EXPLORING THE GENEROUS ARENA

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> So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides *Risk Assessment* for mathematical theories, a *Generous Arena* where the branches of mathematics can be pursued in a unified setting with a *Shared Standard of Proof*, and a *Meta-mathematical Corral* so that formal techniques can be applied to all of mathematics at once.
>
> *What do we want a foundation to do?* Maddy (2019)

We view this analysis as a real step forward in foundational studies. Maddy considers several further criteria and concludes that set theory (specifically ZFC + large cardinals (LC)) serves as an ambient basis for mathematics that meets all the authentic criteria except *Proof Checking* and *Essential Guidance*. Mac Lane articulated a quite different view:

> It is an open scandal that the classical method of applying Zermelo-Fraenkel set theory as foundation for all practice of Mathematics is no longer adequate to the practice of Category Theory. The device of having both large and small categories in some Gödel-Bernays set theory was a convenient arrangement when it was first proposed by Eilenberg Maclane 23 years ago, but it no longer convenes[1] for functor categories (with large domain category) or the category of all categories as used in the theory of fibred categories or Benabou's profunctors. The alternative arrangement of categories in a Grothendieck Universe has been effective for getting on with the development of Mathematics, but it introduces assumptions as to inaccessible cardinals which palpably have nothing to do with the case, and it leaves unsettled (as yet) a variety of questions resulting from a shift of universe.
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> What should we conclude? The happy security provided by one 'monolithic' foundation has been lost. First Principia Mathematica and then Zermelo-Fraenkel had this monolithic character, that all working Mathematics could be formulated within one system. This provided a convenient division of labor, between Mathematicians who just 'used' the system (usually in a naive form) and the Logician who investigated within the system various classical problems.

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[1] OED (1971) gives meaning II.9 of convene as 'to harmonize'.

> The paradise is irretrievably lost; it is high time that open-minded young Mathematicians set to work to construct a new one – perhaps less monolithic." (MacLane, 1969, 130-131)

(Maddy, 2019, 22) thinks that whether set theory or category theory is 'more foundational' does not make for a productive debate. Rather she urges 'a concerted study of the methodological questions raised by category theory'. To begin such a study, we propose a distinction between Foundation and organization (§ 1). We consider category theory and model theory as 'scaffolds' and discuss their complementary virtues for specific mathematical areas (§ 2). We analyze in § 3 comparable claims to be Foundations by material and structural set theories. Finally, in § 4, we summarise the properties of the scaffolds and assess whether the Foundational 'paradise' is lost.

1. Foundations and Organization

Russell and Whitehead (1910) opine 'the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question enables us to deduce ordinary mathematics.' Maddy agrees with that assessment and so we understand Russell and Maddy as taking Foundation with a capital F – a system for deducing ordinary mathematics[2]. And for Maddy, such a Foundation is justified not by an external consistency proof but on its success in grounding mathematics at large. As Maddy stresses in Maddy (2017, 2019), this grounding is not reflected in the daily practice of theorem proving. For her, it is a belief that mathematical research, vaguely thought of as carried out in naive set theory, can be reduced to a formal set theoretic foundation. She calls this belief *shared standard of proof*.

Secondly, Maddy requires a *Generous Arena* that encompasses all of mathematics. (Maddy, 2019, 13) asserts: 'the axioms of set theory imply the existence of (surrogates for) all the entities of classical mathematics – a simple affirmation of set theory's role as Generous Arena.' She stipulates that the surrogates are elements of the universe of sets, V.

Although we focus on these two criteria, Maddy posits three others satisfied by ZFC +LC: risk assessment, meta-mathematical corral, and elucidation; two reasonable (essential guidance, proof checking) not satisfied by ZFC, and rejects two (metaphysical insight, epistemic source). We generally agree with her judgements, but regard the two reasonable criteria as tasks for scaffolds, not Foundations.

In a section entitled 'Foundation or Organization' (MacLane, 1986, 406) is unsatisfied with ZFC (or well-pointed topos) as a **foundation**[3]. On the next page he describes both ZFC and category theory as **not wholly successful organizations** for mathematics. We adopt this distinction and use it to refine Maddy's criteria.

In contrast to a global Foundation, we describe a *scaffold* as an organization that includes both *local* foundations for various areas of mathematics and productive guidance in how to unify them. In a scaffold the unification does not take place by a common axiomatic basis but consists of a systematic ways of connecting results and

[2]This f/F distinction arose on the Foundations of Mathematics listserve in the late 90's. Baldwin (2018) describes other variants (5, fn. 14) and studies small f foundations.

[3]He argues on page 406 that the Gödel incompleteness theorems prevent any 'security blanket'. The longer quote better represents Mac Lane's long term skepticism of the relevance of ZFC to category theory.

proofs in various areas of mathematics. The scaffolds, model theory and category theory, provide local foundations including two flavors of set theory (§ 3.2).

Notation 1.0.1. The familiar *material* set theory takes the notions of element and set as fundamental; *structural* set theory takes function and set as fundamental Shulman (2019). In pure (without ur-elements) ZFC elements are sets. In contrast, the elements of a set X in structural set theory are not sets; they are functions from a terminal object 1 (For every object C, there is a unique morphism from C to 1.) to the object (set) X. Both material (vocabulary: $\epsilon, =$) and structural (vocabulary has symbols for: objects, arrows, domain, codomain, equality, and composition) are first order theories.

Although there are many variants of either sort of set theory, all have an extensional notion of equality. The model theory scaffold gives local foundations for material set theory, ZFC. Various structural set theories extend ETCS (Elementary Theory of the Category of Sets Notation 3.2.1). We will examine whether these alternatives satisfy Maddy's criteria of *Generous Arena* and *Shared Standard of Proof.*

Maddy interprets Mac Lane's complaint (MacLane, 1986, 407) that there is too much 'sand[4]' in ZFC as a proposal that a Foundation should provide *Essential Guidance*, 'that would guide mathematicians toward the important structures and characterize them strictly in terms of their mathematically essential features' (Maddy, 2019, 19). She takes this as a plausible Foundational goal, though not a goal of ZFC. As she cogently argues, this goal conflicts with Generous Arena. So, with Mac Lane, we see guidance as organizational rather than Foundational. Indeed, in (Mathias, 1992, 119) Mac Lane asserts, 'But I see no need for a single foundation — on any one day it is a good assurance to know what the foundation of the day may be — with intuitionism, linear logic or whatever left for the morrow.'

