$^{\perp}N$ AS AN AEC

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Very Preliminary

We show the concept of an Abstract Elementary Class provides a unifying notion for several properties of classes of modules. The important definitions and proof from the study of modules can be found in [GT06] and [EM02]; concepts of AEC are due to Shelah but collected in [Bal]. The easiest result to state is:

Theorem 0.1. (1) *For an abelian group N, the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class if and only if N is a cotorsion module.*
(2) *For any R-module N, over an hereditary ring R, if N is a pure-injective module then the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class.*

Theorem 0.2. *For an abelian group N,*
(1) *In any case$({}^{\perp}N, \prec_N)$ has the amalgamation property.*
(2) *$({}^{\perp}N, \prec_N)$ is stable in all λ with $\lambda^{\omega} = \lambda$ and for some N this is best possible.*

We discuss Theorem 0.1 in Section 1 and Theorem 0.2 in Section 2.

1. When is $({}^{\perp}N, \prec_N)$ an AEC?

We will begin by describing the three main concepts, then prove the theorem. Then we will state several variations. We recall the precise definition of an AEC since checking these axioms is the main content of the note.

Definition 1.1. *A class of τ-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an* abstract elementary class (AEC) *if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If $M \prec_{\boldsymbol{K}} N$ then $M \subseteq N$.*
- **A2**. *$\prec_{\boldsymbol{K}}$ is a partial order on $\boldsymbol{K}$.*
- **A3**. *If $\langle A_i : i < \delta\rangle$ is continuous $\prec_{\boldsymbol{K}}$-increasing chain:*
  (1) *$\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;*
  (2) *for each $j < \delta$, $A_j \prec_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$*
  (3) *if each $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ then $\bigcup_{i<\delta} A_i \prec_{\boldsymbol{K}} M$.*
- **A4**. *If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.*
- **A5**. *There is a Löwenheim-Skolem number $\mathrm{LS}(\boldsymbol{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \boldsymbol{K}$ with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and $|A'| \le |A| + \kappa(\boldsymbol{K})$.*

We use 'module' to mean a right R-module; we will work with various properties of R.

The first author is partially supported by NSF grant DMS-0500841.

Definition 1.2. (1) ${}^{\perp}N = \{A : Ext(A, N) = 0\}$
(2) *For* $A \subseteq B$ *both in* ${}^{\perp}N$, $A \prec_N B$ *if* $B/A \in {}^{\perp}N$.

This notion generalizes the concept of a Whitehead group; that is the special case: ${}^{\perp}\mathbb{Z}$. There is a great deal of work on such classes and on their duals, so-called cotorsion pairs in [GT06] and [EM02]. And the notion $\prec_N$ that we have chosen for our notion of 'strong submodel' arose in abelian group context in the guise of a C-filtration. It was independently developed by the 'Abelian group group' of the AIM workshop on Abstract Elementary Classes in July 2006.

Definition 1.3. *A module* N *over a ring* R *is* cotorsion *if it satisfies the following condition:*
$Ext(F, N) = 0$ *for every flat* F.

If R is a Dedekind (= hereditary) domain then the condition is equivalent to $Ext(J, N) = 0$ for every torsion-free J and to the condition that $\mathrm{Ext}(Q(R), N) = 0$; we will use this condition when studying Abelian groups below.

While the class of cotorsion modules is complex, the torsion free cotorsion Abelian groups are more fully understood; they are the pure-injective Abelian groups.

The Whitehead groups, ${}^{\perp}\mathbb{Z}$ do not form an AEC with $\prec_N$ for $N = Z$; this can be seen directly since they don't satisfy **A.3.3** or by the theorem here, since $\mathbb{Z}$ is not a cotorsion module.

We will begin the proof of the main theorem and return to the notion of cotorsion when it is needed for the argument. We check that each axiom is satisfied.

A1 is trivial.

A2 requires a small observation. We want to show $A \prec_N B$ and $B \prec_N C$ implies $A \prec_N C$; that is that $Ext(C/A, N) \in {}^{\perp}N$. We have an exact sequence:
$$B/A \rightarrow C/A \rightarrow C/B.$$
This induces the exact sequence
$$Ext(B/A, N) \leftarrow Ext(C/A, N) \leftarrow Ext(C/B, N).$$
Since the end terms are 0, so is the middle one.

A3 is much more complicated. The first key point is 'Eklof's Lemma'.

Lemma 1.4. *Let* C *be a module. Suppose that* $A = \bigcup_{\alpha<\mu} A_\alpha$ *with* $A_0 \in {}^{\perp}N$ *and for all* $\alpha < \mu$, $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$ *then* $A \in {}^{\perp}N$.

From this result, **A3.1** is immediate for any R and any N. Lemma 1.4 is proved on page 113 of [GT06]; another proof is XII.1.5 of the second edition of [EM02].

For any R and any N, **A3.2** follows from **A3.1**. To see this, suppose $\langle A_i : i < \alpha\rangle$ is a $\prec_N$ continuous increasing chain with union A. We must show each $A/A_i \in {}^{\perp}N$. Note that $A/A_i = \bigcup_{j>i} A_j/A_i$ so by **A3.1** it suffices to show each $A_j/A_i \in {}^{\perp}N$ and each $A_j/A_i \prec_N A_{j+1}/A_i$. The first follows from the definition of the chain and induction using transitivity and **A3.1** for limit stages. The second requires that
$$(A_{j+1}/A_i)/(A_j/A_i)$$

be in N. But this last is A_{i+1}/A_i which is in $^\perp N$ by hypothesis.

