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# BEYOND FIRST ORDER LOGIC: FROM NUMBER OF STRUCTURES TO STRUCTURE OF NUMBERS PART II

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Abstract. The paper studies the history and recent developments in non-elementary model theory focusing in the framework of *abstract elementary classes*. We discuss the role of syntax and semantics and the motivation to generalize first order model theory to non-elementary frameworks and illuminate the study with concrete examples of classes of models.

This second part continues to study the question of catecoricity transfer and counting the number of structures of certain cardinality. We discuss more thoroughly the role of countable models, search for a non-elementary counterpart for the concept of completeness and present two examples: One example answers a question asked by David Kueker and the other investigates models of Peano Arihmetic and the relation of an elementary end-extension in the terms of an abstract elementary class.

*Beyond First Order Logic: From number of structures to structure of numbers Part I* studied the basic concepts in non-elementary model theory, such as *syntax* and *semantics*, the languages $L_{\lambda\kappa}$ and the notion of a complete theory in *first order logic* (i.e. in the language $L_{\omega\omega}$), which determines an *elementary class* of structures. Classes of structures which cannot be axiomatized as the models of a first-order theory, but might have some other 'logical' unifying attribute, are called *non-elementary.*

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We discussed the categoricity transfer problem and how this lead to the development of a so called stability classification. We emphasized how research questions in counting the number of models of the class in a given cardinality had led to better understanding of the structures of the class, enabled classification via invariants and found out to have applications beyond the original research field.

We mentioned two procedures for proving a categoricity transfer theorem: the *saturation transfer method* and the *dimension method*. Especially, we discussed *types* and how the question whether or how many times certain types are *realized* in a structure was essential. Here we describe how these methods have been applied for Abstract Elementary Classes.

The study of complete sentences in $L_{\omega_1,\omega}$ gives little information about countable models as each sentence is $\aleph_0$-categorical. Another approach to the study of countable models of infinitary sentences is via the study of simple finitary AEC, which are expounded in Subsection 1.1. However, while complete sentences in $L_{\omega_1,\omega}$ is too strong a notion, some strengthening of simple finitary AEC is needed to solve even such natural questions as, 'When must an $\aleph_1$-categorical class have at most countably many countable models?'. In Section 2 we focus on *countable* models and study the concept of *completeness* for abstract elementary classes. Some interesting examples of models of Peano Arithmetic enliven the discussion.

## 1. Abstract elementary classes and Jónsson classes

We recall the definition of an abstract elementary class.

**Definition 1.0.1.** *For any vocabulary $\tau$, a class of $\tau$-structures $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is* an abstract elementary class *(AEC) if*

(1) *Both $\mathbb{K}$ and the binary relation $\preccurlyeq_{\mathbb{K}}$ are closed under isomorphism.*
(2) *If $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$, then $\mathcal{A}$ is a substructure of $\mathcal{B}$.*
(3) *$\preccurlyeq_{\mathbb{K}}$ is a partial order on $\mathbb{K}$.*
(4) *If $\langle \mathcal{A}_i : i < \delta \rangle$ is an $\preccurlyeq_{\mathbb{K}}$-increasing chain:*
    (a) $\bigcup_{i<\delta} \mathcal{A}_i \in \mathbb{K}$;
    (b) *for each $j < \delta$, $\mathcal{A}_j \preccurlyeq_{\mathbb{K}} \bigcup_{i<\delta} \mathcal{A}_i$*
    (c) *if each $\mathcal{A}_i \preccurlyeq_{\mathbb{K}} \mathcal{M} \in \mathbb{K}$, then $\bigcup_{i<\delta} \mathcal{A}_i \preccurlyeq_{\mathbb{K}} \mathcal{M}$.*
(5) *If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{C}$, $\mathcal{B} \preccurlyeq_{\mathbb{K}} \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$.*

(6) *There is a Löwenheim-Skolem number* LS($\mathbb{K}$) *such that if* $\mathcal{A} \in \mathbb{K}$ *and* $B \subset \mathcal{A}$ *a subset, there is* $\mathcal{A}' \in \mathbb{K}$ *such that* $B \subset \mathcal{A}' \preccurlyeq_{\mathbb{K}} \mathcal{A}$ *and* $|\mathcal{A}'| = |B| + \mathrm{LS}(\mathbb{K})$.

Abstract elementary classes arise from very different notions $\preccurlyeq_{\mathbb{K}}$, which do not necessarily have a background in some logic traditionally studied in model theory. If a class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an AEC, many tools of model theory can be applied to study that class. The first essential observation is that an analog of the Chang-Scott-Lopez-Escobar Theorem (See Theorem 3.1.5 in Part I) holds for any AEC. Here, purely semantic conditions on a class imply it has a syntactic definition.

**Theorem 1.0.2.** *(Shelah) Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an abstract elementary class of* $L$*-structures, where* $|L| \leq \mathrm{LS}(\mathbb{K})$*. There is a vocabulary* $L' \supseteq L$ *with cardinality* $|\mathrm{LS}(\mathbb{K})|$*, a first order* $L'$*-theory* $T$ *and a set* $\Sigma$ *of at most* $2^{\mathrm{LS}(\mathbb{K})}$ *partial types such that* $\mathbb{K}$ *is the class of reducts of models of* $T$ *omitting* $\Sigma$ *and* $\preccurlyeq_{\mathbb{K}}$ *corresponds to the* $L'$*-substructure relation between the expansions of structures to* $L'$*.*

This theorem has an interesting corollaries, since it enables us to use the tools available for *pseudoelementary classes*: for example, we can count an upper bound for the Hanf number. To extend the notion of *Hanf number* (See definition 2.1.6 in Part I) to AEC, take $\mathcal{C}$ in the definition as the collection of all abstract elementary classes for a fixed vocabulary and a fixed Löwenheim-Skolem number. (For a more general account of Hanf Numbers see page 32 of [2].) There is an interesting interplay between syntax and semantics: we can compute the Hanf number for AECs with a given LS($\mathbb{K}$), a semantically defined class. But the proof relies on the methods available only for an associated syntactically defined class of structures in an extended vocabulary.

The following properties of an AEC play a crucial role in advanced work:

**Definition 1.0.3** (Amalgamation and Joint embedding)**.**

(1) *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has the* amalgamation property *(AP), if it satisfies the following:*
*If* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$*,* $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$*,* $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{C}$ *and* $\mathcal{B} \cap \mathcal{C} = \mathcal{A}$*, there is* $\mathcal{D} \in \mathbb{K}$ *and a map* $f : \mathcal{B} \cup \mathcal{C} \to \mathcal{D}$ *such that* $f \restriction \mathcal{B}$ *and* $f \restriction \mathcal{C}$ *are* $\mathbb{K}$*-embeddings.*

(2) *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has* the joint embedding property *(JEP) if for every* $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ *there is* $\mathcal{C} \in \mathbb{K}$ *and* $\mathbb{K}$*-embeddings* $f : \mathcal{A} \to \mathcal{C}$ *and* $g : \mathcal{B} \to \mathcal{C}$*.*

The notion of AEC is naturally seen as a generalization of Jónsson's work in the 50's on universal and homogeneous-universal relational systems; we introduce new terminology for those AEC's close to his original notion.