2. Two Scaffolds

Bourbaki wrote,

> Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical theories, and which has led to what is generally known as the 'axiomatic method.'
> (Bourbaki, 1950, 223)

The Bourbaki treatise proceeds by the axiomatization of various fields (local foundations), establishes a common notation, and documents the links among them. This provides one component of a **scaffold** for mathematics. Rather than attempting a precise, full definition of scaffold, we discuss the goals of scaffolding with two examples: category theory and model theory. Along with *local foundations* a scaffold should promote *unity* across mathematics by providing a method for transporting concepts and results from one area to another. There is no requirement that the scaffold encompass all of mathematics but only that it makes connections

[4]In other places MacLane (1988) objects that ZFC allows too many unproductive investigations such as the normal Moore Space conjecture.

among a number of topics. A successful scaffold contributes to generous arena and shared standard of proof. Proofs are built on prior assumptions. One does not prove a new result in mathematics directly from the material or structural set theoretic axioms; there must be a chain of arguments and constructions/definitions from the foundational axioms to the result at hand. A scaffold helps to organize the modules of proofs that involve various areas and provides *Productive Guidance*. I replace essential by productive as various scaffolds will give different advice depending on their framework. The success of the mathematics is the measure of the guidance.

Both scaffolds adopt local foundations by studying the properties of certain classes. The delineation of these areas differs among Bourbaki, model theory, and category theory. Bourbaki axiomatizes classes of structures in the informal way of Euclid or Hilbert (1971). Model theory posits fully formalized theories and Tarski semantics. Category theory gives mathematical definitions (commonly in naive set theory) of its basic classes and morphisms without semantics.

A *formalism-free* (Kennedy (2013)) approach to mathematics defines in *naive* set theory or natural language a class of objects. Often, the exact vocabulary of the class is unclear. In contrast the modern notion of an axiomatic system requires a distinction between a class of structures (defined set theoretically) and a formal language in which axioms are stated. Thus, Bourbaki and Category theory are formalism-free. While, model theory makes essential use of formalization[5].

These are not the only scaffolds. E.g., Descriptive Set Theory scaffolds results in many subjects that are captured by Polish spaces, while the Langlands program expands on Weil's Rosetta stone to unify number theory with harmonic analysis (and points in between) (Frenkel (2013)).

2.1. **The Model Theoretic Approach**

Benis-Sinaceur summarised the source of model theorists' ability to discover analogies across different fields of mathematics.

> Model theory deals specifically with *logical* analogies among mathematical procedures and theories. It proceeds by means of an analysis of the language of theories while exploring the reciprocal relations between this language and the mathematical models that satisfy it. (Benis-Sinaceur, 2000, 282)

In this section we a) describe fundamental model theoretic ideas which clarify the role of ZFC as a generous arena and also as a metamathematical corral, b) explain how the classification of theories serves as a unifying principle to treat different areas of mathematics, c) provides a wild/tame distinction to give productive guidance and d) (§3.1) demonstrates and uses properties of cardinality in exploring Cantor's paradise. See Baldwin (2018) for much more detail.

We concentrate on first order logic, as the so far most successful logic for organizing mathematics by providing and connecting formal local foundations. All the logics are based squarely in ZFC with occasional use of large cardinals and independence results. *Examining a particular mathematical topic*, the investigator selects certain concepts as fundamental. The vocabulary[6] is a set τ of relation symbols,

[5]Remark 2.2.1, (Baldwin, 2018, Chapter 14), and Kennedy (2020) give (non-first order) exceptions to this claim. § 3.2 shows that, for full strength, structural set theory must use formulas.

[6]To avoid ambiguity, I have chosen the word 'vocabulary' rather than such rough synonyms as language, similarity type, signature or, even rougher, logic.

function symbols, and constant symbols chosen to represent these basic concepts. A τ-*structure* with universe A assigns (e.g., to each n-ary relation symbol R an $R^A \subseteq A^n$).

Definition 2.1.1. *A* full formalization *involves the following components.*

(1) Vocabulary: specification of primitive notions.
(2) Logic:
  (a) Specify a class of well formed formulas.
  (b) Specify truth of a formula from this class in a structure.
  (c) Specify a formal deductive scheme for these sentences
(3) Axioms: specify the basic properties of the situation in question by sentences of the logic.

Remark 2.1.2 (The value of formalization)**.** Formalization provides a number of clarifications and solutions for both imagined and real problems.

Category theorists (e.g. Leinster (2014)) often raise tendentious arguments that a mathematician might confusedly ask of two real numbers, 'Is $\pi \in 2$?' The notion of fixing a vocabulary describes exactly when this question makes sense − only if $\in$ is in the vocabulary. Mathematicians don't actually make this mistake; Shulman (2008) makes a more subtle distinction between two uses of $\in$.

Shelah's 1974 proof of the independence (from ZFC) of the famous topologist (J.H.C.) Whitehead (motivated by complex analysis and algebraic topology) conjectured any Whitehead group[7] is free. The conjecture is stated strictly in the vocabulary $(+, 0)$ of Abelian groups. Shelah constructed a specific structure in the vocabulary $(\in, +, 0)$ that under $V = L$ is free as abelian group and under Martin's Axiom is not. Thus the *metamathematical corral* of independence results available in ZFC extends beyond technical problems about ZFC to problems arising in traditional mathematics. Shelah's construction is not directly available in category theory where objects don't have elements (Leinster, 2014, 403). As a further example, consistently, there are regular cardinals (e.g. $\kappa = \aleph_1$) and groups A ($|A| = \kappa$) such that every strictly smaller group is free but A is not free. Shelah's singular cardinal theorem show this fails for any singular cardinal κ and places the result in a general framework Eklof and Mekler (2002); Magidor and Shelah (1994)). Vasey (202x) transfers the result via internal size to accessible categories (Remark 2.2.1).

