Steiner k-systems and the structure of strongly minimal sets
Bogota Logic Seminar

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Overview

1. Quasi-groups and Steiner systems
2. Strongly Minimal Linear Spaces
3. Coordinatization by varieties of algebras
4. The structure of $\mathrm{acl}(X)$

Thanks to Joel Berman, Gianluca Paolini, Omer Mermelstein, and Viktor Verbovskiy.

Quasi-groups and Steiner systems

Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

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Klein 4-group

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Stein 4-quasigroup

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Stein 4-quasigroup

A Latin square is an $n \times n$ square matrix whose entries consist of n symbols such that each symbol appears exactly once in each row and each column.

By definition, this is the multiplication table of a quasigroup.

Definitions

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$.

Steiner triple systems are 'coordinatized' by Latin squares.

Some History

Steiner triple systems were defined for the first time by W.S.B. Woolhouse in 1844 in the Lady's and Gentlemen's Diary and he posed the question.

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

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$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939) Skolem ($6n + 1$, 1958)

Named for Steiner because of his prominence.

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Keevash 2014: for any t and sufficiently large n, if k is not obviously blocked, there are (t, k, n)-Steiner systems.

Linear Spaces

Definition (1-sorted)

The vocabulary τ contains a single ternary predicate R, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems:

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

$\boldsymbol{K}^*$ includes infinite linear spaces.

Groupoids and semigroups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies
   $x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

existentially closed Steiner Systems

Barbina-Casanovas

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are the graphs of a Steiner quasigroup.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T^*_{sq}.
2. T^*_{sqg} has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and locally finite;
   4. T^*_{sqg} has TP_2 and $NSOP_1$.

Strongly Minimal Linear Spaces

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Strongly minimal linear spaces I

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality.

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Corollary

There can be no strongly minimal affine or projective plane, since in such planes the number of lines must equal the number of points.

Strongly minimal linear spaces II

An easy compactness argument establishes

The fundamental corollary of strong minimality

The quantifier $\exists^\infty \phi(x, \mathbf{y})$ is first order definable.

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k.

Specific Strongly minimal Steiner Systems

Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(2, k, \infty)$ Steiner-systems.

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Hrushovski construction for linear spaces

$\mathbf{K}_0^*$ denotes the collection of finite linear spaces in the vocabulary $\tau = \{R\}$.
A line in a linear space is a maximal R-clique
$L(A)$, the lines based in A, is the collections of lines in (M, R) that contain 2 points from A.

Definition: Paolini's δ

[Pao] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

Amalgamation and Generic model

We study classes $\boldsymbol{K}_0$ of finite structures A with $\delta(A') \geq 0$, for every $A' \subset A$.
$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding amalgamation there is unique countable generic.

Theorem: Paolini [Pao]

There is a generic model for $\boldsymbol{K}_0$; it is ω-stable with Morley rank ω.

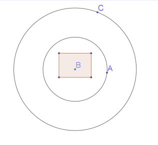
This requires a different notion of 'free amalgamation' than in the Hrushovski construction.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

In Hrushovski's examples the base is unique. But not in linear spaces.

α is the isomorphism type of $(\{a, b\}, \{c\})$,
with $R(a, b, c)$.

Instances of α determines a line in linear spaces.

Overview of construction

Realization of good pairs

1. A good pair C/B *well-placed* by A in a model M, if $B \subseteq A \leq M$ and C is 0-primitive over X.
2. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.

Classes of Structures

I. $\boldsymbol{K}_0^*$: all finite linear τ-spaces.
II. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily ≥ 0.
III. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: $\chi_M(A, B) \leq \mu(A, B)$ μ bounds the number of disjoint realizations of a 'good pair'.
IV. $\boldsymbol{K}_\mu = \mathrm{mod}(T_\mu)$ strongly minimal.

If C/B is well-placed by $A \leq M$, $\chi_M(B, C) = \mu(B/C)$

Basic case

$\boldsymbol{\alpha}$ is the isomorphism class of the good pair $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Context

Let $\mathcal{U}$ be a collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:

(i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;

(ii) a number $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$

T_μ is the theory of a strongly minimal Steiner $(\mu(\boldsymbol{\alpha}) + 2)$-system

If $\mu(\boldsymbol{\alpha}) = 1$, T_μ is the theory of a Steiner triple system bi-interpretable with a Steiner quasigroup.

$\boldsymbol{K}_\mu$

Definition

1. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).
2. X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0$.).
3. Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \boldsymbol{K}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ implies M is d-closed in N.
   Moreover, if $M \in \boldsymbol{K}_d^\mu$, and $B \leq M$, for any good pair (A, B), $\chi_M(A, B) = \mu(A, B)$.

Main existence theorem

Theorem (Baldwin-Paolini)[BP20]

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory T_μ.
If $\mu(\alpha) = k$, all lines in any model of T_μ have cardinality $k + 2$.
Thus each model of T_μ is a Steiner k-system and $\mu(\alpha)$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. α).

Coordinatization by varieties of algebras

Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras V "weakly coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatized

A collection of algebras V "coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct a, b, c:

$$R(a, b, c) \text{ if and only if } a * b = c$$

Theorem

Every strongly minimal Steiner (2,3)-system given by T_μ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroup definable in the system.

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on F by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality k, each $A \in V$ determines a Steiner k-system.
(The 2-generated subalgebras.)
And each Steiner k-system admits a weak coordinatization.

Can this coordinatization be definable in the strongly minimal (M, R)?

Forcing a prime power

Theorem

If a k Steiner systems is weakly coordinatized k must be a prime power q.