**Definition 1.0.4** (A Jónsson class). *An abstract elementary class is a* Jónsson class *if the class has arbitrarily large models and the joint embedding and amalgamation properties.*

The models of a first order theory under elementary embedding form a Jónsson class in which complete first order type (over a model) coincides exactly with the Galois types described below and the usual notion of a monster model is the one we now explain.

A standard setting, stemming from Jónsson's [10] version of Fraïssé limits of classes of structures, builds a 'large enough' *monster model* $\mathfrak{M}$ (or universal domain) for an elementary class of structures via amalgamation and unions of chains. A monster model is *universal* and *homogeneous* in the sense that

- All 'small enough' structures can be elementarily embedded in $\mathfrak{M}$ and
- all *partial elementary maps* from $\mathfrak{M}$ to $\mathfrak{M}$ with 'small enough' domain extend to automorphisms of $\mathfrak{M}$.

Here 'small enough' refers to the possibility to find all structures 'of interest' inside the monster model; further cardinal calculation can be done to determine the actual size of the monster model.

The situation is more complicated for AEC. We consider here *Jónsson classes*, where we are able to construct a *monster model*. However, even then the outcome differs crucially from the monster in elementary classes, since we get only *model-homogeneity*, that is, the monster model for a Jónsson class is a model $\mathfrak{M}$ such that

- For any 'small enough' model $M \in \mathbb{K}$ there is a $\mathbb{K}$-embedding $f : M \to \mathfrak{M}$.
- Any isomorphism $f : M \to N$ between 'small enough' $\mathbb{K}$-elementary substructures $M, N \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ extends to an automorphism of $\mathfrak{M}$.

The first order case has homogeneity over sets; AEC's have homogeneity only over models.

The first problem in stability theory for abstract elementary classes is to define 'type', since now it cannot be just a collection of formulas. We note two definitions of *Galois type*.

**Definition 1.0.5** (Galois type)**.**

(1) *For an arbitrary AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *and models* $M \preccurlyeq_{\mathbb{K}} N \in \mathbb{K}$ *consider the following relation for triples* $(\bar{a}, M, N)$*, where* $\bar{a}$ *is a finite tuple in* $N$*:*
$$(\bar{a}, M, N) \equiv (\bar{b}, M, N')$$
*if there are a model* $N'' \in \mathbb{K}$ *and* $\mathbb{K}$*-embeddings* $f : N \to N''$, $g : N' \to N''$ *such that* $f \restriction M = g \restriction M$ *and* $f(\bar{a}) = \bar{b}$*. Take the transitive closure of this relation. The equivalence class of a tuple* $\bar{a}$ *in this relation, written* $\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, N)$ *is called the Galois type of* $\bar{a}$ *in* $N$ over $M$*.*

(2) *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a Jónsson class and* $\mathfrak{M}$ *is a fixed monster model for the class. We say that the tuples* $\bar{a}$ *and* $\bar{b}$ *in* $\mathfrak{M}$ *have the same Galois type over a subset* $A \subseteq \mathfrak{M}$*,*
$$\mathrm{tp}^{\mathrm{g}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/A),$$
*if there is an automorphism* $f$ *of* $\mathfrak{M}$ *fixing* $A$ *pointwise such that* $f(\bar{a}) = \bar{b}$*.*

Fruitful use of Definition 1.0.5.2 depends on the class having the amalgamation property over the 'parameter sets' $A$. Thus, even with amalgamation, there is a good notion of Galois types only over models and not over arbitrary subsets.

The monster model is $\lambda$-saturated for a 'big enough' $\lambda$. That is, all Galois-types over $\preccurlyeq_{\mathbb{K}}$-elementary substructures $M$ of size $\leq \lambda$, which are realized in some $\preccurlyeq_{\mathbb{K}}$-extension of $M$, are realized in $\mathfrak{M}$. When $M$ is a $\mathbb{K}$-elementary substructure of the monster model $\mathfrak{M}$, the two notions of a Galois type $\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, \mathfrak{M})$ agree. As in the first order case, the set of realization of a Galois-type of $\bar{a}$ (over a model) is exactly *the orbits of the tuple* $\bar{a}$ under automorphisms of $\mathfrak{M}$ fixing the model $M$ pointwise. That is,
$$\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, \mathfrak{M}) = \mathrm{tp}^{\mathrm{g}}(\bar{b}, M, \mathfrak{M})$$
if and only if there is an automorphism $f$ of $\mathfrak{M}$ fixing $M$ pointwise such that $f(\bar{a}) = \bar{b}$. Furthermore, if $N \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ is any $\mathbb{K}$-extension of $M$ containing $\bar{a}$, $\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, N)$ equals $\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, \mathfrak{M}) \cap N$. Hence in Jónsson classes we fix a monster model $\mathfrak{M}$ and use a simpler notation for a Galois type, $\mathrm{tp}^{\mathrm{g}}(\bar{a}/M)$, which abbreviates $\mathrm{tp}^{\mathrm{g}}(\bar{a}, M, \mathfrak{M})$. Since we can also study automorphisms of $\mathfrak{M}$ fixing some *subset* $A$ of $\mathfrak{M}$, also the notion of a Galois type over a set $A$ becomes amenable. But over sets, the two forms are not equivalent.

The notion of Galois type lacks many properties that the compactness of first order logic guarantees for first order types. In the first order case, we can always realize a *union* of an increasing chain of types in the monster model and types have *finite character*: the types of $\bar{a}$ and $\bar{b}$ agree over a subset $A$ if and only if they agree over every finite subset of $A$. Many such nice properties disappear for arbitrary Galois types. But we restrict to better-behaved Jónsson classes. Grossberg and VanDieren [4] isolated the concept of *tameness* that is crucial in the study of categoricity transfer for Jónsson classes.

**Definition 1.0.6** (Tameness). *We say that a Jónsson class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* $(\kappa, \lambda)$*-tame for* $\kappa \leq \lambda$ *if the following are equivalent for a model* $M$ *of size at most* $\lambda$*:*

- $\mathrm{tp}^{\mathrm{g}}(\bar{a}/M) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/M)$,
- $\mathrm{tp}^{\mathrm{g}}(\bar{a}/M') = \mathrm{tp}^{\mathrm{g}}(\bar{b}/M')$ *for each* $M \preccurlyeq_{\mathbb{K}} M$ *with* $|M'| \leq \kappa$.

