.2.1 (Abelian groups)**.** Let $\mathbb{K}$ be the class of all abelian groups. $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an $\aleph_0$-stable AEC with the notion $\preccurlyeq_{\mathbb{K}}$ as the substructure relation.

However, the same class of structures is strictly stable (stable but not superstable) if we take as $\preccurlyeq_{\mathbb{K}}$ the following notion: $M \preccurlyeq_{\mathbb{K}} N$ if and only if M is a subgroup and for each $a \in M$ and $n \in \mathbb{N} \setminus \{0\}$, n divides a in M if and only if n divides a in N.

The model theory of abelian groups is studied in Eklof and Fischer [18], where the latter notion of $\preccurlyeq_{\mathbb{K}}$ is in the focus of study. AECs induced by tilting and co-tilting modules are studied in Baldwin, Eklof, and Trlifaj [5],[53] provides a more semantic notion of $\preccurlyeq_{\mathbb{K}}$ and the classes of Abelian groups are strictly stable except in one degenerate case.

A number of properties in first order classification theory induce 'bad behavior' for an elementary class of structures, signaled by the existence of the maximal number of models in a given cardinality. The most basic of these are instability and unsuperstability. Others include OTOP, 'the omitting types order property' and DOP 'the dimensional order property', with a version ENI-DOP, which gives many countable models. Especially, these play a role in classifying countable complete first order theories; their negations NOTOP, NDOP and ENI-NDOP have 'good' implications, from the viewpoint of classification theory; they aid in the assigning of invariants.

One equivalent definition for unstability is that there is a formula which in the models of a first order theory defines an infinite ordering. Then by compactness, the elementary class must contain models interpreting various different orderings, which (nontrivially) forces the number of models to the maximum. Similarly the properties DOP and OTOP cause certain kind of orderings to appear in the structures; but, the orderings are not defined by a single first order formula. Just as in Example 3.2.1, the unsuperstability of the class of abelian groups is not visible to quantifier-free formulas, the only ones 'seen' by the substructure-relation, OTOP and DOP are a form of instability not visible to first order formulas.

The following two examples illustrate the properties OTOP and DOP. In each case we 'define' an arbitrary graph (e.g. an ordering) on $P \times P$ by describing a column above each point of the plane. The two methods of description, by a type or a single formula, distinguish OTOP and DOP.

Example 3.2.2 (An example with OTOP)**.** Let the vocabulary L consist of two predicates P and Q and ternary relations R_n for each $n < \omega$.

By ternary predicates $R_n(x, y, z)$ we define a decreasing chain of sets $R_n(a, b, z)$ of subsets of Q over each pair (a, b) in $P \times P$. The sets $R_0(a, b, z)$ are disjoint as the pairs (a, b) vary. And there is exactly one element $c_n^{a,b}$ in $R_n(a, b, z)$ but not in $R_{n+1}(a, b, z)$. Thus the types $p_{ab}(x) = \{R_n(a, b, x), x \neq c_n^{a,b} : n < \omega\}$ can be independently omitted or realized.

The resulting elementary class is $\aleph_0$-stable but it has the maximal number of models in each infinite cardinality. Any directed graph (especially any ordering) can be coded by a structure the following way:

$$\text{there exists an edge from } x \text{ to } y \;\Leftrightarrow\; \exists z \bigwedge_{n<\omega} R_n(x, y, z).$$

We can study the same class $\mathbb{K}$ of structures but replace first order elementary substructure by $\preccurlyeq_{\mathbb{K}}$, elementary submodel in a fragment of $L_{\omega_1\omega}$ containing all first order formulas and the formula

$$\phi(x, y) = \exists z \bigwedge_{n<\omega} R_n(x, y, z).$$

The relation $\preccurlyeq_{\mathbb{K}}$ 'sees' the complexity caused by the formula, and the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is unstable in the sense of the fragment. But this means it is also unstable as an abstract elementary class. Galois types always refine syntactic types if the submodel notion has a syntactic definition.

This example also has ENI-DOP and thus DOP. From ENI-DOP, we can define another notion $\preccurlyeq_{\mathbb{K}}$ for that class so that the new AEC is unstable but still has Löwenheim-Skolem number $\aleph_0$. Namely, let $M \preccurlyeq_{\mathbb{K}} N$ if M is an elementary substructure of N and whenever there are only finitely many z such that $M \models \bigwedge_{n<\omega} R_n(x, y, z)$, then the number of such elements z is not increased in N.