Proof: As, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

Strongly minimal block algebras $(M, R, *)$

Theorem: Baldwin

For every prime power q there is a strongly minimal Steiner q-system (M, R) whose theory is interpretable in a strongly minimal block algebra $(M, R, *)$.

We modify the collection of R-structures $\boldsymbol{K}_{\mu}$ to a collection $\boldsymbol{K}_{\mu'}$ of $R, *$ structures so that the generic is a strongly minimal quasigroup that induces a Steiner system.

Conjecture

But for $k > 3$ the coordinatization CAN NOT BE defined in the strongly minimal (M, R).

99%-proved (Almost all cases and almost the last case) conjecture:

For the proof we investigate:

The structure of $\mathrm{acl}(X)$

dcl^* and definability of functions

***-closure**

$\mathrm{dcl}(X)$ and $\mathrm{acl}(X)$ are the definable and algebraic closures of set X.
$b \in \mathrm{dcl}^*(X)$ means $b \notin \mathrm{dcl}(U)$ for any proper subset of X.

dcl^* and definability of functions

*-closure

$\mathrm{dcl}(X)$ and $\mathrm{acl}(X)$ are the definable and algebraic closures of set X.
$b \in \mathrm{dcl}^*(X)$ means $b \notin \mathrm{dcl}(U)$ for any proper subset of X.

Fact

Let I be two independent points in M.
If $\mathrm{dcl}^*(I) = \emptyset$ then no binary function is $\emptyset$-definable in M.

Finite Coding

Definition

A finite set $F = \{\bar{a}_1, \ldots, \bar{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples F is coded by some set S over $\emptyset$.

If $\mathrm{dcl}^*(I) = \emptyset$, T does not have the finite set property.

dcl^* and elimination of imaginaries

Fact: Elimination of imaginaries

A theory T admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no

Fact

If T admits weak elimination of imaginaries then T satisfies the finite set property if and only T admits elimination of imaginaries.

Since every strongly minimal theory with $\mathrm{acl}(\emptyset)$ infinite has weak elimination of imaginaries [Pil99], we have

If a strongly minimal T has no definable binary functions it does not admit elimination of imaginaries.

Group Action and Definable Closure

Fix I as two independent points in the generic model M of T_μ.

2 groups

Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

Note that $\mathrm{dcl}^*(X)$ consists of those elements are fixed by G_I but not by G_X for any $X \subsetneq I$.

symmetric definable closure

The *symmetric definable closure* of X, $\mathrm{sdcl(X)}$, is those a that are fixed by every $g \in G_{\{X\}}$. $b \in \mathrm{sdcl^*(X)}$ exactly when $b \in \mathrm{sdcl(S)}$ but $b \notin \mathrm{sdcl(U)}$ for any proper subset U of X.

No definable binary function/elimination of imaginaries: Sufficient

Lemma

Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and M is a generic model of a strongly minimal theory.

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then I is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter free definable binary function.*

No definable binary function/elimination of imaginaries

Theorem (B-Verbovskiy)

Suppose T_μ has only a ternary predicate (3-hypergraph) R. If T_μ is either in

1. Hrushovski's original family of examples
2. or one of the B-Paolini Steiner systems

and also satisfies:

1. $\mu \in \mathcal{U}$
2. If $\delta(B) = 2$, then $\mu(B/C) \geq 3$ except
3. $\mu(\alpha) \geq 2$ (for linear spaces)

If I is an independent pair $A \leq M \models T_\mu$, then

(I) $\mathrm{dcl}^*(I) = \emptyset$

(II) T_μ does not admit elimination of imaginaries.

In progress

In the Hrushovski case there is an example of the Hrushovski construction with $\mu(C/B) = 2$ and $|B| = 2$
that has non trivial definable closure of a two element set.

The use of $G_{\{I\}}$-decomposition is to show it does have not have elimination of imaginaries.
That is, to eliminate the hypothesis:

If $\delta(B) = 2$, then $\mu(B/C) \geq 3$

from the proof that elimination of imaginaries fails.

G-decomposable sets

Definition

$\mathcal{A} \subseteq M$ is G-decomposable if

1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is G-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

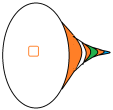
Fact

There are G-decomposable sets.
Namely for any finite U with $d(U/I) = 0$,

$$\mathcal{A} = \mathrm{icl}(\mathrm{I} \cup \mathrm{G}(\mathrm{U}))$$

Constructing a G-decomposition

Linear Decomposition



Constructing a G-decomposition

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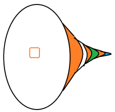
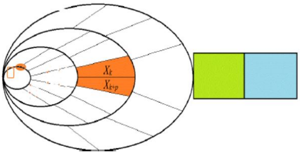


Tree Decomposition



Prove by induction on levels that $\mathrm{dcl}^*(I) = \emptyset$. $(\mathrm{sdcl}^*(\mathrm{I}) = \emptyset)$

Conclusion

Strongly minimal theories with non-locally modular algebraic closure

1. Diversity
   1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
   2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
   3. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
   4. 2-ample but not 3-ample sm sets (not flat) [MT19]
   5. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)
   6. with combinatorial implications

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   6. with combinatorial implications
2. Classifying
   1. discrete
   2. non-trivial but no binary function
   3. non-trivial but no commutative binary function

Do any of the quasigroups (including Hrushovski's) admit elimination of

Combinatorial connections

Unlike many construction in infinite combinatorics these methods give a family of infinite structures with similar properties. Among the properties investigated are:

1. cycle graphs in 3-Steiner systems [CW12] generalized to paths in Steiner k-system; Omitting or demanding finite cycles.
2. preventing or demanding 2-transitivity
3. controlling the lengths of chains.
4. sparse Steiner systems: forbidding specific configurations [CGGW10]

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