*Furthermore, we say that the class is* $\kappa$-tame *if it is* $(\kappa, \lambda)$*-tame for all cardinals* $\lambda$ *and* tame *if it is* $\mathrm{LS}(\mathbb{K})$*-tame.*

Giving up compactness also has benefits: 'non-standard structures' that realize unwanted types, which are forced by compactness, can now be avoided. For example, we might study real vector spaces in a two sorted language and demand that the reals be standard.

The first 'test question' for AECs was to ask if one can prove a categoricity transfer theorem. Shelah stated the following conjecture:

**Conjecture 1.0.7.** *There exists a cardinal number* $\kappa$ *(depending only on* $\mathrm{LS}(\mathbb{K})$*) such that if an AEC with a given number* $\mathrm{LS}(\mathbb{K})$ *is categorical in some cardinality* $\lambda > \kappa$*, then it is categorical in every cardinality* $\lambda > \kappa$*.*

Shelah introduced the notion of a Jónsson class (not the name) in 1999 [18] and proved the following categoricity transfer result. ( Part II [2]).

**Theorem 1.0.8.** *(Shelah) Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be a Jónsson class. Then there is a calculable cardinal* $\mathrm{H}_2$*, depending only on* $\mathrm{LS}(\mathbb{K})$*, such that if* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in some cardinal* $\lambda^+ > \mathrm{H}_2$*, then* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in all cardinals in the interval* $[\mathrm{H}_2, \lambda^+]$*.*

We remark that this almost settles the Categoricity Conjecture for Jónsson classes: for each such AEC with a fixed Löwenheim-Skolem number LS, let $\mu_{\mathbb{K}}$ be the sup (if it exists) of the successor cardinals in which $\mathbb{K}$ is categorical. Since there does not exists a proper class of such AECs, there is a supremum for such $\mu_{\mathbb{K}}$, denote this number $\Lambda(\mathrm{LS})$. Now if a Jónsson class with Löwenheim-Skolem number LS is categorical in some successor cardinal $\lambda > \mu = \sup(\Lambda(\mathrm{LS}), \mathrm{H}_2)$, it is categorical in all cardinals in $[\mathrm{H}_2, \lambda^+]$, and in arbitrarily large successor cardinals, and hence in all cardinals above $\mathrm{H}_2$. Two problems remain in this area. Remove the restriction to successor cardinals in Theorem 1.0.8; this would avoid the completely non-effective appeal to $\Lambda(\mathrm{LS})$. Make a more precise calculation of the cardinal $\mathrm{H}_2$ in the successor case (Problem D.1.5 of [2].)

Shelah proves a downward categoricity transfer theorem and also shows categoricity for $\lambda^+ > H_2$ implies certain kind of 'tameness' for Galois types over models of size $\leq H_2$, which enables the transfer of categoricity up to all cardinals in the interval $[\mathrm{H}_2, \lambda^+]$. Grossberg and VanDieren separated out the upward categoricity transfer argument, and realized that tameness was the only additional condition needed to transfer categoricity arbitrarily high. The downward step uses the *saturation transfer* method, where saturation is with respect to *Galois types*; the upwards induction uses the *dimension method.*

**Theorem 1.0.9.** *(Grossberg and VanDieren) Assume that a $\chi$-tame Jónsson class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is categorical in $\lambda^+$, where $\lambda > \mathrm{LS}(\mathbb{K})$ and $\lambda \geq \chi$. Then $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is categorical in each cardinal $\geq \lambda^+$.*

Lessmann [15] extends the result to $\mathrm{LS}(\mathbb{K})^+$-categoricity in the case $\mathrm{LS}(\mathbb{K}) = \aleph_0$. The restriction to countable Löwenheim cardinal number reflects a significant combinatorial obstacle. In these two results the categoricity transfer is only from successor cardinals and the proof is essentially an induction on dimension. In Subsection 1.1 we discuss further use of the saturation transfer method for simple, finitary AECs by Hyttinen and Kesälä in [11].

1.1. **Simple finitary AECs.** Simple finitary AECs were defined particularly to study independence and stability theory in a framework without compactness. The idea was both to find a common extension for homogeneous model theory and the study of excellent sentences in $L_{\omega_1\omega}$ (See Part I) and also clarify the 'core' properties which support a successful dimension theory. The property *finite character* is essential for this analysis.

**Definition 1.1.1** (Finite character)**.** *We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has* finite character *if for any two models $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ such that $\mathcal{A} \subseteq \mathcal{B}$ the following are equivalent:*

(1) $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathcal{B}$
(2) *For every finite sequence* $\bar{a} \in \mathcal{A}$ *there is a* $\mathbb{K}$*-embedding* $f : \mathcal{A} \to \mathcal{B}$ *such that* $f(\bar{a}) = \bar{a}$.

**Definition 1.1.2** (Finitary AEC)**.** *An abstract elementary class is* finitary *if it is a Jónsson class with countable Löwenheim-Skolem number that has finite character.*

Definition 1.1.2 slightly modifies Hyttinen, Kesälä [6]; in particular the formulation of finite character is from Kueker [14]. Elementary classes are finitary AECs. However, a class defined by an arbitrary sentence in $L_{\omega_1\omega}$, the relation $\preccurlyeq_{\mathbb{K}}$ being the one given by the corresponding fragment, may not have AP, JEP or even arbitrarily large models. A relation $\preccurlyeq_{\mathbb{K}}$ given by any fragment of $L_{\infty\omega}$ will have finite character. Most abstract elementary classes definable in $L_{\omega_1\omega}(Q)$ do not have finite character. An easy example of a class without finite character, due to Kueker [14], is a class of structures with a countable predicate $P$, where $M \preccurlyeq_{\mathbb{K}} N$ if and only if $M \subseteq N$ and $P(M) = P(N)$.

The notion of weak type is just Galois type with built-in finite character: two tuples $\bar{a}$ and $\bar{b}$ have the same *weak type* over a set $A$, written

$$\mathrm{tp}^{\mathrm{w}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{w}}(\bar{b}/A),$$

if they have the same Galois type over each finite subset $A' \subseteq A$. Furthermore, we say that a model $M$ is *weakly saturated* if it realizes all weak types over subsets of size $< M$.

Basic stability theory with a categoricity transfer result for simple finitary AEC's is carried out in the papers [6], [7] and [5]. However, some parts of the theory hold also for arbitrary Jónsson classes; this is expounded in [9]. David Kueker [14] has clarified when AEC admit syntactic definitions and particularly the connection of finite character to definability in $L_{\infty\omega}$ definablity of AEC's; unlike in Theorem 1.0.2, no extra vocabulary is needed for these results.

