The Vaught Conjecture
Do uncountable models count?

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Abstract

We give a model theoretic proof, replacing admisssible set theory by the Lopez-Escobar theorem, of Makkai's theorem: Every counterexample to Vaught's conjecture has an uncountable model which realizes only countably many $L_{\omega_1,\omega}$-types. The following two results are new. **Theorem I.** If a first order theory is a counterexample to the Vaught conjecture then it has $2^{\aleph_1}$ models of cardinality $\aleph_1$. **Theorem II.** $[2^{\aleph_0} < 2^{\aleph_1}]$ If a sentence of $L_{\omega_1,\omega}$ is a counterexample to the Vaught conjecture then it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

In this paper we prove several properties of counterexamples to the Vaught conjecture. Specifically, results concern the number of models the counterexample has in power $\aleph_1$. One of these results was proved 30 years ago using admissible model theory; we give a more straightforward argument. The following question guides our discussion. Is the Vaught Conjecture model theory?

Here are some possible ways in which this question would have a clear answer. Shelah, Buechler, Newelski have shown using rather difficult techniques from stability theory that the conjecture holds for first order theories that are 'simple' from the stability theoretic standpoint: ω-stable or superstable with finite U-rank. If a counterexample were found for a first order theory of slightly greater complexity (e.g. a stable but not superstable first order theory), this would indicate the issue was a model theoretic one. If on the other hand, a uniform proof for sentences of $L_{\omega_1,\omega}$ were given using methods of descriptive set theory, then it would not be a model theoretic problem. The results below give partial answers to the following methodological questions.

What specific model theoretic as opposed to descriptive set theoretic techniques can attack the problem? Can one use more direct model theoretic arguments to obtain some result of admissible model theory?

I would argue the problem is model theoretic if its solution is different for $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$. So we will investigate the differences between properties known about counterexamples to the Vaught Conjecture formulated in $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$. Note that Theorem I of the abstract (for first order logic) is proved in ZFC; our proof of Theorem II (for $L_{\omega_1,\omega}$) relies on additional set theory; we close with thoughts about avoiding the set-theoretic hypothesis.

Much of model theory is concerned with models of arbitrary cardinality and with properties that in some way depend explicitly on cardinality. We pursue the theme, 'Do uncountable models count?' by noticing several results about the Vaught conjecture which revolve around the properties of uncountable models (and even the role of arbitrarily large models). Must a counterexample to VC in $L_{\omega_1,\omega}$ have a model of power $\aleph_2$ or even $\aleph_1$? Hjorth's contribution to this volume provides an answer to the last question– showing that if there is a counterexample to Vaught's conjecture then there is one with no model of cardinality $\aleph_2$. We use his result in the proof of Theorem II.

In Section 1, we provide some background on the nature of 'complete' sentences in $L_{\omega_1,\omega}$ and note that issues arise with both the upward and downward Löwenheim-Skolem theorem when generalizing to infinitary logic. We make a brief excursion into Abstract Elementary clases to illustrate quintessentially 'model theoretic' techniques and then adapt methods of Shelah to provide a model theoretic proof of Makkai's theorem in Section 2. Finally, in Section 3, we prove Theorems I and II from the abstract. The proof of Theorem II is a synthesis of old work of Shelah on constructing models of cardinality $\aleph_2$ with recent work of Hjorth, constructing from an arbitrary counterexample to Vaught's conjecture one that has no model of cardinality $\aleph_2$.

1 Complete Sentences

Using both the upward and downward Löwenheim-Skolem theorem, it is easy to see that a first order theory that is categorical in some infinite cardinality is complete. The analog in the $L_{\omega_1,\omega}$-case requires some analysis. To begin with there are several possible meanings of complete depending on how much of $L_{\omega_1,\omega}$ is considered. Let us formalize what constitutes a useful piece of $L_{\omega_1,\omega}$.