The mantra, 'model theory is the study of definable sets' depends on using ZFC to define structures. Fraïssé and Jónsson constructions further illustrate the significance of internal structure *in ZFC*. Although the properties of the construction are easily expressed in appropriate categories, translating the natural model theoretic proof into structural set theory would seem unnatural[8]. The existence of isomorphic structures that are *not* identical is essential. This framework underlies the notion of saturation (§ 3.1), the theory of random graphs (connection in Blass and Harary (1979)), even with edge probability $n^{-\alpha}$ (Baldwin and Shelah (1997)), Abstract Elementary Classes, the study of exotic strongly minimal sets (Hrushovski (1993)), and Zilber's generalization to the complex exponential field.

[7]Call A a Whitehead group if for any f, B such that $f\colon B \to A$ is a surjective (i.e. onto) group homomorphism whose kernel is isomorphic to the group of integers Z then B is isomorphic to the direct sum of Z and A. Any free Abelian group A is Whitehead (and conversely if A is countable).

[8]Either the notion of universally axiomatized class or the use of set theory to define 'class closed under substructure' is needed.

An informal notion of *tame/wild mathematics* developed during the 20th century. Roughly, wild mathematics includes the 'wilderness' of point set topology; Pillay in (Buss et al. (2001)) includes any area exhibiting the Gödel phenomena, undecidability and coding of pairs (thus, no notion of dimension). Shelah's *Classification theory* divides complete[9] first order theories by syntactical conditions into a small number of classes. Theories in the same class share mathematically significant properties. Bourbaki (1950) posits three great mother-structures: groups, order, and topology and suggest vague notions of particularizing and combining them to study classical mathematics (e.g. topological groups). Shelah's classification refines the Bourbaki program by enabling (by meta-theory or more often by analogy) the transfer of results from one theory to another in the same class and provides guidance to distinguish the wild from the tame. The crudest distinction, between stable and unstable, echoes Bourbaki's mother-structure: order. Stable structures have no infinite subsets that are definably linearly ordered. Theories have increasing degrees of tameness as conditions such as superstability, ω-stability and strong minimality[10] hold. Among the unstable theories *o-minimality* yields a collection of dramatically tame theories. Instability divides into theories that satisfy the strict order property[11] (strop or sop) and those with the independence property (ip). Those with both (strop) and (ip) are thoroughly wild (e.g., Peano arithmetic and ZFC); despite the connotation, the meaning is: *very different methods will be needed to study these topics.*

The wildness of first order arithmetic did not prevent the great twentieth century advances in number theory. But these advances come not from working in the first order theory of arithmetic but embedding the natural numbers in tame structures, such as (stable expansions of) strongly minimal ACF_p. In Baldwin (2018), we describe the classification and in Chapter 6 stress two paths (stable/o-minimal) to pick out tame areas of mathematics. They share a common feature, missed in Bourbaki's list of mother-structures: combinatorial geometry (matroid theory) organizes dimensions of subsets of models of tame theories.

We look first at the stable side. The success of model theory in studying solutions of equations arises as equations define sets. And in theories with elimination of quantifiers (e.g. differentially or algebraically closed fields (*DCF, ACF*) they control all definable sets. The use of first order axiomatizations, interpretations among theories, and stability theory for results in Diophantine geometry, number theory, and automorphic functions demonstrate the value of formalizing these theories (Bertrand and Pillay (2010); Bouscaren (1999); Freitag and Scanlon (2018); Hrushovski et al. (2018)). Note in particular, that while in algebra the domain of a structure is naively thought of as a set of points, the study of the (ω-stable) theory

[9]Incomplete theories such as the theory of R-modules permit closure under natural operations such as product and homomorphism so interact with universal algebra. Concentrating on complete theories enables the classification.

[10]We omit the easily available technical definitions of the stability hierarchy. A strongly minimal set is a definable set such that every definable subset is finite or co-finite. An algebraically closed field ACF is strongly minimal; so too is the set of solutions to the differential equation defining the Weierstrass j-function (Freitag and Scanlon (2018)). Some of the notions arising in abstract stability theory that are now applied across mathematics include: regular type, prime/minimal models, forking/rank, (non)-orthogonality, canonical base and (elimination) of imaginaries, and classification of combinatorial geometries (matroids).

[11]T is unstable and there is a formula $\phi(\mathbf{x}, \mathbf{y})$ that defines a chain on some (likely not definable) subset of n-tuples.

of differentially closed fields (DCF) focuses on models where the elements are functions. Model theory treats morphisms in two ways: i) 'algebraic homomorphisms' between models (specializing to elementary embedding); ii) definable maps between definable sets. Thus, in algebraic geometry *definable* rational functions correspond to the morphisms in the category: Zariski topologies on ACF with continuous maps (Poizat, 2001, 4.3). Recent work on DCF (Nagloo and Pillay (2016)) invokes stability theoretic tools to solve and refine one-hundred year old Painlevé problems of classification and transcendence of solutions of partial differential equations.

Turning to the unstable case, Grothendieck described his motivation (clarifying Mac Lane's reference to 'too much sand' (§ 2)) for a notion of 'tame topology'.

> I would now say, with hindsight, that 'general topology' was developed (during the thirties and forties) by analysts and in order to meet the needs of analysis, not for topology per se, i.e. the study of the topological properties of the various geometrical shapes. That the foundations of topology are inadequate is manifest from the very beginning, in the form of 'false problems' (at least from the point of view of the topological intuition of shapes) such as the 'invariance of domains', even if the solution to this problem by Brouwer led him to introduce new geometrical ideas. (Grothendieck, 1997, 258)

A theory of a linearly ordered structure is *o-minimal* if every definable subset (perhaps in a much richer language than order) is a finite union of intervals. Notably, Wilkie extended the ur-example, real closed fields, to the real exponential field. Wilkie (Wilkie (2007), (Baldwin, 2018, 160))argues that o-minimality is a direct response to Grothendieck's call because o-minimality:

(1) is flexible enough to carry out many geometrical and topological constructions on real functions and on subsets of real Euclidean spaces.
(2) builds in restrictions so that we are a priori guaranteed that pathological phenomena can never arise. In particular, there is a meaningful notion of dimension for all sets under consideration and any constructed by the means of 1)
(3) is able to prove finiteness theorems that are uniform over fibred collections.