**Theorem 1.1.3.** *(Kueker) Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an abstract elementary class with* $\mathrm{LS}(\mathbb{K}) = \kappa$*. Then,*

(1) *The class* $\mathbb{K}$ *is closed under* $L_{\infty,\kappa^+}$*-elementary equivalence.*
(2) *If* $\mathrm{LS}(\mathbb{K}) = \aleph_0$ *and* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *contains at most* $\lambda$ *models of cardinality* $\leq \lambda$ *for some cardinal* $\lambda$ *such that* $\lambda^\omega = \lambda$*, then* $\mathbb{K}$ *is definable with a sentence in* $L_{\lambda^+,\omega_1}$.
(3) *If* $\kappa = \aleph_0$ *and* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has* finite character*, the class is closed under* $L_{\infty,\omega}$*-elementary equivalence.*

(4) *Furthermore, if* $\kappa = \aleph_0$, $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has finite character and at most* $\lambda$ *many models of size* $\leq \lambda$ *for some infinite* $\lambda$, $\mathbb{K}$ *is definable with sentence in* $L_{\lambda^+,\omega}$.

The notion of an *indiscernible sequence* of tuples further illustrates the distinction between the syntactic and semantic viewpoint. Classically a sequence is indiscernible if each increasing n-tupe of elements realize the same (syntactic) type. In AEC, a sequence $(\bar{a}_i)_{i<\kappa}$ is *indiscernible* over a set $A$ (or $A$-indiscernible) if the sequence can be extended to any 'small enough' length $\kappa' > \kappa$ so that any order-preserving partial permutation of the larger sequence extends to an automorphism of the monster model fixing the set $A$.

Note that two tuples lying on the same $A$-indiscernible sequence is a much stronger condition than two tuples having the same Galois type over $A$. However, 'lying on the same sequence' is not a transitive relation and hence not an equivalence relation; the notion of *Lascar strong type* is obtained by taking the transitive closure of this relation.

Using indiscernible sequences we can define a notion of *independence* based on *Lascar splitting*[1]. Furthermore, we say that the class is *simple* if this notion satisfies that each type is independent over its domain. Under further stability hypotheses (Both $\aleph_0$-stability [6],[5] and superstability [7],[9] have been developed.) we get an *independence calculus* for subsets of the monster model. Unlike in elementary stability theory, stability or even categoricity does not imply simplicity; it is a further assumption. However, we show that if *any* reasonable independence calculus exists for arbitrary sets and not just over models, the class must be simple and the notion of independence must agree with the one defined by Lascar splitting, see [5].

[1]The notions are defined 'for weak types' since they are preserved under the equivalence of weak types.

**Definition 1.1.4** (Independence)**.** *A type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ *Lascar-splits over a finite set* $E \subseteq A$ *if there is a strongly indiscernible sequence* $(\bar{a}_i)_{i<\omega}$ *such that* $\bar{a}_0, \bar{a}_1$ *are in the set* $A$ *but*

$$\mathrm{tp}^{\mathrm{w}}(\bar{a}_0/E \cup \bar{a}) \neq \mathrm{tp}^{\mathrm{w}}(\bar{a}_1/E \cup \bar{a}).$$

*We write that a set* $B$ *is* independent *of a set* $C$ *over a set* $A$*, written*

$$B \downarrow_A C,$$

*if for any finite tuple* $\bar{a} \in B$ *there is a finite set* $E \subseteq A$ *such that for all sets* $D$ *containing* $A \cup C$ *there is* $\bar{b}$ *realizing the type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A \cup C)$ *such that* $\mathrm{tp}^{\mathrm{w}}(\bar{b}/D)$ *does NOT Lascar-split over* $E$.

The saturation transfer method was further analyzed for simple, finitary AECs by Hyttinen and Kesälä in [11]. It was noted there, that even without tameness, *weak saturation* transfers between different uncountable cardinalities. Assuming simplicity, they developed much of the stability theoretic machinery for these classes and hence were able to remove the assumption in Theorems 1.0.8 and 1.0.9 that the categoricity cardinal is a successor.

**Theorem 1.1.5.** *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple finitary AEC,* $\kappa > \omega$, *and each model of size* $\kappa$ *is weakly saturated. Then*

(1) *for any* $\lambda > \min\{(2^{\aleph_0})^+), \kappa\}$, *each model of size* $\lambda$ *is weakly saturated.*
(2) *Furthermore, each uncountable* $\aleph_0$*-saturated model is weakly saturated.*

*If in addition* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* $\aleph_0$*-tame, all weakly saturated models with a common cardinality are isomorphic.*

What then is the role of finite character of $\preccurlyeq_{\mathbb{K}}$? If it happens that there are only countably many Galois types over any finite set (this holds for example if the class is $\aleph_0$-stable), the finite character property provides a 'finitary' sufficient condition for a substructure $M$ of $\mathfrak{M}$ to be in $\mathbb{K}$: If all Galois types over finite subsets are realized in $M$, $M$ is back-and forth-equivalent to an $\aleph_0$-saturated $\mathbb{K}$-elementary substructure $N$ of $\mathfrak{M}$ with $|N| = |M|$; a chain argument and finite character give that $N \approx M$. Even without the condition on the number of Galois types, finite character enables many constructions involving building models from finite sequences. It implies, for example, that under simplicity and superstability, two tuples with the same *Lascar type* over a countable set can be mapped to each other by an automorphism fixing the set (i.e. they have the same Galois type over the set), see [9]. These Lascar types (also called *weak Lascar strong types*) are a major tool in geometric stability theory for finitary classes [8], since they have finite character.

## 2. Countable models and completeness

We recall that a theory $T$ in the first order logic $L_{\omega\omega}$ is said to be *complete* if for any sentence $\phi \in L_{\omega\omega}$ either $\phi$ or its negation can be deduced from $T$.

A famous open conjecture for elementary classes was stated by Vaught in [21]:

**Conjecture 2.0.6** (Vaught conjecture)**.** *The number of countable models of a countable and complete first order theory must be either countable or* $2^{\aleph_0}$*.*

The conjecture can be resolved by the continuum hypothesis, which is independent of the axioms of set theory: If there is no cardinality between $\aleph_0$ and $2^{\aleph_0}$, the conjecture is trivially true. The problem is to determine the value in ZFC. Morley [17] proved the most significant result: not just for first order theories but for any sentence of $L_{\omega_1\omega}$ the number of countable models is either $\leq \aleph_1$ or $2^{\aleph_0}$. He used a combination of descriptive set theoretic and model theoretic techniques. There has been much progress using descriptive set theory. The study of this conjecture has also lead to many new innovations in model theory: a positive solution for $\aleph_0$-stable theories was shown by Harrington, Makkai and Shelah in [19] and a more general positive solution for superstable theories of finite rank by Buechler in [3]. However, the full conjecture is still open. [1] provides connections with the methods of this paper.

An easier question for elementary classes is the number of countable models of a theory, which has only one model, up to isomorphism, in some *uncountable* power. Morley [16] showed that the number of countable models of an uncountably categorical elementary class must be countable. We consider as a useful 'motivating question'.

