TAMENESS

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TOPICS

1. Some history
2. background on tameness
3. A new sufficient condition and application
4. Non-tameness

HISTORY AND CONTEXT

In 1970 logic was assimilating striking new methods:

recursion theory: priority method

set theory: forcing, diamonds, descriptive set theory

model theory: Morley's theorem

proof theory: ?

FORCING FESTIVAL 1973?

East Coast Model Theory (Yale): small models; focus on first order theories of 'real' structures; definability in individual models

tools: compactness, quantifier elimination in natural languages

West Coast Model Theory (Berkeley, Madison, and Cornell): Models of arbitrary cardinality; arbitrary theories: first order and infinitary

tools: compactness, Erdos-Rado, Ehrenfeucht-Mostowski; quantifier elimination by fiat; combinatorial and sometimes axiomatic set theory; Morley rank; indiscernibles

SHELAH'S REVOLUTION

1. Clsssification of theories (not models 1974):

   (a) frees first order model theory from axiomatic set theory

   (b) systematizes model theoretic investigations by providing a general tool

2. monster model – universal domain

3. dependence relation (forking)

4. analyze types locally

   (a) geometry of types

   (b) nonorthogonality, p-regular types, internality,

5. decompose all models into models of size $|L|$

6. uncountable languages

Marxist Synthesis

Pillay (2000): ‘There is only one model theory’.

LET MANY FLOWERS GROW

Models of Arithmetic,

finite model theory,

universal algebra,

model theory of alternative logics,

abstract model theory,

admissible model theory

IDEOLOGY

Can the study of large ($> 2^{\aleph_0}$) structures seriously impact the understanding of structures discovered in the 19th century?

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Can the study of large ($> 2^{\aleph_0}$) structures seriously impact the understanding of structures discovered in the 19th century?

METHODOLOGICALLY: The answer is decisively yes. The tools of classifying theories (stability, superstability) orthogonality, p-regular decompositions, etc., invented by Shelah to study the uncountable spectrum of *arbitrary* first order theories (in possibly uncountable languages), were converted through geometric stability theory as fundamental tools used in the modern model theoretic investigations of algebraic structures.

Properly axiomatized categoricity implies quantifier elimination.

Can the connection be stronger? In a very rough sense, Morley's theorem says no. At least for categoricity, in first order logic the story is over at $\aleph_1$.

Is the answer different for infinitary logic?

STRONGER IDEOLOGY

For Shelah and probably Tarski, this is not an interesting question. Their goal is to understand mathematics not just the mathematics that has already been long investigated.

ABSTRACT ELEMENTARY CLASSES

AEC's are 'theories' not logics.

The structures are 'ensembliste'; Tarski Structures.

Methodological Problem Understand the exact connection between AEC and the cats-positive bounded-continuous logic configuration.

MONSTER MODEL

Definition. The model M is κ-$(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$-*homogeneous* if $A \leq M, A \leq B \in \boldsymbol{K}$ and $|B| \leq \kappa$ implies there exists $B' \leq M$ such that $B \cong_A B'$.

Standard arguments show:

Theorem 1 *If a class $(\boldsymbol{K}_0, \preceq_{\boldsymbol{K}})$ has the amalgamation property and the joint embedding property, for every κ, there is a structure M_κ which is κ-$(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$-homogeneous.*

For sufficiently large κ, this is the **Monster model**.

GALOIS TYPES

Define:

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

‘The Galois type of a over M in N’ is the same as ‘the Galois type of a' over M in N'’

if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.

AMALGAMATION PROPERTY

If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ has the amalgamation property:

1. $\cong$ is an equivalence relation.

2. $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ has a monster model.

$\mathcal{S}(M)$ denotes the set of Galois types over M.

GALOIS TYPES IN THE MONSTER

If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ has a monster model $\mathcal{M}$,

$$\mathrm{ga-tp}(a/M) = \mathrm{ga-tp}(a'/M)$$

if a and a' are conjugate under an automorphism of $\mathcal{M}$ fixing M.

SYNTACTIC TYPES

Fix a logic $\mathcal{L}$. (E.g. $L_{\omega,\omega}, L_{\omega_1,\omega}, L_{\omega_1,\omega}(Q)$.

Then

$$\mathrm{tp}(a/M, N) = \mathrm{tp}(a'/M, N')$$

if a satisfies the same $\mathcal{L}$-formulas with parameters from M in N as

a' satisfies in N'.