Definition 1.1 *A* fragment Δ *of* $L_{\omega_1,\omega}$ *is a subset of* $L_{\omega_1,\omega}$ *closed under subformula, substitutions of terms, finitary logical operations and such that: whenever* $\Theta \subset \Delta$ *is countable and* $\phi, \bigvee \Theta \in \Delta$ *then* $\bigvee\{\exists x\theta : \theta \in \Theta\}$*,* $\bigvee\{\phi \wedge \theta : \theta \in \Theta\}$*, and* $\bigvee(\{\phi\} \cup \Theta)$ *are all in* Δ*.*

Definition 1.2 *Let* $\phi \in \Delta \subset L_{\omega_1,\omega}$*.*

1. ϕ *is* complete for $L_{\omega_1,\omega}$ *(or just* complete*) if for every sentence* ψ *of* $L_{\omega_1,\omega}$*, either* $\phi \to \psi$ *or* $\phi \to \neg\psi$*.*

2. *For any countable fragment Δ, ϕ is* complete for Δ *if for every sentence $\psi \in \Delta$, either $\phi \rightarrow \psi$ or $\phi \rightarrow \neg\psi$.*

This is an important distinction because the downward Lowenheim Skolem theorem is true for theories in any countable Δ and thus for sentences in $L_{\omega_1,\omega}$ but it is not true for arbitrary theories in $L_{\omega_1,\omega}$. In particular it easy to find examples of uncountable structures which have no countable $L_{\omega_1,\omega}$-elementary submodel. Note that a sentence is complete if and only if it is a Scott sentence (a sentence of $L_{\omega_1,\omega}$ which completely describes a (countable) model).

A complete sentence of $L_{\omega_1,\omega}$ is $\aleph_0$-categorical, trivializing Vaught's conjecture. In Section 2 we will use of Δ-complete counterexamples. And in Section 3 we will make crucial use of sentences that are complete.

Definition 1.3 *Let Δ be a fragment of $L_{\omega_1,\omega}$.*

1. *A model is* Δ-small *if it realizes only countably many Δ-types over the empty set.*
2. *A model is* small *if it realizes only countably many $L_{\omega_1,\omega}$-types over the empty set. That is, it is Δ-small for $\Delta = L_{\omega_1,\omega}$.*

Note that M is small if and only M is Karp-equivalent (i.e. $L_{\omega_1,\omega}$-equivalent) to a countable model. Recall:

Definition 1.4 *A first order theory is* small *if for every n, it has only countably many n-types over the empty set.*

Note that by the downward Löwenheim-Skolem theorem every model of a complete sentence of $L_{\omega_1,\omega}$ is small. So every $L_{\omega_1,\omega}$-complete sentence is scattered in the following sense.

Definition 1.5

1. *$S_n(\sigma, \Delta)$ denotes the collection of n-types in Δ that are realized in models of σ.*
2. *A sentence σ of $L_{\omega_1,\omega}$ is* scattered *if for every countable fragment Δ of $L_{\omega_1,\omega}$, $S_n(\sigma, \Delta)$ is countable for each n.*

Of course, there are scattered sentences which are not complete. But we show in Section 2 the following partial converse. If ϕ is scattered, there exists a complete ϕ' with an uncountable model such that $\phi' \rightarrow \phi$. More precisely, we prove below that:

Theorem 1.6 *If the $L_{\omega_1,\omega}$-sentence ψ has a model of cardinality $\aleph_1$ which is Δ-small for every countable fragment Δ of $L_{\omega_1,\omega}$, then ψ has a small model of cardinality $\aleph_1$.*

If σ is scattered and $\sigma' \to \sigma$, then σ' is scattered. In his landmark proof that a counterexample to Vaught's conjecture has at most $\aleph_1$ models of cardinality $\aleph_1$, Morley [10] established, by essentially descriptive set theoretic arguments:

Theorem 1.7 (Morley) *If σ is a counterexample to VC, σ is scattered.*

Shelah 'reduces' Morley's categoricity theorem for $L_{\omega_1,\omega}$ to complete sentences. This reduction involves a crucial model theoretic technique. Prove a theorem for arbitrary vocabularies τ. In fact, this reduction applies to more general questions concerning the number of models in $\aleph_1$ if the sentence has few models in $\aleph_1$.