**Question 2.0.7.** *Must an AEC categorical in* $\aleph_1$ *or in some uncountable cardinal have only countably many countable models?*

As asked, the answer is opposite to the first order case. For example, we can define a sentence $\psi$ in $L_{\omega_1\omega}$ as a disjunct of two sentences, one totally categorical and one having uncountably many countable models but no uncountable models. This problem does not occur in the first order case because categoricity implies completeness. $L_{\omega_1,\omega}$ poses two difficulties to this approach. First, deducing completeness from categoricity is problematic; there are several completions. Secondly, $L_{\omega_1,\omega}$-completeness is too strong; it implies $\aleph_0$ categoricity and there are interesting $\aleph_1$-categorical sentences that are not $\aleph_0$-categorical. But sentences like $\psi$ lack 'good' semantic properties such as joint embedding. We might ask a further question: are there some semantic properties that allow the dimensional analysis of the Baldwin-Lachlan proof for an abstract elementary class? For example, does the question have a negative answer for, say, finitary AECs? (See Subsection 2.1.) What can we say on the number of countable models in different frameworks? Some results and conjectures were stated for *admissible* infinitary logics already by Kierstead in 1980 [12].

For a non-elementary class with a better toolbox for dimension-theoretic considerations it might be possible to say more on such questions. For example, *excellent* sentences of $L_{\omega_1\omega}$ have a well-behaved model theory; but such sentences are *complete*, so their countable model is unique up to isomorphism. An essential benefit of the approach of *finitary abstract elementary classes* is that the framework also enables the study of incomplete sentences of $L_{\omega_1\omega}$. The Vaught conjecture is false for finitary abstract elementary classes: Kueker [14] gives an example, well-orders of length $\leq \omega_1$, where $\preccurlyeq_{\mathbb{K}}$ is taken as end-extension. This example has exactly $\aleph_1$ many countable models. The example is categorical in $\aleph_1$, but is not a finitary AEC since it does not have arbitrarily large models. However, we can transform the example to a finitary AEC, by adding a sort with a totally categorical theory; but we lose categoricity.

Contrast the semantic and syntactic approach. If we require definability in some specific language, $L_{\omega\omega}$ or $L_{\omega_1\omega}$, the Vaught conjecture is a hard problem, but it has an 'easy' solution under the 'semantic' requirements we have suggested, such as, a finitary AEC. Is there a similar difference for Question 2.0.7, maybe in the opposite direction? David Kueker had a special reason for asking question 2.0.7 for *finitary AECs*. Recall that by Theorem 1.1.3 (4) that if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an AEC with finite character, $\mathrm{LS}(\mathbb{K}) = \aleph_0$, and $\mathbb{K}$ contains at most $\lambda$ models of cardinality $\leq \lambda$, then it is definable in $L_{\lambda^+\omega}$. Hence if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is $\aleph_1$ categorical and has only countably many countable models, it is definable in $L_{\omega_1\omega}$. But under what circumstances can we gain this? Clearly if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is $\aleph_0$-categorical, this holds. Kueker asks the following, refining Question 2.0.7:

**Question 2.0.8.** *(Kueker) Does categoricity in some uncountable cardinal imply that a finitary AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is definable with a sentence in* $L_{\omega_1\omega}$*?*

Answering the following question positively would suffice:

**Question 2.0.9.** *(Kueker) Does categoricity in some uncountable cardinal imply that a finitary AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has only countably many countable models?*

Unfortunately, Example 2.1.1 gives a negative answer Question 2.0.9, leaving the first question open.

Kueker's results illuminate the distinction between semantic and syntactic properties. Abstract elementary classes were defined with only semantic properties in mind, Kueker provides additional semantic conditions which imply definability in a specific syntax. Thus, the ability to choose a notion of $\preccurlyeq_{\mathbb{K}}$ for an AEC to make it finitary has definability consequences. The concept of *finite character* concerns the relation $\preccurlyeq_{\mathbb{K}}$ between the models in an AEC; Kueker's results conclude definability for the class $\mathbb{K}$ of structures. He does prove some, but remarkably weaker, definability results without assuming finite character.

2.1. **An example answering Kueker's second question.** The following example is a simple, finitary AEC, which is categorical in each uncountable power but has uncountably many countable models. Hence the example gives a negative answer to Kueker's second question.

**Example 2.1.1.** *We define a language $L = \{Q, (P_n)_{n<\omega}, E, f)$, where $Q$ and $P_n$ are unary predicates, $E$ is a ternary relation and $f$ is a unary function. We consider the following axiomatization in $L_{\omega_1\omega}$:*

(1) *The predicates $Q$ and $\langle P_n : n < \omega\rangle$ partition the universe.*
(2) *$Q$ has at most one element.*
(3) *If $E(x, y, z)$ then $x \in Q$ and $z, y$ are not in $Q$.*
(4) *If $Q$ is empty, we have that for each $n < \omega$, $|P_{n+1}| \leq |P_n| + 1$.*
(5) *If $P_0$ is nonempty, then $Q$ is nonempty.*
(6) *For all $x \in Q$, the relation $E(x, -, -)$ is a equivalence relation where each class intersects each $P_n$ exactly once.*
(7) *$f(x) = x$ for all $x \in Q$ and $y \in P_n$ implies $f(y) \in P_{n+1}$.*
(8) *$f$ is one-to-one.*
8. *For $x \in Q$, $y \in P_n$ and $z \in P_{n+1}$, $E(x, y, z)$ if and only if $f(y) = z$.*

*Now we define the class $\mathbb{K}$ be the $L$-structures satisfying the axioms above and the relation $\preccurlyeq_{\mathbb{K}}$ to be the substructure relation.*

The example has two kinds of countable models. When there is no element in $Q$, the predicate $P_n$ may have at most $n$ elements, and either $|P_{n+1}| = |P_n|$ or $P_{n+1}$ is one element larger. If any $P_n$ has more than $n$ elements, the predicate $Q$ gets an element. When there is an element $x$ in $Q$, all predicates $P_n$ have equal cardinality, since the relation $E(x, -, -)$ gives a bijection between the predicates.

Thus we can characterize the countable models of $\mathbb{K}$: There are countably many models with nonempty $Q$: one where each $P_n$ is countably infinite and one where each $P_n$ has size $k$ for $1 \leq k < \omega$. If $Q$ is empty, the model is characterized by a function $f : \omega \to \{0, 1\}$ so that $f(n) = 1$ if and only if $|P_{n+1}| > |P_n|$. Hence there are $2^{\aleph_0}$ countable models.

This example is an AEC with $\mathrm{LS}(\mathbb{K}) = \aleph_0$. The key to establish closure under unions of chains is to note that if the union of a chain has a nonempty $Q$, some model in the chain must already have one. This example clearly has finite character, joint embedding and arbitrarily large models. Furthermore, the class is categorical in all uncountable cardinals.