KEY PROPERTIES

Compactness: If $\langle p_i : i < \delta \rangle$ is an increasing sequence of syntactic types $\bigcup_{i<\delta} p_i$ is a syntactic type.

Locality: If $M = \bigcup_{i<\delta} M_i$, $p \neq q \in S(M)$ then there exists an i such that $p|M_i \neq q|M_i$.

Tameness: If $p, q \in S(M)$ and $p|N = p|N$ for every small submodel N, then $p = q$.

TAMENESS PARAMETERS

If for every model M of cardinality μ,

if $p|N = q|N$ for every submodel N with $|N| \leq \kappa$ implies $p = q$;

we say $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is (μ, κ)-tame.

No assumption of amalgamation.

HANF NUMBERS

Notation 2 *1. Let $H(\lambda, \kappa)$ be the least cardinal μ such that if a first order theory T with $|T| = \lambda$ has models of every cardinal less than μ which omit each of a set Γ of types, with $|\Gamma| = \kappa$, then there are arbitrarily large models of T which omit Γ.*

2. Write $H(\kappa)$ for $H(\kappa, \kappa)$.

3. For a similarity type τ, $H(\tau)$ means $H(|\tau|)$.

4. $H_1 = H(\tau)$; $H_2 = H(H(\tau))$.

PRESENTATION THEOREM

Theorem 3 (The Presentation Theorem) *If K is an AEC with Lowenheim number* $\mathrm{LS}(\boldsymbol{K})$ *(in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$), there is a vocabulary τ' with cardinality $|\mathrm{LS}(\boldsymbol{K})|$, a first order τ'-theory T' and a set Γ of $2^{\mathrm{LS}(\boldsymbol{K})}$ types such that:*

$$\boldsymbol{K} = \{M'|\tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is a τ'-substructure of N' where M', N' satisfy T' and omit Γ then $M'|\tau \preceq_{\boldsymbol{K}} N'|\tau$.

By Shelah's presentation theorem:

Corollary 4 *If $\boldsymbol{K}$ is an AEC with Löwenheim-Skolem number κ and $\boldsymbol{K}$ has a model of cardinality at least $H(\kappa, 2^\kappa)$ (which is at most $\beth_{(2^\kappa)^+}$), then $\boldsymbol{K}$ has arbitrarily large models.*

When $\mathrm{LS}(\boldsymbol{K}) = |\tau(\boldsymbol{K})| = \kappa$, $H(\kappa) = H_1$ is the Hanf number for all AEC with the same Löwenheim number.

Same Hanf number for omitting Galois types.

TWO THEMES

What is the eventual behavior of categoricity? So assume there exist arbitrarily large models?

Does categoricity in small cardinals (below $\beth_{\omega_1}$ or better $\aleph_\omega$ imply **existence** and categoricity eventually?

Yes, for excellence. Excellence yields upward Löwenheim-Skolem.

CONSEQUENCES I

Theorem (Shelah) If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ has amalgamation and is categorical in $\lambda^+ > H_2$ and $\boldsymbol{K}$ is then $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is categorical on the interval $[H_2, \lambda^+)$.

This uses the derivation of tameness from categoricity described above.

CONSEQUENCES II

Grossberg and Vandieren began the investigation of stability spectrum for tame AEC with:

Theorem 5 $\boldsymbol{K}$ *is stable in* $\mu > H_1$ *and* (∞, χ)*-tame for some* $\chi < H_1$*, then* $\boldsymbol{K}$ *is stable in* λ *if* $\lambda^{\mu} = \lambda$*.*

Baldwin, Kueker, VanDieren proved the following:

Theorem 6 *Let $\boldsymbol{K}$ be an AEC with Löwenheim-Skolem number $\leq \kappa$. Assume that $\boldsymbol{K}$ satisfies the amalgamation property and is κ-weakly tame and Galois-stable in κ. Then, $\boldsymbol{K}$ is Galois-stable in κ^{+n} for all $n < \omega$.*

With one further hypothesis we get a very strong conclusion in the countable case.

