What is a geometric proof? Reflections on De Zolt's axiom
Notre Dame PhilMath Workshop

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Hartshorne's question

> *We avoid the use of the words 'greater' and 'lesser' because these imply the existence of an order relation among figures which we have not yet established. In fact, the existence of an order relation for content depends on (Z) (Exercise 22.7). [We will also see that 'if squares are equal then their sides are equal' follows from (Z) (Exercise 22.6).]*
>
> *I do not know of any purely geometric proof of axiom(Z)* from the definition of content [area] we have given. *... (Z) holds however, whenever there is a measure of area function defined in the geometry.*
> *([Har00, p.202])*

He repeats the same sentiment in more detail on page 210 with additional detail. 'The proof [of De Zolt and of area function] is analytic in that it makes use of the field of segment arithmetic and similar triangles.' ([Har00, p.210]).

Questions

These passages raises several questions. What is De Zolt's axiom (Z)?

What is De Zolt good for?

What is a geometric proof?

Answers

Hartshorne's version of De Zolt (Z)

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De Zolt version

If a polygon is divided into parts in a given way, it is not possible, when one of these parts is omitted to recompose the remaining parts in such a way that they cover entirely the polygon.
(De Zolt 1881 p. 12)

One distinction is that De Zolt is referring to scissors congruence (dissection) which we call *equidecomposable*.

While, Hartshorne uses *equal content* which we call equicomplementable. In this case subtraction is allowed as in Euclid I.35.

Equicomplimentablity

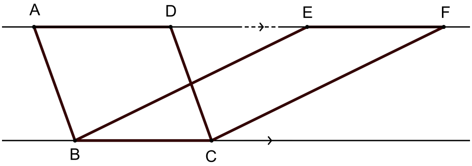

Figure: Euclid I.35

Confronting the Dilemma

Two notions of geometric proof

1. A proof in 'set theory' about geometry
2. A proof formalizable from a set of axioms for geometry.

19th century rejected Euclid's general assumption.
CN5: The whole is greater than the part.
De Zolt's axiom is an attempt to give a precise assertion of CN5 for geometry.
We investigate 'same magnitude' in several geometric contexts.
Properties of such equivalence relations depend on a number of factors:

- background theory
- definition of figures
- precise description of the equivalence relation
- dimension of the space

Background Theories

Axiom systems for geometry

1. First-order axioms:

   HP, HP5 We write HP (Hilbert plane) for Hilbert's incidence, betweenness, and congruence axioms. We write HP5 for HP plus the parallel postulate. HP is often known as absolute or neutral geometry.

   EG The *axioms for Euclidean geometry*, *HP*5 + circle-circle intersection.

2. Hilbert's group continuity axioms, must be formalized in infinitary and second-order logic

   Archimedes The sentence in the logic $L_{\omega_1,\omega}$ expressing that any segment is contained in a finite number of copies of any other.

   Dedekind Every Dedekind cut is realized.

Relevant equivalence relations

Definition

1. Two figures P, P' are *equidecomposable* if they each can be written as a non-overlapping union of the same number of pairwise congruent atoms.
2. Two figures P, P' are *equicomplementable* if there are other figures Q, Q' such that:
   1. P and Q are non-overlapping;
   2. P' and Q' are non-overlapping;
   3. Q and Q' are equidecomposable
   4. $P \cup Q$ and $P' \cup Q'$ are equidecomposable.
3. Two figures P, P' are *equimeasured* (by α) if there is a measure of content function α such that $\alpha(P) = \alpha(P')$
4. For a subgroup G of the group of rigid motions of the space, two figures P, P' are *G-equivalent* if there is $g \in G$ with $g(P) = P'$.

Area Assumptions to be formalized

Among Hartshorne's list of

Euclid's implicit assumptions about area

1. The whole is greater than the part.
2. equal squares have equal sides.

Hartshorne proves

In models of *EG*, de Zolt for equicomplementation implies equal squares have equal sides.

The same argument works replacing square by 'rectangles with the same height'.

We prove the converse

In models of *HP*5, 'equal rectangles with the same height have equal sides' implies de Zolt for equicomplementation.

General framework

Definition: Admissible equivalence relations

Let $n = 2$ or 3.

1. An atom is an n-dimensional convex hull of $n + 1$ points in n-space.
2. A figure in n-space is a non-overlapping (intersection cannot have an interior) union of atoms.
3. An equivalence relation E on figures is admissible if
   1. Congruent atoms are E-equivalent;
   2. For disjoint figures P, P' and Q, Q', if $E(P, P')$ and $E(Q, Q')$, then $E(P \cup P', Q \cup Q')$.
4. The 'De Zolt order' with respect to an admissible equivalence relation on figures with non-empty interior is defined by $[P] \leq [Q]$ if $(\exists R)[Q] = [P + R]$.