Example 3.2.3 (An example with DOP)**.** Let the vocabulary L consist of predicates X_1, X_2 and P and two binary relation symbols π_1 and π_2. We define a theory in first order logic, with definable projections from P to each X_i and study the dimensions of pre-images of pairs in $X_1 \times X_2$. We require that

- The universe of a structure consists of three disjoint infinite predicates X_1, X_2 and P,
- the relations π_i determine surjective functions $\pi_i : P \to X_i$ for $i = 1, 2$ and
- for each $x \in X_1$ and $y \in X_2$ there are infinitely many $z \in P$ such that $\pi_1(z) = x$ and $\pi_2(z) = y$.

Again we get an $\aleph_0$-stable elementary class, which is $\aleph_0$-categorical but has the maximal number of models in each uncountable cardinality. Now we can code an ordering $(I, <)$ on the pairs $(x_i, y_i)_{i\in I}$ in an uncountable model so that $(x_i, y_i) < (x_j, y_j)$ if and only if $\{z \in P : \pi_1(z) = x_i \text{ and } \pi_2(z) = y_j\}$ is uncountable.

Furthermore, we get an unstable abstract elementary class for the same class of structures $\mathbb{K}$ as follows: strengthen $\preccurlyeq_{\mathbb{K}}$ so that $M \preccurlyeq_{\mathbb{K}} N$ implies that for all pairs (x, y) in the set $X_1 \times X_2$ of the structure M, if there are only countably many z in the set P of M such that $\pi_1(z) = x$ and $\pi_2(z) = y$, then no such z is added to the set P of the structure N. Since automorphisms must preserve the cardinalities of sets described on the right hand side of the above displayed equivalence, the class is unstable for Galois types. This notion $\preccurlyeq_{\mathbb{K}}$ does not have *finite character* (See Part II) and the new $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has Löwenheim-Skolem number $\aleph_1$.

Similar phenomena appear in *differentially closed fields* of characteristic zero, whose elementary theory is $\aleph_0$-stable with ENI-DOP, and thus DOP. They have the maximal number of models in each infinite cardinality. See the survey articles by Marker [39],[40].

The following examples exhibit the difference between a traditional first order approach and a non-elementary approach.

Example 3.2.4 (Exponential maps of abelian varietes)**.** Martin Bays, Misha Gavrilovich, Anand Pillay, and Boris Zilber [10, 9, 19] study 'exponential maps' or 'universal group covers' $\pi : (\mathbb{C}^g, +) \to A(\mathbb{C})$, where $(\mathbb{C}^g, +)$ is the additive group of the complex numbers to power g and $A(\mathbb{C})$ is an abelian variety. The kernel Λ of π is a free abelian subgroup of $(\mathbb{C}^g, +)$. Two approaches appear in the work: the structures modeling the first order theory of such a map and the structures modeling the $L_{\omega_1\omega}$-theory. The $L_{\omega_1\omega}$-sentence describing the map is quasi-minimal excellent and so categorical in each uncountable cardinality. All the models of the sentence share the same Λ and are determined up to the transcendence degree of the field interpreted in $A(\mathbb{C})$. However, the first order theory is also 'classifiable', it is superstable with NDOP and NOTOP and is 'shallow', although not categorical. Each model of the first-order theory is described by choosing a lattice Λ and a transcendence degree for the field in $A(\mathbb{C})$.

In this case, the non-elementary framework was understood first; the elementary class gives a little more information. Both depend on rather deep algebraic number theory. This topic is an offshoot of trying to understand the model theory of the complex exponentiation $\exp : (\mathbb{C}, +, \times) \to (\mathbb{C}, +, \times)$, which has a very ill-behaved theory in first order logic; see [3, 58] for more discussion on the subject.

Example 3.2.5 (Valued fields)**.** The recent book by Haskell, Hrushovski and Macpherson [23] greatly develops the first order model theory of *algebraically closed valued fields.* The elementary class is unstable and not even simple, and hence the structure theory has involved developing new extensions of the stability-theoretic machinery investigating the class of theories without the independence property.

A valued field consists of a field K together with a homomorphism from its multiplicative group to an ordered abelian group Γ, which satisfies the ultrametric inequality. The problems in the elementary theory of valued fields reduce to that of the value group Γ and the so called *residue field*.

However, we can study valued fields as an AEC fixing the value group as $(\mathbb{R}, +, <)$ and taking all substructures as elementary substructures, requiring also that the value group stays fixed. This class is stable and contains those valued fields that are of most interest. The cases where $(\Gamma, +, <)$ is not embeddable to $(\mathbb{R}, +, <)$ are often called *Krull valuations*. They are forced to be in the scope of study in the first order approach since first order logic cannot separate them from the usual ones. The non-elementary class fixing the value group can be seen as 'almost compact'; see the work of Itaï Ben Yaacov [55].