Theorem 1.8 *Let ψ be a* complete *sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ and a first order τ'-theory T such that reduct is a* 1-1 *map from the* atomic *models of T onto the models of ψ.*

If ψ is not complete, the reduction is only to 'finite diagrams' [12]. This is a very important distinction, the arguments given in Section 3 depend heavily on working in an atomic class. This 'reduction' is not direct. In order to deduce categoricity for an arbitrary $L_{\omega_1,\omega}$-sentence, stronger results than transfer of categoricity must be proved for complete $L_{\omega_1,\omega}$ sentences ([15] expounded in [3, 1]).

There are two different arguments to obtain this reduction. If the sentence ψ has arbitrarily large models the result is is a fairly straightforward argument with Ehrenfeucht-Mostowski models.

Theorem 1.9 *Let ψ be an $L_{\omega_1,\omega}(\tau)$-sentence which has arbitrarily large models. If ψ is categorical in some cardinal κ, ψ is implied by a consistent complete sentence ψ', which has a model of cardinality κ.*

Without this assumption, the argument is more difficult and relies on a theorem of Keisler [7]; but this argument needs only few models in $\aleph_1$, not $\aleph_1$-categoricity.

Theorem 1.10 *For any $L_{\omega_1,\omega}$-sentence ψ and any fragment Δ containing ψ, if ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then for any $M \models \psi$ of cardinality $\aleph_1$, M realizes only countably many Δ-types over the empty set*

Theorem 1.11 *Let ψ be an $L_{\omega_1,\omega}(\tau)$-sentence If ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$, ψ is implied by a consistent complete sentence ψ', which has a model of cardinality $\aleph_1$.*

Proof. By Theorem 1.10, there is a model of power $\aleph_1$ which is Δ-small for every countable fragment Δ. But then by Theorem 1.6, there is an uncountable small model N of ψ and the Scott sentence of N is as required. $\square_{1.11}$

2 Constructing models of larger power

In this section we reprove (in one case much more simply) old theorems showing that a counterexample to Vaught's conjecture must have two models in power $\aleph_1$. But we first take a brief excursion through Abstract Elementary Classes to see what I take as the essence of 'model theoretic' methods – arguments involving the direct constructions of models.

Vaught's conjecture concerns the set of countable models of a 'theory'. An Abstract Elementary Class (AEC) is one of the most abstract formulations of 'theory' [19, 17, 4, 1]. A class of L-structures and a notion of 'strong submodel' $\prec$, $(\boldsymbol{K}, \prec)$, is said to be an *abstract elementary class* if both $\boldsymbol{K}$ and the binary relation $\prec$ are closed under isomorphism and satisfy a collection of conditions generalizing those of Jonsón's for constructing homogeneous universal models. In particular, the class must be closed under $\prec$-increasing chains. The class is presented as a collection of models and a further crucial requirement is the existence of a Löwenheim number for the class.

So a more extreme form of 'Vaught's conjecture is model theory' would be to prove it for any AEC. But this fails. The set $\boldsymbol{K} = \{\alpha : \alpha \leq \aleph_1\}$ with $\prec$ as initial segment is an AEC with $\aleph_1$ countable models. But the counterexample has no large models. (The Löwenheim number requirement forbids using all ordinals as the example.) The upward Lowenheim-Skolem theorem is true for $L_{\omega,\omega}$ but not $L_{\omega_1,\omega}$. So this excursion into the abstract leads us to some more precise questions.

In the absence of the upward Löwenheim-Skolem theorem, how can one construct models of larger cardinality? For the moment we continue in the context of AEC.