How can one measure area?

global method Fix a unit; say, a square; tile the plane with congruent squares. Then to measure a figure, continually refines the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure.

local method (Hilbert) Triangulate a figure with finitely many triangles, which are each assigned area

Euclidean Geometry $\frac{bh}{2}$

Hyperbolic Geometry $(0, \delta)$ or $(1, \delta)$ depending on the size of the defect δ

and the area of the figure is the sum of the areas of the triangles.

representative method Fix a representative of each equivalence class.

The first two are described in [Bol78]; the third in [Har00, §36]; we introduce the fourth here.

Measure of Area function

Hartshorne provides a general definition that applies to all three.

Definition

A *measure of area function* on figures in n-space is a function α with values in *an ordered semigroup G*[a] satisfying:

1. Congruent *atoms* have the same value.
2. For disjoint figures P, Q, $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.

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[a] [Hil62] does not introduce the term semigroup. This is unsurprising since the term only came into use in the next decade [Hol14]. But Hilbert was avoiding such an algebraic slant. An ordered semigroup is a structure $(*, <)$ such that $*$ is associative and satisfies $(\forall x, y, z) x < y \rightarrow (xz < yz \wedge zx < yz)$.

Scales

Definition

1. A *figure type* $\boldsymbol{A}$ is an explicit description of a figure given by a first order formula $\phi(\overline{x})$; e.g. square, triangle, equiangular n-gon, regular pentagon, rectangle with a fixed height.
   Suppose P and P' realize $\boldsymbol{A}$; fix an enumeration of a_i of P and b_i of P'.
2. A figure type $\boldsymbol{A}$ is a *scale* if whenever P, P' each satisfy $\boldsymbol{A}$, if either
   1. for some i, $a_i a_{i+1} \approx b_i b_{i+1}$
      then P and P' are congruent.
      Example: for some n, $\boldsymbol{A}$ is the collection of regular n-gons.
   2. Suppose there is a fixed segment AB such that $AB \approx x_0 x_1$ is subformula of ϕ so $AB \approx a_0 a_1 \approx b_0 b_1$ and a fixed i such that for any P, P' satisfying $\boldsymbol{A}$ if $x_i x_{i+1} \approx y_i y_{i+1}$ then $P \approx P'$.
      Example: Rectangle of fixed height.

Scaled and well-scaled

Definition

1. An equivalence relation on figures is *scaled* (by a scale $\boldsymbol{A}$) if every equivalence class contains at least an instance of $\boldsymbol{A}$.
2. An equivalence relation E on figures is *well-scaled* (by a scale $\boldsymbol{A}$) if P, P' satisfy $\boldsymbol{A}$ and $E(P, P')$ implies $P \approx P'$. (Here and below $P \approx P'$ means P and P' are congruent.)

Main Theorem

The Well-Scaled Theorem

Work in neutral geometry. Suppose E is scaled by a scale $\boldsymbol{A}$.

1. If E is well-scaled by $\boldsymbol{A}$ then there is measure function for E, the equivalence classes are linearly ordered, and E satisfies De Zolt.
2. if E satisfies De Zolt, E is well-scaled.

Corollary

If the plane π satisfies *HP*5 (*EG*) and the rectangle (squares) property for equicomplementation then it satisfies De Zolt
and there is a measure of area function on π.

Proof. By II.14 of [Euc56], equicomplementation is scaled by squares (EG) and by I.44 by rectangles with fixed height. It is then well-scaled by the square (rectangle) property.

Why is De Zolt a worry?

Euclid implicitly gives a formula for the area of triangle.

But he uses the method of exhaustion to show ‘triangular pyramids of the same angle are to each other as to their bases.’

Removing such limit processes is one of the goals of 19th century rigorizing.

fact

[Wallace-Bolyai-Gerwien Theorem] Two polygons *in an Archimedean plane* are equidecomposable (scissors congruent) if and only if they have the same measure of area.

Fact (Dehn-Snyder Theorem)

Two polyhedra in $\Re^3$ *are equidecomposable iff they have the same volume and the same Dehn invariant.*

Non-Archimedean planes

A crucial contribution of the Grundlagen is to show the geometrical results of Euclid do NOT depend on Archimedes.

Fact

There is a non-Archimedean plane satisfying HP5.