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References

[1] H. Adler. An introduction to theories without the independence property. To appear in Archive for Mathematical Logic.

[2] J. T. Baldwin. *Fundamentals of Stability Theory*. Perspectives in Mathematical Logic. Springer-Verlag, 1988.

[3] J. T. Baldwin. The complex numbers and complex exponentiation: Why infinitary logic is necessary! 2006.

[4] J. T. Baldwin. *Categoricity*, volume 50 of *University Lecture Series*. AMS, 2009.

[5] J. T. Baldwin, P. C. Eklof, and J. Trlifaj. ${}^{\perp}N$ as an abstract elementary class. *Annals of Pure and Applied Logic*, 149(1-3):25–39, 2007.

[6] J. T. Baldwin and A. Kolesnikov. Categoricity, amalgamation, and tameness. *Israel Journal of Mathematics*, 170:411–443, 2009.

[7] J. T. Baldwin and A. H. Lachlan. On strongly minimal sets. *The Journal of Symbolic Logic*, 36:79–96, 1971.

[8] J. Barwise and S. Feferman. *Model-Theoretic Logics*. Springer-Verlag,New York, 1985.

[9] M. Bays. Model theory of exponential maps of abelian varieties. Talk at the meeting Geometric Model Theory, 25th-28th March 2010, Oxford.

[10] M. Bays and B. Zilber. Covers of multiplicative groups of an algebraically closed field of arbitrary characteristic. preprint: arXive math.AC/0401301, 200?

[11] T. Beke and J. Rosick. Abstract elementary classes and accessible categories. Preprint, 2010.

[12] I. Ben-Yaacov, A.J.Berenstein, C.W.Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, editors, *Model theory with Applications to Algebra and Analysis Vol. 2*, London Math Soc. Lecture Note Series Nr 350. Cambridge Univ Press, Almaty, 2008.

[13] E. Bouscaren, editor. *Model Theory and Algebraic Geometry : An Introduction to E. Hrushovski's Proof of the Geometric Mordell-Lang Conjecture*. Springer-Verlag, 1999.

[14] S. Buechler. *Essential stability Theory*. Springer-Verlag, 1991.

[15] S. Buechler and O. Lessmann. Simple homogeneous models. *Journal of the American Mathematical Society*, 16(1):91–121, 2003.

[16] C.C.Chang. Some remarks on the model theory of infinitary languages. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, pages 36–64. Springer-Verlag, 1968.

[17] C. Chang and H. Keisler. *Model theory*. North-Holland, 1973.

[18] P. C. Eklof and E. R. Fischer. The elementary theory of abelian groups. *Annals of Pure and Applied Logic*, 4:115–171, 1972.

[19] M. Gavrilovich. *Model Theory of universal covering spaces of complex analytic varieties*. PhD thesis, Balliol college Oxford.

[20] R. Grossberg and B. Hart. The classification of excellent classes. *The Journal of Symbolic Logic*, 54(4):1359–1381, 1989.

[21] R. Grossberg and O. Lessmann. Shelah's stability spectrum and homogeneity spectrum in finite diagrams. *Archive of Mathematical Logic*, 41(1):1–31, 2002.

[22] B. Hart and S. Shelah. Categoricity over P for first order T or categoricity for $\phi \in \mathrm{L}_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$. *Israel J Math*, 70:219–235, 1990.

[23] D. Haskell, E. Hrushovski, and D. Macpherson. *Stable domination and independence in algebraically closed valued fields*. Lecture notes in Logic. Cambridge University Press, 2008.

[24] Å. Hirvonen and T. Hyttinen. Categoricity in homogeneous complete metric spaces. *Arch. Math. Logic*, 48(3-4):269–322, 2009.

[25] E. Hrushovski. Almost orthogonal regular types. *Annals of Pure and Applied Logic*, 45(2):139–155, 1989.

[26] T. Hyttinen and M. Kesälä. Interpreting Groups and Fields in Simple, Finitary AECs . To appear in the Journal of Symbolic Logic, 2011.

[27] T. Hyttinen and O. Lessmann. Simplicity and uncountable categoricity in excellent classes. *Annals of Pure and Applied Logic*, 139(1-3):110–137, 2006.

[28] T. Hyttinen, O. Lessmann, and S. Shelah. Interpreting groups and fields in some nonelementary classes. *Journal of Mathematical Logic*, 5(1):1–47, 2005.