Some of the successes of this program, beyond the basic Dries (1999), include a case of the Andre-Oort conjecture (Karp Prize 2013) (Pila (2011)) (relying on Peterzil and Starchenko (2010); Pila and Wilkie (2006)), the developing subfields in o-minimal algebraic topology Berarducci and Otero (2002), and now o-minimal Hodge theory (Bakker et al. (2020)). The 2018 Karp prize winning book, Aschenbrenner et al. (2017), on asymptotic analysis brings to fruition notions of Hardy. The universal domain is the proper class of surreal numbers (Dries and Ehrlich (2001); Ehrlich (2012)).

The developments outlined above are guided by the following strategies.

Remark 2.1.3 (Model Theory Strategies)**.**

(1) Fix an appropriate vocabulary τ to study the subject.
(2) Give a (first-order) axiomatization T of the area involved.
(3) Study definable relations on the structure to obtain tameness.
(4) Modify your vocabulary to reduce quantifier complexity of formulas.
(5) Use syntactic conditions (stability hierarchy, o-minimality) and the dividing line strategy to guide your search for analogies.

Problem 1. *Give a philosophically meaningful definition of* tame. *What is sand? Does structural set theory, even as weak as ETCS, actually eliminate sand and if so how? What are examples?*

2.2. **Category Theoretic Approach**

Category theory describes classes of structures (which have no internal anatomy) by maps (morphisms) between them. It provides an axiomatic operational description of constructions which apply across mathematics and sufficient conditions for a category to be closed under them. We examine certain results and discuss why they are only awkwardly justified in ZFC + LC. Our analysis is limited to the 'first generation' ($\leqslant$ 1975) of general category theory. *Dependent type theory* and the HOTT program raise important issues (The Univalent Foundations Program (2015) and Voevodski (2014) lecture 3 slide 11) not dealt with here.

While model theory relies heavily on material set theory, the objects of a category have no elements. Thus a group is defined as a category[12] with one object and whose morphisms are all isomorphisms (i.e. invertible). §2.1 described the benefits of internal structure. The gain here is that the properties of constructions used in many areas are defined axiomatically without involving 'irrelevant' internal information.

We indicate some underlying causes of the wide influence of category theory in homological algebra, algebraic topology, and algebraic geometry[13]. Lou Kauffman explains[14], 'We would never have regarded topological spaces as morphisms in a monoidal category, as is part and parcel of TQFT [topological quantum field theory] and Quantum Topology, without a category theoretic point of view[15]. *Categories make possible conceptual shifts that a strictly set theoretic point of view would not see.*'

Category theory began with the conceptual shift engendered by the realization that problems Eilenberg was studying on the continuous side (homology theories in topology) were reflected in problems Mac Lane was studying on the discrete side (group extensions). They described their goals:

> In a metamathematical sense our theory provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines. In particular, it provides opportunities

[12]A usual group $(G, \times, 1)$ induces a category $\hat{G}$ and the elements of G acting as morphisms of $\hat{G}$ by left-multiplication. Conversely, take each morphism as an element and 1_G as the unit. Isomorphisms between material groups become functors.

[13] Other areas include K-theory, scheme theory, Langlands program, and model categories. Krömer (2007) and Marquis (2009) insightfully unite historical, philosophical and mathematical expositions of category theory.

[14]Foundations of Mathematics Listserve: January 23, 2020

[15]Blass and Gurevich (2018) give a more detailed example. Lawvere and Schanuel (1997), a high school text written from a 'function-first' viewpoint, is built around age-appropriate 'real world' examples. The book is slow reading because 'almost familiar' ideas appear in a strange guise – conceptual shift.

> for the comparison of constructions and of the isomorphisms occurring in different branches of mathematics; in this way it may occasionally suggest new results by analogy (Eilenberg and MacLane, 1945, 236).

Category theory describes families of objects and morphisms between them. Morphisms implement the intuition of a function as a rule while hiding the actual rule but studying only the patterns of composition of morphisms like homotopy type. Thus f is a 1-1 function becomes: for every g, h: $fg = gh$ implies $g = h$ (monomorphism). De Toffoli (2017) argues that commutative diagrams and the technique of 'diagram chasing' are an effective hybrid of diagrammatic and linguistic elements that supports reasoning. The operational definition enables the discovery of common constructions (e.g. adjoint functor § 3.1) for diverse subjects.

Problem 2. *Must formal reasoning be represented linearly? What is lost/gained in transforming diagramatic to linear reasoning?*

In contrast to model theory, equations are not studied as 'nice' definable sets. As, there are no definable sets. Rather, for example, the family of possible equations is considered as a vector space of linear transformations and the kernel (in the dual space) of such a transformation contains the solutions. The same tools are now available for both the 'syntax' and the semantics. Moreover, the same abstract formulation applies in many different areas.

The definitions of categories and functor exemplify *par excellence* 'formalism-free' mathematics (§ 2). Yet the *informal* Eilenberg-Steenrod axioms (ES-axioms) for homology required more complex formalizations than known in the 1940's. A *homology theory* on $\mathcal{A}$ (the category of pairs (X, A) of topological spaces with continuous maps as morphisms) assigns to each (X, A) a countable sequence of abelian groups, and group homomorphisms. There were at least 5 such theories by 1945 when (Eilenberg and Steenrod, 1952, Chapter I) announced the ES-axioms that *unify* those theories by provide a common basis, satisfied by each of the examples. That is, axiomatizing functors from $\mathcal{A}$ to (sequences from) **Abgrp** (abelian groups) and natural transformations among these functors. They can be formalized as a theorem schemata in ZFC (or a theory in Gödel-Bernays). A key theorem asserts that two homology theories fulfilling the axioms (and certain additional conditions) are naturally equivalent and yield isomorphic homology groups for a given space from an appropriate category of topological spaces. The ES axioms yield without calculation important results, such as Brouwer's fixed point theorem ('without any appeal to a concretely defined homology theory' (Eilenberg and Steenrod, 1952, 298)). They remark (page viii), 'Heretofore this [homology theory] has been an imprecise picture which the expert could use in his thinking but not in his exposition.' This exemplifies what Maddy later calls *Elucidation* [16].