We prove that the class has amalgamation. For this, let $M, M'$ and $M''$ be in $\mathbb{K}$ such that $M'$ and $M''$ extend $M$. We need to amalgamate $M'$ and $M''$ over $M$. The case where $Q(M)$ is nonempty is easier and we leave it as an exercise. Hence we assume that $Q(M)$ is empty. By taking isomorphic copies if necessary we may assume that the intersection $P_n(M'') \cap P_m(M')$ is $P_n(M)$ for $n = m$ and empty otherwise. Furthermore, we extend both $M'$ and $M''$ if necessary so that $Q(M')$ and $Q(M'')$ become nonempty and each $P_n(M')$ and $P_n(M'')$ become infinite. We amalgamate as follows: For two elements $x \in P_n(M')$ and $y \in P_n(M'')$, if there is $k < \omega$ such that $f^k(x) = f^k(y)$ in $P_{n+k}(M)$, then we identify $x$ and $y$. Otherwise, we take a disjoint union.

We prove that the class is simple. For this, define the following notion of independence for $A, B, C$ subsets of the monster model:

$$A \downarrow_C B \quad \Leftrightarrow \quad \begin{array}{l}\text{For any } a \in A, b \in B \text{ if we have that } E(x, a, b),\\ \text{then there is } c \in C \text{ with } E(x, a, c).\end{array}$$

This notion satisfies invariance, monotonicity, finite character, local character, extension, transitivity, symmetry and uniqueness of free extensions. furthermore, $\bar{a} \not\downarrow_C B$ if and only if for some $D \supseteq B$ and every $\bar{b} \models \mathrm{tp}^{\mathrm{w}}(\bar{a}/C \cup B)$, the type $\mathrm{tp}^{\mathrm{w}}(\bar{b}/D \cup C)$ (Lascar-)splits over $C$. Hence the notion is the same as the independence notion defined for general finitary AECs. This ends the proof.

We can divide this AEC into two disjoint subclasses, both of which are AECs with the same Löwenheim-Skolem number. The class of models where there is no element in $Q$ has uncountably many countable models and is otherwise 'badly-behaved'; all models are countable and the amalgamation property fails. However, the class of models where $Q$ is nonempty, is an uncountably categorical finitary AEC with only countably many countable models. This resembles the example of the sentence in $L_{\omega_1\omega}$, mentioned in the beginning of this section, which was a disjunction of two sentences, a totally categorical one and one with uncountably many countable models and no uncountable ones. Is this 'incompleteness' the reason for categoricity not implying countably many countable models? Can we obtain the conjecture if we require the AEC to be somehow 'complete'? These concepts and questions are explored in the next section.

Jonathan Kirby recently suggested another example with similar properties. This example might feel more natural to some readers, since it consists of 'familiar' structures.

**Example 2.1.2.** Let $\mathbb{K}$ be the class of all fields of characteristic 0 which are either algebraically closed or (isomorphic to) subfields of the complex algebraic numbers $\mathbb{Q}^{alg}$. Let $\preccurlyeq_{\mathbb{K}}$ be the substructure relation. Then $\mathbb{K}$ is categorical in all uncountable cardinalities and has $2^{\aleph_0}$ countable models which all embed in the uncountable models. Also $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple finitary AEC. Further, this class can be divided into smaller AEC's. For example, we can take all algebraically closed fields of characteristic 0, *except* those isomorphic to subfields of $\mathbb{Q}^{alg}$ as one class and all fields isomorphic to a subfield of $\mathbb{Q}^{alg}$ as the other.

2.2. **Complete, Irreducible and minimal AECs.** We define several concepts to describe the 'completeness' or 'incompleteness' of an abstract elementary class. A nonempty collection $\mathbb{C}$ of structures of an AEC $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a *sub-AEC* of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, if

- $\mathbb{C}$ is an abstract elementary class with $\preccurlyeq_{\mathbb{C}} = \preccurlyeq_{\mathbb{K}} \cap \mathbb{C}^2$
- $\mathrm{LS}(\mathbb{K}) = \mathrm{LS}(\mathbb{C})$, that is, the Löwenheim-Skolem numbers are the same.

This allows both 'extreme cases' that $\mathbb{C}$ is $\mathbb{K}$ or that $\mathbb{C}$ consists of only one structure, up to isomorphism. The latter can happen if the only structure in $\mathbb{C}$ is of size $\mathrm{LS}(\mathbb{K})$ and is not isomorphic to a proper $\preccurlyeq_{\mathbb{K}}$-substructure of itself.

**Definition 2.2.1** (Minimal AEC)**.** *We say that an AEC is* minimal*, if it does not contain a proper sub-AEC.*

**Definition 2.2.2** (Irreducible AEC)**.** *We say that an AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* irreducible *if there are no two proper sub-AECs* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *of* $\mathbb{K}$ *such that* $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K}$.

**Definition 2.2.3** (Complete AEC)**.** *We say that an AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* complete *if there are no two sub-AECs* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *of* $\mathbb{K}$ *such that* $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K}$ *and* $\mathbb{C}_1 \cap \mathbb{C}_2 = \emptyset$*.*

Example 2.1.1 is not complete, not irreducible and not minimal. The sub-AEC of Example 2.1.1, which contains the models where $Q$ is nonempty, is also not complete: One abstract elementary class can be formed by taking all such models where each $P_n$ is of equal size $\leq M$ for some finite $M$, and the rest of the models of the class form another AEC.

We make a few remarks that follow from the definitions.

**Remark 2.2.4.**
1. *Minimality implies Irreducibility, which implies Completeness.*
2. *Minimality implies the joint embedding property for models of size* LS($\mathbb{K}$)*.*
3. *Completeness and the amalgamation property imply joint embedding.*
4. *If* $T$ *is a complete first order theory, then the elementary class of models of* $T$ *is not necessarily complete in the sense above.*

Item 1 is obvious. Item 2 holds, since if there are a pair $M_0, M_1$ of models in $\mathbb{K}$ with size LS($\mathbb{K}$), which do not have a common extension, those structures of $\mathbb{K}$ which $\mathbb{K}$-embed $M_0$ form a proper sub-AEC. For item 3, note that if the class has the amalgamation property, the following classes are disjoint sub-AECs: $\{M \in \mathbb{K} : M$ can be jointly embedded with $M_0\}$ and $\{M \in \mathbb{K} : M$ cannot be jointly embedded with $M_0\}$. Furthermore, the amalgamation property gives that joint embedding for models of size LS($\mathbb{K}$) implies joint embedding for all models. Note that an $\aleph_1$ but not $\aleph_0$-categorical countable first order theory is not complete as an AEC.