Definition 2.1 *$M \in \boldsymbol{K}$ is* maximal *if there is no $N \neq M$ with $M \prec N$.*

We begin my mentioning some fairly easy principles. Much more technical arguments are need to obtain the hypotheses of these lemmas. Obviously,

Lemma 2.2 *In any AEC, if there is no maximal model of size λ, there is a model of size λ^+.*

As,

Lemma 2.3 *In any AEC, if there a strictly increasing $\prec$-sequence M_α, $\alpha < \lambda^+$ of models of size λ, there is a model of size λ^+.*

Even more,

Lemma 2.4 *If the AEC $\boldsymbol{K}$ is λ categorical and there exists N a proper extension of M with cardinality λ, there is a model of cardinality λ^+.*

Now we specialize to $L_{\omega_1,\omega}$. We sketch the analysis of Harnik and Makkai [5] to show every counterexample to VC has an uncountable 'large' (not small) model. For this they introduce another technical meaning for large; now describing a sentence rather than a model.

Definition 2.5 *A sentence σ of $L_{\omega_1,\omega}$ is* large *if it has uncountably many countable models. A large sentence σ is* minimal *if for every sentence ϕ either $\sigma \wedge \phi$ or $\sigma \wedge \neg\phi$ is not large.*

By a tree argument [5] show:

Lemma 2.6 (Harnik-Makkai) *For every counterexample σ to Vaught's conjecture, there is a minimal counterexample ϕ such that $\phi \models \sigma$.*

Our goal is to show any counterexample to Vaught's conjecture has an uncountable model which is not small. Fix a minimal counterexample σ to Vaught's conjecture. For any countable fragment Δ containing σ, define

$$T_\Delta = \{\sigma \wedge \phi : \phi \in \Delta \text{ and } \sigma \wedge \phi \text{ is large }\}.$$

Note that T_Δ is consistent and complete for Δ. Keisler [7] with no use of admissible model theory shows that the 'prime' part of Vaught's fundamental paper on countable models of complete first order theories [20] goes through for scattered σ. This translation is fairly straightforward without any appeal to admissible model theory. In particular,

Fact 2.7 *A theory T that is complete for a countable fragment of $L_{\omega_1,\omega}$ and has only countably many types over the empty set has a prime model.*

Since σ is scattered, each T_Δ has a prime model (for Δ).

Lemma 2.8 *If σ is a counterexample to the Vaught Conjecture and Δ is smallest fragment containing σ, there is a strictly increasing $\prec_\Delta$-sequence M_α, $\alpha < \aleph_1$ of countable models.*

Proof. Fix a minimal counterexample σ to Vaught's conjecture and let Δ_0 be a countable fragment containing σ ($\{\sigma\} = T_0$). Define by induction $\langle \Delta_\alpha, T_\alpha, M_\alpha \rangle$ such that

1. If $\beta < \alpha$, the Scott sentence ψ_β of M_β is in Δ_α.
2. $T_\alpha = T_{\Delta_\alpha}$
3. M_α is the Δ_α prime model of T_α.

For this, let Δ_α be the minimal fragment containing $\bigcup_{\beta<\alpha} \Delta_\beta$ and the Scott sentence of each M_β for $\beta < \alpha$. The M_α are as required. The chain is strictly increasing since $M_\alpha \models \neg\psi_\beta$ if $\beta < \alpha$. And each $M_\alpha \prec_{\Delta_0} M_\beta$ for $\alpha < \beta$ since the Δ_i and T_i are increasing. That is, M_α is the prime model of T_α and $M_\beta \models T_\alpha$. $\square_{2.8}$

Theorem 2.9 (Harnik-Makkai) *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to VC then it has a model N of cardinality $\aleph_1$ which is not small.*

Proof. We continue the argument from Lemma 2.8. Now if $M = \bigcup_\alpha M_\alpha$, M does not satisfy any complete sentence of $L_{\omega_1,\omega}$, as any sentence θ true on M is true on a cub of M_α; so it has more than one countable model and cannot be complete. $\square_{2.9}$

Our goal now is to show that any counterexample to Vaught's conjecture has small uncountable models. This was first obtained by Makkai, using (in contrast to Keisler's study of prime models) notions of saturated models in admissible set theory and some reasonably elaborate machinery devised by Ressayre [11] (basic to admissible model theory but much more than we will use here).