Maddy dubs as *surrogates*, the constructions described by Burgess:

> [this] *common, unified starting point will have to be such as to make provisions for all types of constructions by which new, auxiliary spaces, or number systems or whatever are manufactured out of old, traditional ones ....* There are a handful of basic types of

[16]Elucidation replaces an *imprecise, pre-theoretic* notion which interferes with mathematical practice with a precise, 'set-theoretically defined one' ((Maddy, 2017, 293).)

> constructions that keep being used over and over, in ever more elaborate combinations (Burgess, 2015, 62)

Why does Mac Lane doubt that ZFC is adequate to support categorical reasoning? A category is *concrete* if it is equipped with a forgetful functor[17] to the category of sets. Applying this functor allows one to think of the objects of the category as sets with additional structure, and of its morphisms as structure-preserving *functions* (Adámek et al. (1990)). The most naive interpretation of finding surrogates would consider only concrete categories. The fundamental classes studied in 20th century mathematics: groups, topological spaces, etc. are concrete categories that are usually defined in naive set theory. But there are categories which, *intrinsically*, are not concrete.

Let $\mathcal{H}$ be the category, whose objects are topological spaces and morphisms are homotopy classes of continuous functions. Freyd wrote in 1970, '$\mathcal{H}$ is not concrete. There is no interpretation of the objects of $\mathcal{H}$ so that the maps may be interpreted as functions (in a functorial way, at least). $\mathcal{H}$ has always been the best example of an abstract category, historically and philosophically. Now we know that it was of necessity abstract, mathematically' (Freyd, 2004, 1).

It is easy to see that a morphism in $\mathcal{H}$ may be a proper class. Namely for any cardinal κ, let X_κ be the κ-pointed star consisting of κ copies of the unit interval which are disjoint except for one point common to all. Each space is contractible (it can be continuously shrunk to a point), and so any pair of continuous maps from one of these spaces into another (including the same space) are homotopic to a constant map. Thus, this morphism has a proper class of members. Freyd shows there is no other representation that avoids this problem.

Beyond products, unions, direct sums etc., Burgess points to quotients[18] as a tool for building the generous arena. Taking such quotients enables finding surrogates in Gödel-Bernays set theory. Kucera (Kucera (1971) (Adámek, 1983, Chapter 6)) proved that *every (possibly abstract) category can be represented as a quotient of a (possibly large) concrete category.* As in the homotopy case, the equivalence classes may well be proper classes even though each object is a set. Suppose, as in MacLane (1971), we work in one 'universe', i.e. assume a single inaccessible[19] κ exists. Now one can modify the Kucera argument by considering categories to have objects in V_κ. Then, the surrogates for the morphisms in Kucera's argument become sets of cardinality κ. Thus, to guarantee surrogates are realized in a concrete category one uses ZFC + an inaccessible cardinal.

The issue is even more complicated. The forgetful functor $U\colon \mathbf{Hilb_r} \to \mathbf{Set}$ gives the category $\mathbf{Hilb_r}$ of Hilbert spaces with linear isometries the structure of a concrete category. However, the necessity of taking the completion means that the category is not closed under increasing unions. Moreover, Lieberman, Rosický, and Vasey (Lieberman et al. (2019b)) prove there is no faithful functor from $\mathbf{Hilb_r}$ to **Set** which preserves directed co-limits (the appropriate abstraction of 'union'). The difficulty arises because, while one can choose representatives for the equivalences

[17]A forgetful functor is a map from a category $\boldsymbol{C}$ into sets that forgets the structure on the elements of $\boldsymbol{C}$ and maps the *morphisms to functions.* A functor is *faithful* if it is injective (which can defined in category theory terms) on morphisms.

[18]Recall the quotient group of a group G by a normal subgroup H is constructed by putting a natural group structure on the collection of E-equivalence classes where aEb if $ab^{-1} \in H$.

[19]A cardinal κ is *inaccessible* if the V_κ is closed under power set and under unions indexed by member of V_κ (i.e. κ is regular).

classes, one *cannot* select one representative of each class in such a way that the composition of two selected representatives is always a selected representative.

Remark 2.2.1. Accessible Categories/Abstract Elementary Classes (Adámek and Rosický (1994), Baldwin (2009)) are treated by both scaffolds. AEC are Shelah's solution to the complications of syntax in infinitary logic. They provide a formalism-free axiomatic treatment of classes of structures. That is, there are (naive set-theoretic) axioms which describe the property of a pair $(\boldsymbol{K}, \leqslant)$, where $\boldsymbol{K}$ is a class of τ-structures for some vocabulary τ, $\leqslant$ is a notion of *strong substructures* where the properties of 'strong' is specified axiomatically but include being a subset. Although this definition depends heavily on the set theoretic notion of subset, the axioms specify conditions on closure under union of chains which are such that the class is closed under colimits and thus it becomes a special kind of *accessible category*[20] (Lieberman (2013)). Lieberman et al. (2020) exploit the connection between (set theoretically defined) cardinality and *internal size*, defined in any accessible cardinal to propose an exciting new generalized 'eventual categoricity problem'. Lieberman et al. (2019a) show how to generalize the fundamental study of abstract independence relations from model theory to category theory.

(Riehl, 2016, 11) emphasizes the unifying aspect of category theory, 'the action of packaging each variety of objects into a category shifts one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them.'

3. Foundations

In this section we explore how the category and model theoretic viewpoints differ regarding size/cardinality. Then we study two approaches to the formalization of set theory: material and structural set theory.