Example 2.1.1 has joint embedding and amalgamation but is not complete or minimal, hence the implications of items 2 and 3 are not reversible. Is one or both of the implications of item 1 of Remark 2.2.4 reversible? If $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an $\aleph_0$-stable elementary class which is not $\aleph_0$-categorical, the class of $\aleph_0$-saturated models of $T$ is a proper sub-AEC, so the class is not minimal. Example 2.3.7 below gives a class which is complete but not irreducible, minimal or $\aleph_0$-categorical. However, this example is not finitary: it does not have finite character or even arbitrarily large models.

To discuss the relationship between minimality and LS($\mathbb{K}$)-categoricity, it is important to specify the meaning of LS($\mathbb{K}$)-categoricity. We define an AEC to be LS($\mathbb{K}$)-categorical, if it has only one model up to isomorphism, of size *at most* LS($\mathbb{K}$). We have forbidden smaller models because models of an AEC which are strictly smaller than the number LS($\mathbb{K}$) can cause quite irrational and one could say insignificant changes to the class. We could add, say, one finite model which is not embeddable in any member of the class; this would give non-minimality, since the one model constitutes an AEC. However, an AEC with one model of size LS($\mathbb{K}$) and no smaller models, is automatically minimal: For any sub-AEC $\mathbb{K}'$ we can show by induction on the size of the models in $\mathbb{K}$, using the union and Löwenheim-Skolem axioms, that all models of $\mathbb{K}$ are actually contained in $\mathbb{K}'$.

Here are some further questions:

**Question 2.2.5.** (1) *If an AEC is uncountably categorical and* complete*, can it have uncountably many countable models?*
(2) *Is there a minimal AEC which is not* LS($\mathbb{K}$)*-categorical?*
(3) *Is there an irreducible AEC which is not minimal?*

2.3. **An example of models of Peano Arithmetic - Completeness does not imply Irreducibility.** In this section we present an example of a class of models of Peano Arithmetic suggested by Roman Kossak. The example shows that completeness does not imply irreducibility. The properties of the class are from Chapters 1.10 and 10 of the book *The Structure of Models of Peano Arithmetic* [13].

A model $M$ of Peano Arithmetic (PA) is *recursively saturated* if for all finite tuples $\bar{b} \in M$ and recursive types $p(v, \bar{w})$, if $p(v, \bar{b})$ is finitely realizable then $p(v, \bar{b})$ is realized in $M$. Clearly an elementary union of recursively saturated models is recursively saturated. For $M$, a nonstandard model of $PA$, define $SSy(M)$, the *standard system* of $M$, as follows:

$$SSy(M) = \{X \subseteq \mathbb{N} : \exists Y \text{ definable in } M \text{ such that } X = Y \cap \mathbb{N}\}$$

**Lemma 2.3.1.** *(Proposition 1.8.1 of [13]) Let $N, M$ be two recursively saturated models of Peano Arithmetic. Then $M \equiv_{\infty\omega} N$ if and only if $M \equiv N$ and $SSy(M) = SSy(N)$.*

It follows that any countable recursively saturated elementary end-extension of a recursively saturated $M$ is isomorphic to $M$.

We say $N \models PA$ is $\omega_1$*-like*, if it has cardinality $\aleph_1$ and every proper initial segment of $N$ is countable. We say that $N \models PA$ is an *elementary cut* in $M$ if $M$ is an elementary end-extension of $N$.

**Theorem 2.3.2.** *(Corollary 10.3.3 of [13]) Every countable recursively saturated model $M \models PA$ has $2^{\aleph_1}$ pairwise non-isomorphic recursively saturated $\omega_1$-like elementary end-extensions.*

The following abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, has one countable model, $2^{\aleph_1}$ models of size $\aleph_1$ and no bigger models. We will use it to generate the counterexample.

**Example 2.3.3.** *Let $M$ be a countable recursively saturated model of Peano Arithmetic. Let $\mathbb{K}$ be the smallest class, closed under isomorphism, containing $M$ and all $\omega_1$-like recursively saturated elementary end-extensions of $M$. Let $\preccurlyeq_{\mathbb{K}}$ be elementary end-extension.*

**Lemma 2.3.4.** *The AEC 2.3.3 does not have finite character.*

*Proof.* Let $M$ be a recursively saturated countable model of $PA$. Let $M'$ be a recursively saturated elementary substructure of $M$ (not necessarily a cut) and let $\bar{a}$ be a finite tuple in $M'$. We construct a $\preccurlyeq_{\mathbb{K}}$-map $f : M' \to M$ fixing $\bar{a}$. When $M'$ is not a cut we contradict finite character. For this, we will find an elementary cut $M''$ of $M$ and an isomorphism $f : M' \to M''$ such that $f(\bar{a}) = \bar{a}$. Since $M$ and $M'$ are recursively saturated, both $(M, \bar{a})$ and $(M', \bar{a})$ are recursively saturated. Furthermore, $(M, \bar{a})$ is elementarily equivalent to $(M', \bar{a})$. Now let $M''$ be an elementary cut in $M$ such that $(M, \bar{a})$ is an elementary end-extension of $(M'', \bar{a})$ and $(M'', \bar{a})$ is recursively saturated. Then $(M', \bar{a}) \cong (M'', \bar{a})$. □

From now on, let $M$ be a fixed countable recursively saturated model of PA.

Now we construct a complete but not irreducible AEC. Let $\prec_{end}$ denote elementary end-extension. We define

$$M(a) = \bigcap \{K \prec_{end} M : a \in K\},$$

$$M[a] = \bigcup \{K \prec_{end} M : a \notin K\},$$

where $M[a]$ can be empty. Then let $gap(a)$ denote $M(a) \setminus M[a]$.

It is easy to see that an equivalent definition is the following: Let $\mathcal{F}$ be the set of definable functions $f : M \to M$ for which $x < y$ implies $x \leq f(x) \leq f(y)$. Let $a$ be an element in $M$. The $gap(a)$ in $M$ is the smallest subset $C$ of $M$ containing $a$ such that whenever $b \in C$, $f \in \mathcal{F}$ and $b \leq x \leq f(b)$ or $x \leq b \leq f(x)$, then $x \in C$.

We say that $N \models PA$ is *short* if it is of the form $N(a)$ for some $a \in N$. Equivalently, $N$ has a *last gap*. A short model $N(a)$ is not recursively saturated, since it omits the type

$$p(v,a) = \{t(a) < v : t \text{ a Skolem term}\}.$$

If $N$ is not short, it is called *tall*. The following three properties are exercises in [13].

(1) The union of any $\omega$-chain of end-extensions of short elementary cuts in $M$ is tall.
(2) Any tall elementary cut in $M$ is recursively saturated and hence isomorphic to $M$.
(3) If $K$ is an elementary cut in $M$ and is NOT recursively saturated, then $K = M(a)$ for some $a \in M$.

It follows also that the union of any $\omega$-chain of elementary end-extensions of models isomorphic to short elementary cuts in $M$ is isomorphic to $M$. For the following theorem, see [20].