Corollary 7 *Let $\boldsymbol{K}$ be an AEC that satisfies the amalgamation property with Löwenheim-Skolem number $\aleph_0$ that is ω-local and $\aleph_0$-tame. If $\boldsymbol{K}$ is $\aleph_0$-Galois-stable then $\boldsymbol{K}$ is Galois-stable in all cardinalities.*

CONSEQUENCES III

Theorem (Grossberg-VanDieren) If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is categorical in $\lambda^+ > \mathrm{LS}(\boldsymbol{K})^+$ and $\boldsymbol{K}$ is $(\infty, \mathrm{LS}(\boldsymbol{K})$-tame, then $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is categorical on the interval $[\lambda^+, \infty)$.

Theorem (Lessmann) If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is categorical in $\lambda^+ > \mathrm{LS}(\boldsymbol{K}) = \aleph_0$ and $\boldsymbol{K}$ is $(\infty, \mathrm{LS}(\boldsymbol{K})$-tame, then $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is categorical on the interval $[\lambda^+, \infty)$.

SUFFICIENT CONDITIONS I

If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is quasiminimal excellent in the sense of Zilber then Galois types = syntactic types.

(Corollary to proof of categoricity transfer.)

SUFFICIENT CONDITIONS II

Theorem 8 *If $\boldsymbol{K}$ is λ-categorical for $\lambda > H_1$, then for any $\mu < \lambda$, $\boldsymbol{K}$ is (μ, H_1)-weakly tame.*

(Weakly means the large model is assumed to be saturated.)

SUFFICIENT CONDITIONS III

The Hrushovski construction (even with infinitary input) gives rise to an abstract elementary class;

$(\boldsymbol{K}, \preceq_{\boldsymbol{K}}, d)$

d is dimension function which imposes a combinatorial geometry and a notion of strong submodel which gives an abstract elementary class.

GEOMETRY

Assumption 9 *1. δ and so d map into the integers.*

2. d_N satisfies $d_N(X) \leq |X|$.

3. If $X \subseteq Y$, $d_N(X) \leq d_N(Y)$.

Definition 10 *Let $a \in \mathrm{cl}_N(X)$ if $d_N(a/X) = 0$.*

Much more restrictive hypotheses than general AEC or Grossberg-Kolesnikov (who earlier deduced tameness from similar hypotheses assuming the existence of an abstract independence relation).

For each $\overline{a}$ in N, there is a finite $\overline{m_{\overline{a}}}$ with

$$d(\overline{a}/M) = d(\overline{a}/\overline{m_{\overline{a}}}).$$

Lemma 11 *Let $M \preceq_{\boldsymbol{K}} N_1, N_2$ with $|N_j - M| = \kappa$. There are N_1', N_2', M' with $M' \subset N_j$ and $|N_j - M| = |M'| = \kappa$ such that each N_j' is independent from M over M'.*

Theorem 12 *(B,Villaveces,Zambrano) If $(\boldsymbol{K}, \preceq_{\boldsymbol{K}}, d)$ in a countable language satisfies*

1. *free amalgamation*

2. *free extension over independent pairs*

3. *weak 3-amalgamation.*

then $(\boldsymbol{K}, \preceq_{\boldsymbol{K}}, d)$ is $(\infty, \aleph_0)$-tame.

Note hypotheses are about arbitrarily large models.

UNIVERSAL COVERS

When is the exact sequence:

$$0 \to Z \to V \to A \to 0. \tag{1}$$

categorical where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety.

Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \mathrm{ln}(\mathrm{acl}(\exp(\mathrm{X})))$$

Moreover by direct basic algebraic argument one can show the three hypotheses. Just using linear disjointness. This is free amalgamation so there are arbitrarily large models.

By Lessmann $\aleph_1$ categoricity implies categoricity in all powers.

Zilber had proved: categoricity in all uncountable powers

iff ‘arithmetic’ properties on semiabelian variety V’.

We improve this to:

categoricity in $\aleph_1$

iff ‘arithmetic’ properties on semiabelian variety V

GOAL

The goal of this line of research is to identify those properties of the Hrushovski construction which are ‘automatic’ and those which must be verified in individual cases.

Next task: Check the status of the hypotheses of Theorem 12 for pseudoexponentiation.

NONTAMENESS

We want to find a monster model $\mathcal{M}$ and a submodel N with elements a, b such that for every small $N_0 \preceq_{\boldsymbol{K}} N$, there is an automorphism of $\mathcal{M}$ fixing N_0 and taking a to b but there is no such automorphism which fixes N.