[29] T. Hyttinen and S. Shelah. Strong splitting in stable homogeneous models. *Annals of Pure and Applied Logic*, 103:201–228, 2000.

[30] T. Hyttinen and S. Shelah. Main gap for locally saturated elementary submodels of a homogeneous structure. *J Symbolic Logic*, 66:1286–1302, 2001.

[31] H. J. Keisler. Logic with the quantifier "there exists uncountably many". *Annals of Mathematical Logic*, 1(1):1–93, 1970.

[32] H. J. Keisler. *Model theory for infinitary logic*. Studies in logic and the foundations of mathematics. North-Holland Publishing Company, 1971.

[33] J. Kennedy. Gödel and formalism freeness. Preprint, 2010.

[34] J. Kirby. Abstract elementary categories. Preprint, 2008.

[35] J. Kirby. On quasiminimal excellent classes. *J. Symbolic Logic*, 75(2):551–564, 2010.

[36] O. Lessmann. An introduction to excellent classes. In A. Blass and Y. Zhang, editors, *Logic and its applications*, volume 380 of *Contemporary Mathematics*, pages 231–259. AMS, 2005.

[37] M. Lieberman. *Topological and category-theoretic aspects of abstract elementary classes*. PhD thesis, University of Michigan, 2009.

[38] P. Lindström. On extensions of elementary logic. *Theoria*, 35:1–11, 1969.

[39] D. Marker. Model theory of differential fields. In *Model theory, algebra, and geometry*, pages 53–63. Cambridge Univ. Press, Cambridge, 2000.

[40] D. Marker. The number of countable differentially closed fields. *Notre Dame Journal of Formal Logic*, 48(1):99–113, 2007.

[41] D. Marker, M. Messmer, and A. Pillay. *Model Theory of Fields*. Springer-Verlag, 1996.

[42] A. H. Mekler and S. Shelah. $L_{\infty\omega}$-free algebras. *Algebra Universalis*, 26:351–366, 1989.

[43] M. D. Morley. Categoricity in power. *Transactions of the American Mathematical Society*, 114:514–538, 1965.

[44] A. Mostowski. On a generalization of quantifiers. *Polska Akademia Nauk. Fundamenta Mathematicae*, 44:12–36, 1957.

[45] A. Pillay. *Geometric Stability Theory*. Oxford University Press, 1996.

[46] S. Shelah. Finite diagrams stable in power. *Annals Math Logic*, 2:293–300, 1970.

[47] S. Shelah. Categoricity of uncountable theories. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. of California, Berkeley, Calif., 1971)*, pages 187–203, Providence, R.I., 1974. Amer. Math. Soc.

[48] S. Shelah. The lazy model-theoretician's guide to stability. *Logique et Analyse. Nouvelle Série*, 18(71-72):241–308, 1975.

[49] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models.*. North-Holland, 1978. Second Revised Edition 1990.

[50] S. Shelah. Classification theory for for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Parts A and B.. *Israel J Math*, 46:212–273, 1983.

[51] S. Shelah. Classification of Non Elementary classes II, Abstract elementary classes. In J. T. Baldwin, editor, *Classification theory, Proceedings, Chicago 1985*, pages 419–497. Springer-Verlag, Berlin, 1987.

[52] S. Shelah. *Classification Theory for Abstract Elementary Classes*, volume 18 of *Studies in Logic, Mathematical Logic and Foundations*. College Publications, 2009.

[53] J. Trlifaj. Abstract elementary classes induced by tilting and cotilting modules have finite character. *Proceedings of the American Mathematical Society*, 137(3):1127–1133, 2009.
[54] F. O. Wagner. *Simple theories*. Springer, 2000.
[55] I. B. Yaacov. Model theoretic properties of metric valued fields. arXiv:0907.4560, 2009.
[56] B. Zil'ber. Strongly minimal countably categorical theories III. *Siberian Mathematics Journal*, 25:559–571, 1984.
[57] B. Zilber. *Uncountably categorical theories*, volume 117 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993.
[58] B. Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic 0. *Annals of Pure and Applied Logic*, 132:67–95, 2004.
[59] B. Zilber. A categoricity theorem for quasi-minimal excellent classes. In *Logic and its applications*, volume 380 of *Contemp. Math.*, pages 297–306. Amer. Math. Soc., Providence, RI, 2005.
[60] B. Zilber. Covers of the multiplicative group of an algebraically closed field of characteristic zero. *Journal of the London Mathematical Society.*, 74(1):41–58, 2006.

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