3.1. **Size**

Most mathematical theorems apply either to structures of bounded cardinality (e.g., classifying 3-manifolds) or to structures of arbitrary cardinality. Category theory aims at uniform results for objects (e.g. topological spaces) of all cardinalities. But the techniques require recourse to proper classes[21]. So, Eilenberg and MacLane (1945) introduced the concept of category theory with foundations in Gödel-Bernays set theory. 'Cardinality' does not appear in the index to Mac Lane's text, MacLane (1971). He considers three sizes: finite, small, and large. (Enayat et al., 2017, 3/4) distinguish relative and absolute solutions to the foundational issues arising from the small/large problem. An *absolute solution*, e.g. one inaccessible (MacLane (1971)) or Gödel-Bernays (Eilenberg and MacLane (1945)), fixes the distinction once and for all. Grothendieck adopted a *relative* view and built his general cohomology theory on the basis of *universes*, essentially the existence of a (proper class) of inaccessible cardinals. Countably many universes suffices for most purposes. As argued in McLarty (2010), Grothendieck uses universes to provide a

[20]For regular λ, a category is λ-accessible if it has λ-directed colimits, has only a set (up to isomorphism) of λ-presentable objects , and every object can be written as a λ-directed colimit of λ-presentable objects.

[21]Recall that in ZFC classes are definable pieces of the universe which are equinumerous with the universe. Gödel-Bernays set theory treats these as a different kind of object.

smooth general framework. Authors presume, but rarely detail, that this assumption is not needed in specific applications. In contrast, comparing cardinalities is a central feature of the model theoretic study of algebraic phenomena. In this section we study the necessity of the small/large distinction in category theory and the impact of cardinality on mathematics as mediated by model theory.

One of the main achievements of category theory is to prove theorems (not meta-theorems), e.g., the *adjoint functor theorem*, that apply in many areas of mathematics. To see how proper classes arise in even formulating that theorem we need a few technical definitions but avoid others not essential here.

Definition 3.1.1. *(Awodey, 2010, 24) A category is* small *if it has only a set of objects and there is only a set of morphisms between every pair of objects. Otherwise it is* large *and* locally small *if each 'hom-set' is a set. It is complete if it has all* small limits. *(*For the moment just think, 'all unions indexed by a set are included[22]'.)

We do not define the notion[23]: 'the functor U has a left adjoint'. The three page list of examples in (MacLane, 1971, 85) includes: the left adjoint to the forgetful functor[24] from groups to sets is a free group; from fields to integral domains is the field of quotients, and from complete metric spaces to metric spaces is the completion. An *adjoint functor theorem* provides sufficient conditions on a category $\boldsymbol{C}$ and a functor $\boldsymbol{F}$ for the existence of a left adjoint to $\boldsymbol{F}$. A natural candidate for a general sufficient condition to obtain adjoints is a result of Freyd:

Theorem 3.1.2. *If $\boldsymbol{C}$ is a complete small category then every functor from $\boldsymbol{C}$ to another category $\boldsymbol{D}$ that preserves limits has a left adjunct.*

The *hypotheses* of Theorem 3.1.2 is too strong in two ways. First, as a short Cantor diagonal argument shows, any such category satisfies a quite restrictive condition[25]. (Otherwise a set of size $2^{|\boldsymbol{C}|}$ is imbedded in $\boldsymbol{C}$.) This well-known result is expounded in (Enayat et al., 2017, Theorem 1) or in more detail (Awodey, 2010, Chap. 9).) Moreover, the hypothesis doesn't apply to large categories such as Set or Grp. The standard solution is to weaken the hypothesis by considering *locally small* categories and replacing 'preserves limits' by a more technical *solution set condition* that allows multiple morphisms between a pair of objects. Further applicable sufficient conditions for left adjoints apply in many contexts. Yet, the small/large dichotomy is inevitable.

(Baldwin, 2018, Chapter 8) describes the intricate history of interactions among first order model theory, ZFC, cardinality, and cardinal arithmetic. One of the most influential problems in modern model theory asked about the function, $I(T, \aleph_\alpha)$, the number of (non-isomorphic) models a countable theory T has with cardinality $\aleph_\alpha$. Morley conjectured that this function was non-decreasing for every T (except possibly from $\aleph_0$ to $\aleph_1$). Shelah (1990) solved the conjecture using his *dividing line strategy* ((Baldwin, 2018, 13.3), Baldwin (2021), Shelah (2020)). This strategy successively splits theories by syntactic properties until arriving at two classes: i) *unclassifiable theories* have the maximal number of models in every uncountable

[22]See § 2.2 and the text Awodey (2010) for details. Shulman (2008) and Enayat et al. (2017) give a fuller explanation of the category terminology and different resolutions of the issue.

[23]See Chapters 4 and 5 of Marquis (2009) for the impact of Kan's work on adjoint functors.

[24]See footnote 17.

[25]It is a preorder; that is, it has at most one morphism between any two objects.

cardinality and there is no uniform way to assign invariants determining isomorphism; ii) *classifiable theories* have a local and sometimes global dimension theory; each model is determined by a tree of invariants with countable height; and $I(T, \aleph_\alpha)$ is bounded by a function of $|\alpha|$. Both the classification of theories and the crucial tools for the main gap proof permeate applications across mathematics.

The notion of a universal object for a diagram is fundamental in category theory (MacLane, 1971, page 2). (MacLane, 1971, 123) observes in a particular case that the size of the universal object is bounded in terms of the size of the diagram and the objects in it. But no actual cardinal calculations appear in the book. In contrast, (Hausdorff, 2005, H 1908) introduces a form of universality that is seminal for model theory: M is κ-universal for a class $\boldsymbol{K}$ if $|M| = \kappa$, if $N \in \boldsymbol{K}$ and $|N| \leqslant \kappa$ then there is an embedding of N into M. He proves the existence of $\aleph_{n+1}$-universal linear orders if $\aleph_{n+1} = 2^{\aleph_n}$.

Generalizing ideas of Hausdorff, Fraïssé, and Jónsson, Morley and Vaught (1962) proved the existence of saturated[26] models of any complete theory in all regular cardinals using the GCH. A minor variant (special) extends the result to singular cardinals. An important consequence of Shelah's classification theory is that GCH can be replaced by model theoretic hypotheses for 'tame' theories: A stable theory T has a saturated model in cardinality κ if and only if $\kappa^\omega = \kappa$ and a superstable one has saturated models in any cardinal $\geqslant 2^{\aleph_0}$. Intriguing problems in in axiomatic set theory arise from determining which theories have universal models in which cardinals (Shelah (2020); Baldwin (2021)).