Proof. Fix points A and C on a plane satisfying $HP5$. Consider the set of sentences[1] $\phi_n(A, B, C)$ where $\phi_n(A, B, C)$ asserts the B_n are decreasing, $B_1C \approx AB_1$ and for each i, $2 \leq i \leq n-1$ $B_iB_{i-1} \approx AB_i$. Since each ϕ_n is true for an appropriate choice of B_{n+1} to witness B, the compactness theorem for first order logic implies there is a B_∞ such that $\phi_n(A, B_\infty, C)$ for every n. Then B_∞ is an infinitesimal. $\square_2$

[1] In symbols, using B as an antisymmetric (to avoid extra notation for order) strict betweenness predicate:

$$\exists B_1, \ldots B_n(B(C, B_1, B_2) \wedge \bigwedge_{2 \leq i \leq n-2} B(B_i, B_{i+1}, B_{i+2}) \wedge (B(B_{n-1}, B_n, A) \wedge B(B_n, B, A)$$

Archimedes matters

Lemma

There is a model of HP5 where Equidecomposition is not scaled by squares.

Proof. We show there is a model of $HP5$ with a parallelogram $EBCF$ that is not equidecomposable with a square. Consider Hilbert's example in a Cartesian plane π over a non-archimedean extension F of the reals.

Fix a copy of the natural numbers in F as natural number multiples of of a fixed segment. We say a point A in π is finite, if both coordinates of A are less than some natural number n.

Proof

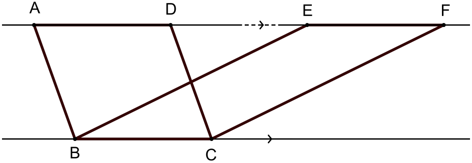


Figure: Euclid I.35

Using the diagram for Euclid I.35, suppose A, B, C, D are finite (standard) but E and F are not;the length of EF must be standard since $EF \approx BC$. We know $ABCD$ is equidecomposable with a (standard) square, since all its sides are finite. Since $\pi \models HP5$, Hilbert's measure of area function gives the same finite area to both $ABCD$ and $EBCF$. But $EBCF$ is not equidecomposable with any finite square, as it has a side of infinite length. But then $EBCF$ is not equidecomposable with any square. Since de Zolt implies that at most one congruence class of squares a square appears in an equivalence class.

Geometric Proof

Three topics

1. area by equicomplementation
2. multiplication
3. proportionality

Euclid's path is from 1) to 3) (using Archimedes) and 2) is a corollary (for Descartes).
19th century worries about the rigor of 1).
Hilbert's path is from 2) to 3) to 1).
But the only use of 3) in Hilbert's area theory is

Theorem

Any of the three choices of base for a triangle give the same value for the product of the base and the height.

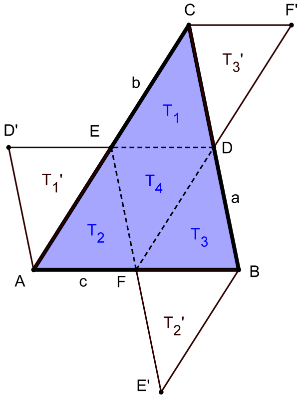
So Hilbert is actually going from 2) to 1).

Independence of base

We prove the following without use of proportionality.

Theorem

Any of the three choices of base for a triangle give the same value for the product of the base and the height.



Proof

Take triangle ABC. Draw midpoints D, E, F of a, b, c respectively. Let h_a, h_b, h_c be the heights on a, b, c respectively. The rigid motions needed below are described and proved to exist in any Hilbert plane in [Har00, Section 17].

Rotating T_1, T_2, T_3 about E, F, D respectively gives us parallelograms having equal content with triangle ABC and with each other. Therefore the area formulas for each parallelogram are equal.

Rotating T_1 about E results in parallelogram ABDD' with height 1/2 h_c and base c and area 1/2 (h_c)(c).

Rotating T_2 about F results in parallelogram BCEE' with height 1/2 h_a and base a and area 1/2 $(h_a)(a)$.

Rotating T_3 about D results in parallelogram CAFF" with height 1/2 h_b and base b and area 1/2 (h_b)(b).

Hence 1/2 (h_a)(a), 1/2 (h_b)(b), 1/2 (h_c)(c) are all formulas for the area of the triangle, i.e. 1/2 (base)(height) is independent of the choice of base.

Conclusion

Hartshorne asks about the significance of the existence of a measure function in establishing the theory of area.

Equidecomposablity and thus equicomplementability are described by infinite *disjunctions* of first order formulas. Thus, (as grasped by De Zolt), non-equidecomposablity is a treacherous notion; to establish it requires checking infinitely many possibilities. Moreover, these possibilities are too wild to support an induction.

The independence of Hilbert's measure of area on triangulation shows that a figure with area g with respect to some triangulation is equicomplementable with a triangle of height 1 and base g.

Thus, two triangles are equicomplementable if and only if some/ any calculation of their areas give the same value.

This immediately yields either the rectangle property or De Zolt.

Any well-scaled equivalence relation will work. But, like Hartshorne, we see a complete proof only using Hilbert's function.

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