We find examples by translating problems from Abelian group theory.

Hart-Shelah

There is an AEC $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ which is

1. $\aleph_1$ and $\aleph_0$ categorical.

2. Ap fails in $\aleph_0$.

3. Many models in all cardinals $\geq \lambda$ for some $\lambda < 2^{\aleph_1}$.

The argument that $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is not $(\aleph_1, \aleph_0)$-tame needs some work. It may be that one has to go to a larger example. E.g. the Hart-Shelah example which is $\aleph_3$-categorical.

WHITEHEAD GROUPS

A is a *Whitehead group* (W-group) if $Ext(Z, A) = 0$.

That is, every short exact sequence:

$$0 \to Z \to H \to A \to 0. \tag{2}$$

splits.

BACKGROUND

J.H.C. Whitehead conjectured that every Whitehead group of cardinality $\aleph_1$ is free. Shelah proved:

1) (ZFC) There is an $\aleph_1$-free (every countable subgroup is free) Abelian group of cardinality $\aleph_1$.

2) The Whitehead Conjecture is independent of ZFC.

THE EXAMPLE

Let $\boldsymbol{K}$ be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\boldsymbol{G}, \boldsymbol{Z}, \boldsymbol{I}, \boldsymbol{H})$.

G is a torsion-free Abelian Group. Z is a copy of $(Z, +)$. I is an index set and H is a family of infinite groups.

The vocabulary also includes function symbols $\boldsymbol{F}, \boldsymbol{k}$ and $\boldsymbol{\pi}$, naming functions F, k, and π. There is a binary $+$ on G and on $\mathcal{Z}$ and a ternary $+$.

F maps H onto I and for $s \in I$, $+(\_, \_, s)$ is a group operation on $H_s = F^{-1}(s)$. Finally, π maps H onto G so that $\pi_s = \pi | H_s$ is a projection from H_s onto G.

The kernel of each π_s is isomorphic to Z via a map $\boldsymbol{k}(\_, s)$ where $k : \mathcal{Z} \times I \mapsto H$.

$M_0 \preceq_{\boldsymbol{K}} M_1$ if

M_0 is a substructure of M,

but $\boldsymbol{Z}^{M_0} = \boldsymbol{Z}^{M}$

and $\boldsymbol{G}^{M_0}$ is a pure subgroup of $\boldsymbol{G}^{M_1}$.

FACTS

Definition 13 *We say the AEC* $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ admits closures *if for every* $X \subseteq M \in \boldsymbol{K}$*, there is a minimal closure of* X *in* M*. That is,* $M | \bigcap\{N : X \subseteq N \preceq_{\boldsymbol{K}} M\} = \mathrm{cl}_M(X) \preceq_{\boldsymbol{K}} M$*.*

The class $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is an abstract elementary class that admits closures and has the amalgamation property.

NOT LOCAL

Lemma 14 $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ *is not* $(\aleph_1, \aleph_1)$*-local. That is, there is an* $M^0 \in \boldsymbol{K}$ *of cardinality* $\aleph_1$ *and a continuous increasing chain of models* M_i^0 *for* $i < \aleph_1$ *and two distinct types* $p, q \in \mathcal{S}(M^0)$ *with* $p|M_i^0 = q|M_i$ *for each* i*.*

Let G be an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. There is an H such that,

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow G \rightarrow 0$$

is exact but does not split.

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

Let $p = \text{tp}(t_1/M^0, M^1)$ and $q = \text{tp}(t_2/M^0, M^2)$.

Since the exact sequence for $\boldsymbol{H}^{M^2}$ splits and that for $\boldsymbol{H}^{M^1}$ does not, $p \neq q$.

NOT $\aleph_1$-LOCAL

But for any countable $M'_0 \preceq_{\boldsymbol{K}} M_0$, $p|M'_0 = q|M'_0$, as

$$0 \to Z \to H'_i \to G' \to 0. \tag{3}$$

splits.

$G' = \boldsymbol{G}(M'_0)$, $H' = \pi^{-1}(t_i, G')$.

NOT $\aleph_0$-TAME

It is easy to see that if $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is $(\aleph_1, \aleph_0)$-tame then it is $(\aleph_1, \aleph_1)$-local, so

$(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is not $(\aleph_1, \aleph_0)$-tame.