Before turning to that proof which passes through the existence of an uncountable model that may not be small, we detour to note a very natural approach.

Lemma 2.10 *A sentence σ of $L_{\omega_1,\omega}$ has an uncountable small model iff it has a pair of countable models such that M_0 is a proper substructure of M_1, M_0 and M_1 are isomorphic and $M_0 \prec_L M_1$, where L is the smallest fragment containing the Scott sentence of M_0.*

Proof. If N is an uncountable small model of σ, let ψ be the Scott sentence of N and L the fragment generated by ψ. Then take M_0 an L-elementary submodel of N and M_1 an L-elementary submodel of N which properly extends M_0. Conversely, construct an chain $\langle M_i : i < \aleph_1\rangle$ where (M_i, M_{i+1}) is isomorphic to (M_0, M_1). This construction goes through limits by taking unions since for countable δ, all M_δ are isomorphic. Then every type realized in M_{ω_1} is realized in M_0 so it is a small uncountable model.ψ. $\square_{2.10}$

Continuing our methodological queries, is there any direct way (using only countable models) to deduce the existence of such a pair of countable models directly from the failure of Vaught's conjecture?

During the conference Sacks sketched a positive reply to this question by a nice argument using admissible sets and Barwise compactness which gave the result via a construction on countable models. In essence Makkai's original argument [9] also provides a positive answer using the technology of admissible set theory.

Now we modify an argument of Shelah to provide a proof which trades the mechanism of admissible sets for a model theoretic coding to analyze models of cardinality $\aleph_1$. We rely on the following result which combines results of Lopez-Escobar, Morley, and Keisler. The ingredients are in [7].

Theorem 2.11 *Let τ be a similarity type which includes a binary relation symbol $<$. Suppose ψ is a sentence of $L_{\omega_1,\omega}$, $M \models \psi$, and the order type of $(M, <)$ imbeds ω_1. There is a model N of ψ with cardinality $\aleph_1$ such that the order type of $(N, <)$ imbeds $\mathbb{Q}$.*

Now we describe the proof (Shelah [13]; see Section 7.3 of [1]) of Theorem 1.6. Note that the hypothesis is satisfied by any scattered $L_{\omega_1,\omega}$-sentence that has an uncountable model.

Proof of Theorem 1.6: If the $L_{\omega_1,\omega}$-τ-sentence ψ has a model of cardinality $\aleph_1$ which is Δ-small for every countable τ-fragment Δ of $L_{\omega_1,\omega}$, then ψ has a τ-small model of cardinality $\aleph_1$.

Add to τ a binary relation $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any τ-fragment, write $L_{\omega_1,\omega}(\tau)$ as a continuous increasing chain of fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$.

Extend the similarity type to τ' by adding new $2n+1$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $n+1$-ary functions f_n.

Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type.

Let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ implies $f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \mathbf{b})$.

Notice the following facts.

1. $E_n(\beta, \mathbf{y}, \mathbf{z})$ refines $E_n(\alpha, \mathbf{y}, \mathbf{z})$ if $\beta > \alpha$;
2. $E_n(0, \boldsymbol{a}, \mathbf{b})$ implies $\boldsymbol{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \boldsymbol{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\boldsymbol{a}, c_2\mathbf{b})$ and
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \mathbf{y}, \mathbf{z})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau‘)$ sentence ϕ. Let Δ^* be the smallest τ‘-fragment containing $\phi \wedge \psi$. Now by Lopez-Escobar (Theorem 2.11) there is a structure N of cardinality $\aleph_1$ satisfying $\phi \wedge \psi \wedge \chi$ such that $<$ is not well-founded on N. Fix an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N. For each n, define $E_n^+(\mathbf{x}, \mathbf{y})$ if for some i, $E_n(d_i, \mathbf{x}, \mathbf{y})$. Now using 1), 2) and 3) prove by induction on the quantifier rank of ϕ for every $L_{\omega_1,\omega}(\tau)$-formula ϕ that $N \models E_n^+(\boldsymbol{a}, \mathbf{b})$ implies $N \models \phi(\boldsymbol{a})$ if and only if $N \models \phi(\mathbf{b})$.