**Theorem 2.3.5.** *Two short elementary cuts $M(a)$ and $M(b)$ are not isomorphic if and only if the sets of complete types realised in gap$(a)$ and gap$(b)$ respectively are* disjoint. *There are countably many pairwise non-isomorphic short elementary cuts in $M$.*

**Lemma 2.3.6.** *If $a \notin M(0)$, the model $M(a)$ is isomorphic to some proper initial segment $M(a')$ of $M(a)$, which is an elementary cut of $M(a)$.*

*Proof.* Define the recursive type

$$p(x,a) = \{\phi(x) \leftrightarrow \phi(a) : \phi(x) \in L\} \cup \{t(x) < a : t \text{ is a Skolem term}\}.$$

Any finite subset of $\text{tp}(a/\emptyset)$ is realized in $M(0)$ since $M(0) \prec M$. Thence $p(x,a)$ is consistent as $M(0)$ is closed under the Skolem terms. Let $a' \in M$ realize $p(x,a)$. Then $\text{tp}(a') = \text{tp}(a)$ and $M(a') < a$. Hence $M(a)$ is isomorphic to $M(a')$ by Theorem 2.3.5. Furthermore, $M(a')$ is an elementary cut in $M(a)$. □

Lemma 2.3.6 implies elementary $\prec_{end}$-chains can be formed from isomorphic copies of one $M(a)$, when $a \notin M_0$. Hence, each of the following classes $\mathbb{K}_\alpha$ is an abstract elementary class extending the $\aleph_0$-categorical class $\mathbb{K}$ from the Example 2.3.3 and $\mathbb{K}_\alpha$ has $\alpha$ many countable models, where $\alpha \in \omega \cup \{\omega\}$.

**Example 2.3.7.** *Let $\alpha$ be a finite number or $\omega$. Choose $(M(a_i))_{i<\alpha}$ to be pairwise non-isomorphic short elementary cuts in $M$, where each $a_i$ is non-standard. Let $\mathbb{K}_\alpha$ be the smallest class, closed under isomorphism,* *containing* $\mathbb{K}$ *and* $M(a_i)$ *for all* $1 \le i < \alpha$. *Let* $\preccurlyeq_{\mathbb{K}}$ *be elementary end-extension.*

The countable models of $\mathbb{K}_\alpha$ are exactly $M$ and $M(a_i)$ for $1 \le i < \alpha$. This class is closed under $\preccurlyeq_{\mathbb{K}}$-unions: if $\langle M_j,\, j < \beta\rangle$, is a $\preccurlyeq_{\mathbb{K}}$-chain of models in $\mathbb{K}_\alpha$, we have that for every countable limit ordinal $\beta$, $\bigcup_{j<\beta} M_\beta$ is tall and hence isomorphic to $M$, and if $\beta$ is uncountable, the union is isomorphic to some $\omega_1$-like recursively saturated model in $\mathbb{K}$. (Note that the union is also an end-extension of $M$.)

Any abstract elementary class containing a short elementary cut $M(a)$ for some $a \in M$ must contain $M$, as $M$ is a union of models isomorphic to $M(a)$ elementarily end-extending each other. Hence any abstract elementary class containing $M(a)$ contains $M$.

It follows that $\mathbb{K}_\alpha$ is complete since it has no disjoint sub-AECs. Furthermore, the class $\mathbb{K}_\alpha$ is not irreducible for $\alpha > 2$, since we can divide it into two classes, one containing $M(a_i)$ but not $M(a_j)$ and one vice versa, for any $i \neq j < \alpha$.

However, Example 2.3.7 is neither a Jónsson class (all models have cardinality below the continuum) nor a finitary AEC. We ask:

**Question 2.3.8.** *Is there a Jónsson class which is complete but not irreducible or minimal? Furthermore, is there such a finitary AEC?*

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## References

[1] J. T. Baldwin. The Vaught conjecture: do uncountable models count? *Notre Dame Journal of Formal Logic*, 48(1):79–92 (electronic), 2007.
[2] J. T. Baldwin. *Categoricity*, volume 50 of *University Lecture Series*. AMS, 2009.
[3] S. Buechler. Vaught's conjecture for superstable theories of finite rank. *Annals of Pure and Applied Logic*, 155(3):135–172, 2008.

[4] R. Grossberg and M. VanDieren. Galois-stability in tame abstract elementary classes. *Journal of Mathematical Logic*, 6(1):25–49, 2006.
[5] T. Hyttinen and M. Kesälä. Categoricity transfer in simple finitary abstract elementary classes. To appear in the Journal of Symbolic Logic.
[6] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Annals of Pure and Applied Logic*, 143(1-3):103–138, 2006.
[7] T. Hyttinen and M. Kesälä. Superstability in simple finitary AEC. *Fundamenta Mathematicae*, 195(3):221–268, 2007.
[8] T. Hyttinen and M. Kesälä. Interpreting Groups and Fields in Simple, Finitary AECs . To appear in the Journal of Symbolic Logic.
[9] T. Hyttinen and M. Kesälä. Lascar types and Lascar automorphisms in Abstract Elementary Classes. *Notre Dame Journal of Formal Logic*, 52(1):39–54, 2011.
[10] B. Jónsson. Homogeneous universal relational systems. *Mathematica Scandinavica*, 8:137–142, 1960.
[11] M. Kesälä. *Finitary Abstract Elementary Classes*. PhD thesis, University of Helsinki, Department of Mathematics and Statistics, 2006.
[12] H. A. Kierstead. Countable models of $\omega_1$-categorical theories in admissible languages. *Annals of Mathematical Logic*, 19(1-2):127–175, 1980.
[13] R. Kossak and J. Schmerl. *The Structure of Models of Peano Arithmetic*. Number 50 in Oxford Logic Guides. Oxford University Press, 2006.
[14] D. W. Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156(2-3):274–286, 2008.
[15] O. Lessmann. Upward categoricity from a successor cardinal for tame abstract classes with amalgamation. *The Journal of Symbolic Logic*, 70(2):639–660, June 2005.
[16] M. Morley. Countable models of $\aleph_1$-categorical theories. *Israel J. Math*, 5:65–72, 1970.
[17] M. Morley. The number of countable models. *The Journal of Symbolic Logic*, 35:14–18, 1970.
[18] S. Shelah. Categoricity of abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. Shelah [Sh:394].
[19] S. Shelah, L. Harrington, and M. Makkai. A proof of Vaught's conjecture for $\omega$-stable theories. *Israel Journal of Mathematics*, 49(1-3):259–280, 1984.
[20] C. Smoryński. Elementary extensions of recursively saturated models of arithmetic. *Notre Dame Journal of Formal Logic*, 22(3):193–203, 1981.
[21] R. L. Vaught. Denumerable models of complete theories. In *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959)*, pages 303–321. Pergamon, Oxford, 1961.

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