So in fact, $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is not $(\chi, \aleph_0)$-tame for any χ.

NOT κ-TAME

With some use of diamonds, for each successor cardinal κ, there is a κ-free but not free group of cardinality κ which is not Whitehead. This shows that, consistently,

For arbitrarily large κ,

$(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is not (κ^{+}, κ)-tame for any κ.

BECOMING TAME

Grossberg and Van Dieren asked for $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$, and $\mu_1 < \mu_2$ so that $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is not $(\geq \mu_1, \mu_1)$-tame but is $(\geq \mu_2, \mu_2)$-tame.

This requires finding a way to bound the cardinality of G without imposing too much structure on G.

Even more interesting. Is it possible to find categorical examples which only become tame near the Hanf number.

LOCALITY AND COMPACTNESS

The existence of a sup for an increasing chain of Galois types is also problematic. Every ω-chain has a sup.

Locality up to κ implies compactness at κ^+.

There are NO known sufficient conditions for locality.

Variants of the Whitehead construction should give (consistently) a class that is not $(\aleph_2, \aleph_2)$-compact.

TAMENESS AND AMALGAMATION

How closely are tameness and amalgamation intertwined?

TWO PROPERTIES

$\boldsymbol{K}$ is *model complete* if $N \subset M$ and $N \in \boldsymbol{K}$, implies $N \preceq_{\boldsymbol{K}} M$.

Definition 15 *We say the AEC* $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ admits closures *if for every* $X \subseteq M \in \boldsymbol{K}$*, there is a minimal closure of* X *in* M*. That is,* $M | \bigcap\{N : X \subseteq N \preceq_{\boldsymbol{K}} M\} = \mathrm{cl}_M(X) \preceq_{\boldsymbol{K}} M$.

Non-tame with A.P.

Theorem 16 *For any AEC* $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ *which admits closures, we can assign* $(\boldsymbol{K}', \preceq'_{\boldsymbol{K}})$ *which has the amalgamation property and and is no more tame than* $\boldsymbol{K}$*.*

IDEA: Expand the language by adding a new binary relation. Form a complete graph on each $\boldsymbol{K}$-structure. $\boldsymbol{K}'$ is the smallest AEC containing these expansions such that each finite complete graph is extended to a minimal model that expands to a member of $\boldsymbol{K}$ (which is also a complete graph).

$N' \preceq_{\boldsymbol{K}} M$ if $\mathrm{cl}'_{N'}(A) = \mathrm{cl}'_{M'}(A)$ for every finite A.

Get ap by no edges amalgamation of the underlying graphs.

ulary τ. The vocabulary τ' of $\boldsymbol{K}'$ is obtained by adding one additional binary relation R. We say the domain of a τ'-structure A is an R-set if R defines a complete graph on A.

1. *The class $\boldsymbol{K}'$ is those τ'-structures M which assign a τ'-structure $M_A = \mathrm{cl}'_M(A)$ to each finite subset A of M such that:*

   (a) *If the finite subset A of M is an R-set there is a τ'-structure M_A such that $A \subseteq M_A \subseteq M$ with $M_A \leq \mathrm{LS}(\boldsymbol{K})$, M_A is an R-set, and $M_A|\tau \in \boldsymbol{K}$.*

   (b) *If $N \subset M$ satisfies the conditions on M_A in requirement 1), then $M_A|\tau \preceq_{\boldsymbol{K}} N|\tau$.*

2. *If $M_1 \subseteq M_2$ are each in $\boldsymbol{K}'$, then $M_1 \preceq'_{\boldsymbol{K}} M_2$ if for each finite R-set A in M_1, $\mathrm{cl}'_{M_1}(A) = \mathrm{cl}'_{M_2}(A)$.*

Motivated by issues in databases Baldwin-Benedikt introduced the notion of 'small' to study embedded finite model theory.

Casanovas and Ziegler extended and clarified this work; providing a framework which included Poizat's theory of 'belles paires'.

Baldwin and Baizhanov refined this further:

Theorem 18 *If* (M, A) *is uniformly weakly benign and the #-induced theory on* A *is stable then* (M, A) *is stable.*

The beautiful work of Van den Dries and Ghuryikan could have applied these results.

REFERENCES

My monograph at http://www2.math.uic.edu/ jbaldwin/model.html

contains much of this. The actual papers discussed here may not appear on the web for a couple of months.