For each n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E_n^+(\mathbf{x}, \mathbf{y})$ and by 4) $E_n(d_0, \mathbf{x}, \mathbf{y})$ has only countably many classes; so N is small. $\square_{1.6}$

We conclude the result proved by Makkai[9] using admissible model theory.

Theorem 2.12 (Makkai) *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to VC then it has an uncountable model N which is small.*

Proof. By Lemma 1.7, ψ is scattered. By Theorem 2.9, it has a model of power $\aleph_1$ and then by Lemma 1.6, it has a small uncountable model. $\square_{2.12}$

We have shown:

Corollary 2.13 *There is no $\aleph_1$-categorical counterexample to Vaught's conjecture.*

3 The number of models in $\aleph_1$

We have shown that any counterexample to Vaught's conjecture has at least two models of cardinality $\aleph_1$. Why stop there? The following result seems to be new.

Theorem 3.1 *If a first order theory is a counterexample to the Vaught conjecture then it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

But it is easy from two well-known but difficult theorems:

Theorem 3.2 (Shelah) *If a first order T is not ω-stable T has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

This argument uses many descriptive set theoretic techniques. See Shelah's book [14] or Baldwin's paper [2].

Theorem 3.3 (Shelah) *An ω-stable first order theory satisfies Vaught's conjecture.*

Proof of 3.1: If T has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then by Theorem 3.2, it is ω-stable and then by Theorem 3.3, it satisfies Vaught's conjecture. $\square_{3.1}$

We now show that, assuming the weak continuum hypothesis ($2^{\aleph_0} < 2^{\aleph_1}$), the previous theorem extends to $L_{\omega_1,\omega}$. This will require some preparation. Shelah observed that under the weak continuum hypothesis, Theorem 1.10, few models in $\aleph_1$ yields few types over the empty set, implies:

Fact 3.4 *($2^{\aleph_0} < 2^{\aleph_1}$) If a sentence $\psi \in L_{\omega_1,\omega}$ is not ω-stable it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

For first order logic, few models in $\aleph_1$ implies ω-stable. And it even holds (in ZFC) for sentences in $L_{\omega_1,\omega}$, which have arbitrarily large models. The arbitrarily large models give us access to Ehrenfeucht-Mostowski models. But for an arbitrary sentence in $L_{\omega_1,\omega}$, to show few models in $\aleph_1$ implies ω-stable, requires weak CH. Shelah [13] first provided a counterexamples in $L_{\omega,\omega}(Q)$ using Baumgartner's order. But examples can be found in $L_{\omega_1,\omega}$ [19, 1].

This leads us to some natural generalization of Theorem 3.3. The notion of an excellent class [15, 16, 3, 21] plays a crucial role in the model theory of infinitary logic.