The notion of a saturated (homogeneous-universal) model M plays a fundamental role in model theory for finding 'universal domains'. Model theory papers routinely begin, 'we work in a suitable *monster model*'. This means choose a domain (necessarily, for an arbitrary theory, whose cardinality is strongly inaccessible) large enough to encompass any construction from the objects actually under study (Baldwin, 2018, 5.2). As in the last paragraph, for 'sufficiently tame' theories, these inaccessible cardinals are not necessary.

Problem 3. *Further explore the distinctions between relative and absolute 'solutions of size'. What are the mathematical consequences of the choices; what are the philosophical justifications?*

3.2. Material and Structural Set Theories as Foundations

Notation 3.2.1 (Weak set theories)**.** *A well-known[27] family of bi-interpretable weak set theories (Mathias (2001); Shulman (2019)) include these four.*

1. *structural set theories:*
   1. *a well-pointed topos with a natural numbers object and with the axiom of choice;*
   2. *ETCS, Elementary Theory of the Category of Sets; (Lawvere (1964)).*
2. *material set theories:*
   1. *BZC, bounded Zermelo with choice[28];*
   2. *Mac Lane set theory (finite order arithmetic McLarty (2020)).*

[26] κ saturated means κ-universal and κ-homogeneneous: isomorphic strictly smaller submodels of M are automorphic in M. M is saturated means $|M|$-saturated.

[27] These equivalences date back to Mitchell (1972).

[28] BZC has comprehension only for Σ_0-formulas. We add C to any such abbreviation when choice is added. The Bourbaki foundation omitted the axioms of foundation and replacement.

Each of these weak theories omit the axiom of replacement. Thus they do not have the cardinals[29] $\aleph_\omega$ *or* $\beth_\omega$.

From a set theoretic standpoint ETCS is very weak. Mathias (2001) identifies $V_{\omega+\omega}$ as a natural model for Mac Lane set theory and gives the axioms in ϵ-style and a chart organizing various weak set theories. Shulman is a precise reference for the known ability to extend the axioms of ETCS by considering axioms which vastly strengthen ETCS: including full separation, collection, and replacement axioms. These extensions leave the formalism-free world and require explicit set theoretic syntax to formulate axiom schemes. He concludes that ETCS plus 'structural replacement', is equi-consistent and indeed mutually interpretable[30] with ZFC ((Shulman, 2019, Cor. 8.53)).

Remark 3.2.2 (Large Cardinals)**.** As a first approximation, any cardinal whose existence implies the consistency of ZFC is termed 'large'. Thus, any inaccessible cardinal is large. From the set-theoretic standpoint merely inaccessible cardinals are quite small. Since large cardinal axioms (e.g. measurable cardinals) are properties of a particular cardinal κ they can usually be easily rephrased as category theoretic properties of the cardinal κ. There is no real difficulty in *extending structural set theories by large cardinal axioms.*

Unlike category theory, set theory and model theory use the gradations of large cardinals. In particular, (Maddy, 2011, 47-51) expounds the equivalences between appropriate large cardinals, the projective hierarchy on first order definable subsets of $(R, +, \times, 0, 1, N)$, and determinacy axioms.

In a little known example (Shelah (1982) (Baldwin, 2021, 2.3)), Shelah invokes a smallish large cardinal (a beautiful cardinal[31]) to give an account of the main gap that, by counting non-mutually embeddible rather than non-isomorphic models, is more resistant to forcing. Several recent results on eventual categoricity in AEC, (e.g., Boney (2014); Shelah and Vasey (2018), Remark 2.2.1), require the existence of a proper class of strongly compact cardinals, while other AEC results require much smaller large cardinals. Thus, large cardinals are essential to placing all of model theory in the generous arena.

Remark 3.2.3 (CCAF)**.** Lawvere (1966) introduces the 'category of categories' as a foundation for mathematics' as a first order theory with no ambient set theory. Enayat et al. (2017) lay out in detail two distinct goals for category theory, as Feferman had articulated in a series of works on Lawvere's proposal, labeled R: ('unrestricted existence', e.g. to have the category of all categories[32] (CAT) exist) and S: (the desire to contrast small and large sets: e.g. Grp vs. GRP).

[29]More algebraically, (Mathias, 2001, 9.32) notes that Mac Lane set theory *cannot* prove that for every n, one can iterate the process of taking the dual vector space n-times (starting with $\mathbf{R}[x]$). Mathias (1992) has an interesting dialog between Mathias and Mac Lane.

[30](MacLane and Moerdijk, 1992, 343), citing Mitchell (1972) write that the basic interpretations are actually invertible and so a bi-interpretation of (a weak version) of replacement holds.

[31]A beautiful cardinal is much larger than ω universes, but still smaller than a weakly compact.

[32]Expanding on Enayat et al. (2017), Gorbow regards the following definition of CAT as appropriate for (R): 'The category determined by (1) having the set of all categories as objects, (2) for any categories A, B, having the set of all functors from A to B as morphisms from A to B, (3) having identity functors as identity morphisms, and (4) having composition of morphisms defined by composition of functors.' Contra to Lawvere (1966); McLarty (1991), he does not require that CAT admit both finite products and exponentiation.

The first suggests Russell's paradox (but to establish the paradox one needs information (included in R) about the ability to construct from given categories). The precise meaning of property R is central. In Ernst (2015)'s understanding, such a system must be inconsistent. Enayat et al. (2017) summarise and extend the different interpretations (Feferman (2004) and Shulman (2008)) and show consistency of certain meanings of (R) with respect to various material set theories e.g. NFU (New Foundations with ur-elements), NFUA etc. Note that while the consistency of NF remains contentious, Jensen proved the extension NFU (by allowing ur-elements) is consistent relative to ZFC. NFU has a stratified notion of membership and admits a universal set. Thus Enayat et al. (2017) characterize various notions of *natural implementability* depending on the image of the usual category theoretic notions (Cat, Set) in the interpreting theory. Under their weakest notion of natural interpretability they show NFUA interprets ZFC plus n Mahlo cardinals, for each standard n, on a part of its domain and a small/large distinction. Therefore, the category theory of MacLane (1971) can be implemented in NFUA on that subdomain. But the salience of the precise meaning of natural implementation is emphasized as McLarty (1992) showed that under any reasonable definition of function', after such an interpretation the category SET of *all* sets is not closed under direct products and exponentiation. On the other hand, Forster (2007) points out that such issues have frequently appeared with NF and solved by finding a different definition of the troublesome concept (that is equivalent in ZFC). Gorbow (in correspondence) has emphasized the need for further investigation of the methodological gain for category theoretical consideration of CAT and SET.