Question 3.5 *Does Vaught's conjecture hold for ω-stable sentences in $L_{\omega_1,\omega}$? For excellent classes?*

These questions pose two difficulties. As Rami Grossberg pointed out the questions are not really well-formed. The work in [15, 16] on ω-stable and excellent classes is restricted to atomic classes – the translation of *complete* sentences of $L_{\omega_1,\omega}$. All such classes are $\aleph_0$-categorical. So the first step is to adapt the stability theory machinery for the translations of arbitrary sentences in $L_{\omega_1,\omega}$. These are finite diagrams in the sense of [12]. But the machinery of that paper is primarily directed at the study of uncountable models and makes the additional assumption that there is a homogeneous model. Once an appropriate framework is found that circumvents these difficulties, the real task begins. The proof that an ω-stable first order theory has either $\aleph_0$ or $2^{\aleph_0}$ countable models has two parts. On the one hand various conditions are shown to imply the existence of $2^{\aleph_0}$ countable models; on the other the conjunction of the negations of these properties are shown to allow such control over the structure of models that the theory has only countably many models. This second part might be easier with the greater expressive power of $L_{\omega_1,\omega}$. But the loss of compactness may greatly complicate the first.

Many of the difficulties in studying $L_{\omega_1,\omega}$ stem from the difficulty of proving the amalgamation property. Recall that a sentence σ in a fragment Δ of $L_{\omega_1,\omega}$ satisfies the amalgamation property if $M_0 \prec_\Delta M_1, M_2$ implies M_1 and M_2 have a common Δ-elementary extension.

Theorem 3.6 (Shelah) *($2^{\aleph_0} < 2^{\aleph_1}$) If a sentence σ in $L_{\omega_1,\omega}$ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then the countable models of σ have the amalgamation property.*

The argument for this can be found in [19, 4, 1]. The weak CH is used to apply the Devlin-Shelah diamond; this use is necessary and counterexamples are in the same place. Our theme of the need to study larger models continues in proving the extension of Theorem 3.1 to $L_{\omega_1,\omega}$. Consider the following theorem of Shelah.

Theorem 3.7 *An $\aleph_1$-categorical sentence ψ in $L_{\omega_1,\omega}$ has a model of power $\aleph_2$.*

This result actually was first proved in more generality for $L_{\omega_1,\omega}(Q)$ (adding the quantifier, 'there exists uncountably many'), but for Vaught conjecture considerations we restrict to $L_{\omega_1,\omega}$. The original proof [13] used diamond and developed a considerable amount of stability theory for $L_{\omega_1,\omega}$. In ([19], see also [1]) a beautiful proof of Theorem 3.7 is given in ZFC. The crux is to use another application of Lopez-Escobar to construct a properly increasing pair of cardinality $\aleph_1$. Then as in Lemma 2.4, categoricity in $\aleph_1$ yields a model in $\aleph_2$. The argument below weakens categoricity to few models in $\aleph_1$. The condition that there is some proper pair in $\aleph_1$ is strengthened to showing there is no maximal model of power $\aleph_1$ and then the model of power $\aleph_2$ follows as in Lemma 2.2.

By the reductions of Section 2.2, we may work with an atomic class: the class of atomic models of a complete first order theory. In the next theorem, which appears to be newly remarked (although of course implicit in [15] if not [13]), we weaken the hypothesis of $\aleph_1$-categoricity in Theorem 3.7 to ω-stability; we are still working in ZFC.

As in [13, 15] and expounded in [1, 8], we develop the notion of an ω-stable atomic class. (Warning, many words (type, ω-stable, independent etc.) have subtly different meanings in this context. So new arguments are needed for what at first appear to be old results.) Most crucially, all amalgamation questions are slippery. A notion of independence, $M \underset{N}{\perp} P$, is defined (based on splitting), which has many of the properties of the first order notion of 'nonforking'. One is able to show that *countable models* in $\boldsymbol{K}$ admit a form of free amalgamation. See the chapter on independence in ω-stable atomic classes of [1] for a recent detailed exposition of the next few theorems.

Definition 3.8 *A and B are* freely amalgamated *over C in N, if* $A \underset{C}{\perp} B$ *and* $AB \subset M \in \boldsymbol{K}$.

Fact 3.9 *If* $M_0 \prec M_1, M_2$ *then there exists* $M_1' \approx M_1$ *and* M_3 *with* M_1' *and* M_2 *freely amalgamated over* M_0 *in* M_3.

Theorem 3.10 *If the atomic class is* ω*-stable then it has a model of power* $\aleph_2$.

Proof. As in Lemma 2.2, it suffices to show every model N in $\boldsymbol{K}$ of cardinality $\aleph_1$ has a proper elementary extension M in $\boldsymbol{K}$. Write N as a continuous increasing chain $\langle N_i : i < \aleph_1 \rangle$. By Theorem 3.2, $\boldsymbol{K}$ is ω-stable. Now define an increasing sequence $\langle M_i : i < \aleph_1 \rangle$ such that $N_i \prec M_i$, M_i is freely amalgamated with N_{i+1} over N_i in M_{i+1}. Since independent sets intersect only where they have to, M_0 properly extends N_0. The union of the M_i is the required proper extension of N. The construction is routine taking unions at limits. The successor stage is also easy from the following claim, replacing $0, 1, 2$ by $\alpha, \alpha + 1, \alpha + 2$ but keeping N fixed.

Claim 3.11 *Let* $N_0 \prec N_1 \prec N_2 \prec N$. *Given* $M_0 \underset{N_0}{\perp} N_2$, *with* M_0 *and* N_1 *freely amalgamated over* N_0 *in* M_2', *we can choose* M_2 *and* M_3' *so that* $N_2, M_1 \prec M_2$ *and* M_2 *and* N_2 *are freely amalgamated over* N_1 *in* M_3'.

$\square_{3.10}$.

Now we can strengthen Theorem 3.7 replacing categoricity in $\aleph_1$ by few models in $\aleph_1$ at the cost of assuming $2^{\aleph_0} < 2^{\aleph_1}$. The following corollary is immediate since with this set-theoretic hypothesis, few models in $\aleph_1$ implies ω-stability (Lemma 3.2).

Corollary 3.12 (Shelah) *[$2^{\aleph_0} < 2^{\aleph_1}$] If the atomic class* $\boldsymbol{K}$ *has fewer than* $2^{\aleph_1}$ *models of cardinality* $\aleph_1$ *then it has a model of power* $\aleph_2$.

Recall Hjorth [6] proved:

Theorem 3.13 (Hjorth) *If there is a counterexample to Vaught's conjecture, there is one with no model of size* $\aleph_2$.

Theorem 3.14 (ZFC + $2^{\aleph_0} < 2^{\aleph_1}$) *If an* $L_{\omega_1,\omega}$ *sentence is a counterexample to the Vaught conjecture then it has* $2^{\aleph_1}$ *models of cardinality* $\aleph_1$.

Proof. If not, it has a model of power $\aleph_2$. But by Theorem 3.13, if Vaught's conjecture fails there is such a counterexample ϕ with no model of power $\aleph_2$. $\square_{3.14}$

Naturally, one would like to remove the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$ from Theorem 3.14, and to avoid going through Theorem 3.12 and Theorem 3.13. Since the deduction of ω-stability from few models definitely requires extending ZFC ([1, 18], a very different attack is needed. A natural strategy for this is to return to the initial Harnik-Makkai argument, Lemma 2.9, and code stationary sets into the construction of the tree. But this requires some notion of how different 'tops' are put on the limits of countable chains and there is nothing of this sort evident (to me) in the proof. And such an argument might not avoid set the set theory since Devlin-Shelah diamond is used in many such arguments. Moreover, the proof of the first order case involves a deep analysis of the models; it would be very striking to avoid this.

This conference exhibited a striking interaction among logicians of various stripes. This paper is one example; I raised the question of whether a counterexample to Vaught's conjecture necessarily had a model of cardinality $\aleph_2$ early in the conference; Sacks elaborated on the question in his second presentation; Hjorth heard the problem in Sack's lecture and had the tools to solve it. And in writing up my contribution, I saw that Hjorth's solution solved a second question, the number of models in $\aleph_1$. The tools were available in the various communities but even the statement of the results depended on the connections made in the conference.

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