Remark 3.2.4 (Metamathematical Corral)**.** Most work in the corral is driven by the goal of finding implications among various principles and truly sufficient conditions which are not overly broad. Set theory itself is driven by internal problems such as the singular cardinals problem and problems arising in combinatorial set theory. Friedman's Friedman (1998) program produces concrete combinatorial properties of the natural numbers which require large cardinals. Examples arising from ordinary mathematical practice include the Whitehead problem and others (Section 2.1). Farah's ICM address (Farah (2014)) indicates the impact of model theory, descriptive set theory, and forcing constructions on the study of operator algebras. In these examples, material set theorists have the advantage; despite the interpretablity discussed above, ZFC provides a friendlier workplace.

4. Comparisons

We quickly assess how our two scaffolds meet three characteristics of a scaffold: i) local foundations, ii) unifying methods, iii) productive guidance. Local foundations are found by specifying in first order logic either a complete theory or a category with appropriate morphisms. Unity[33] and Guidance are provided by the hierarchical classification[34] of first order theories or by Grothendieck style axiomatization of families of categories such as abelian category, additive category, topos.

[33]Unity arises in several ways: Bi-iterpretablity in model theory and natural equivalences in category theory connect different areas. Category theorem may view the same concept from different perspectives; (Ashfaque, 2020, 1.6) gives twelve categories for studying elliptic curves.

[34]See the map of the universe at `http://www.forkinganddividing.com`.

Major differences are: a) explicit syntax/semantics vs. formalism-free and b) use of cardinality vs. large/small. Independence results in algebra and topology are naturally expressed through ZFC and model theoretic formalization (Remark 2.1.2). Reformulating these in category theory would not only require mimicking the forcing arguments in material set theory but formulating categories where the 'internal structure' of algebras was revealed.

Particular structural and material set theories are bi-interpretable at all levels from BZ to $ZFC + LC$ (§ 3.2). Thus, if we consider *Shared Standard of Proof* to be a question of which statements are theorems in the system[35], and *Generous Arena* as giving surrogates for all mathematical entities there is nothing to choose between the approaches. The level within each approach makes a huge difference. Both model theory and set theory apply replacement, large cardinals, and forcing to obtain results in traditional mathematics (§ 2.1, Remark 3.2.4).

As noted in our introduction, Mac Lane proclaimed, 'inaccessible cardinals palpably have nothing to do with [category theory]'. We argued (§2.2) that Kucera's theorem exhibits the connection if the foundation is to be in set theory rather than class theory. Either material or structural set theories admit conservative extensions to class theory at each level. Recall that Mac Lane set theory sees no cardinal numbers $\geqslant \aleph_\omega$. As Mclarty has observed, there is a proper class containing exactly the $\aleph_n$ in the natural class extension of Mac Lane set theory. So Mac Lane requires proper classes (§2.2, 3.1) that are seen in ZFC as much smaller than inaccessibles. The McLarty (2020) proof that Fermat's conjecture is provable in Mac Lane set theory, using a suitable conservative class theory extension, demonstrates that only very weak class theory is needed for a classical problem. To obtain all of contemporary mathematics the *Generous Arena* requires a set theoretic bases of both replacement (Borel determinacy Friedman (1971)) and large cardinals.

Mathematicians of various stripes might object to the naturalness/accessibility of surrogates or proofs in either of the proposed foundations. Indeed, Tao suggests the Ax-Grothendieck theorem[36] supports the following:

> I have always been fond of the idea that model-theoretic connections between objects (e.g. relating two objects by comparing the sentences that they satisfy) are at least as important in mathematics as the more traditional category-theoretic connections (where morphisms are the fundamental connective tissue between objects) or topological connections (where the objects are gathered into some common topological space or metric space in order to compare them). Terry Tao [37]

A great advantage of ZFC is the simple set-up in terms of a single binary membership relation. In Lemma 2.1.2 we noted that fixing a vocabulary suitable to the topic studied resolves purported ambiguities about membership. This is a very soft and uniform form of typing. As Shulman (2013) points out, category theory provides a more rigid type system.

Alternatively, (Scott, 1974, 10) provocatively asserted, 'there is only one satisfactory way of avoiding the paradoxes, use some form of the *theory of types*. That

[35]This is a mathematical question as opposed to the 'sociological' ((Maddy, 2017, 296)) standard of *belief* in (§1) of proof from a common foundation (ZFC).

[36]A 1-1 polynomial map between algebraic varieties is onto (e.g. (Baldwin, 2018, 102))

[37]`https://quomodocumque.wordpress.com/2012/09/03/mochizuki-on-abc/?#comment-10506`

was the basis of both Russell's and Zermelo's intuitions.' He goes on to outline a *cumulative* theory of types where all the sets up to a certain level form a partial universe which is a set. And the type of a set is the level at which it appears.

Problem 4. We have given descriptions of the mathematical and methodological differences between two scaffolds to engage the following problems. *Provide a more systematic philosophical analysis and justification of criteria for a successful scaffold. Find a notion of 'grounding' which accounts for the unease category theorists feel with ZFC (and model theorists with structural set theory) as the Foundation despite the equivalences between material and structural set theories.*

Celluci (2000) distinguishes between open and closed systems. He requires a closed system to have an immutable set of axioms. Maddy's notion of *metamathematical corral*, recognizes the essence of Gödel incompleteness; *there must be a continuing search for new axioms.* All of mathematics may be an open system consisting of cooperating (and perhaps changing) closed systems.

We described two Foundational systems. They pursue different conceptions of 'set'; each allows revision of axioms. Why choose between them? Rather than a Foundationalist view, one may just consider several mutually reinforcing Foundations as an open system of Mathematics.

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