To check **A3.3**, we need the hypothesis that N is cotorsion for the first time. In the following we mean that the class $^\perp N$ is closed under arbitrary direct limits of *homomorphisms*, not just under direct limits of strong embeddings which is a well-known consequence of **A3.1**.

Lemma 1.5. *For an abelian group N, N is cotorsion if and only if $^\perp N$ is closed under direct limits.*

Proof. Suppose $^\perp N$ is closed under direct limits. Since every free groups is in $^\perp N$ (for any N) and every torsion free group is a direct limit of free groups, every torsion free group is in $^\perp N$, so by characterization 1) of the definition N is cotorsion.

Conversely, Theorem 3.2.7 of [GT06], (see also [ET00]), if N is cotorsion then $^\perp N$ is closed under arbitrary direct limits.

There will be extensions of this result depending on the ring R.

Now to verify **A3.3**, suppose $\langle A_i : i < \alpha \rangle$ is a $\prec_N$ continuous increasing chain with union A and each $A_i \prec_N B$. We must show $B/A \in {^\perp N}$. But B/A_i is the direct limit of the family of surjective homomorphism $f_{i,j} : B/A_i \to B/A_j$, for $i < j$.

A4 is rather straightforward but this is where we require that the ring R is hereditary. One of the many equivalent definition of an hereditary ring is exactly what we need. Since the ring of integers is hereditary, the results here apply to abelian groups.

Definition 1.6. *R is hereditary if and only if for every pair $A \subset B$ of R-modules and any N, $Ext(B, N) = 0$ implies $Ext(A, N) = 0$.*

Lemma 1.7. *If R is a hereditary ring,* **A4** *holds for* $({^\perp N}, \prec_N)$.

Suppose $A \subseteq B \subseteq C$ with $A \prec_N C$ and $B \prec_N C$. To show $A \prec_N B$, we need only show $B/A \in {^\perp N}$. But this is immediate since $B/A \subset C/A$ from the following fact.

Remark 1.8. *We didn't use the hypothesis $B \prec_N C$. The key monotonicity of Ext property holds because $\mathbb{Z}$ is a hereditary ring.*

Verifying **A5** again relies on important concept from homological algebra. We modify the notation in [ET00].

Definition 1.9. *For any right R-module A and any cardinal κ, a (κ, N)*-refinement *of length σ of A is a continuous chain $\langle A_\alpha : \alpha < \sigma \rangle$ of submodules such that $A_0 = 0$, $A_{\alpha+1}/A_\alpha \in {^\perp N}$, and $|A_{\alpha+1}/A_\alpha| \leq \kappa$ for all $\alpha < \sigma$.*

It is not clear that both the notion of refinement and filtration are needed. But for this version I am using both, so that I can quote directly from both [ET00] and [vT]

Definition 1.10. *A admits a $\mathbb{C}$-filtration if A can be written as the union of continuous chain A_i where A_{i+1}/A_i is in $\mathbb{C}$ and $A_0 = 0$.*

I state now Theorem 6 of [vT]. I omit some of the conclusions which are not needed here.

Lemma 1.11 (Generalized Hill Lemma)**.** *Suppose M admits a $\mathbb{C}$-filtration: $\langle M_\alpha : \alpha \leq \sigma\rangle$, where $\mathbb{C}$ is a set of $< \kappa$-presented modules. There is a family $\mathbb{F}$ of submodules of $M = M_\sigma$ such that:*

1. $M_\alpha \in \mathbb{F}$ *for all* $\alpha \leq \sigma$.
2. $\mathbb{F}$ *is closed under arbitrary intersections and unions.*
3. *Let* $N \subset P$ *with both in* $\mathbb{F}$*. Then* P/N *admits a* $\mathbb{C}$*-filtration.*
4. *If* $N \in \mathbb{F}$ *and* $X \subset M$ *with* $|X| < \kappa$*, then there is a* $P \in \mathbb{F}$ *with* $N \cup X \subset P$ *and* P/N *is* $< \kappa$ *presented.*

Lemma 1.12. *If every module in $^\perp N$ admits a (κ, N)-refinement then $(^\perp N, \prec_N)$ has Löwenheim-Skolem number κ.*

Proof. Note that a (κ, N)-refinement yields a filtration by the $< \kappa$-generated elements of $^\perp N$. Note also that if $A \in \mathbb{F}$, then by Eklof's lemma and 2) of Theorem 1.11, $A' \prec_N M$. Write an arbitrary $X \subset M$ as $\bigcup_{i<\mu} X_i$ where $|X_i| < \kappa$ and $\mu = |X|$. Then inductively, using condition 4, construct N_i so that:

1. $N_0 = 0$;
2. $N_{i+1} \supseteq N_i \bigcup X_i$;
3. N_{i+1}/N_i is $< \kappa$-presented.

Then N_μ is as required.

Remark 1.13 (Summary)**.** *We have shown that if $^\perp N$ admits refinements and satisfies A3.3, then $(^\perp N, \prec_N)$ is an AEC.*

But the question of when refinements exist is rather complicated.

Lemma 1.14. *Each member of $^\perp N$ admits a $(|R| + \aleph_0, N)$-refinement under any of the following conditions.*

1. N *is pure-injective and* R *is arbitrary.*
2. N *is cotorsion and* R *is a Dedekind domain.*
3. *(V=L)* N *is arbitrary and* R *is hereditary.*

Proof. These results are in [ET00]. 1) is Theorem 8; 3) is Theorem 14; 2) follows from (ii) of Theorem 16 and Theorem 8.

In view of Remark 1.13, the classes described in (1) when R is hereditary and (2) of Lemma 1.14, $(^\perp N, \prec_N)$ are AECs. This leads to several questions.

Question 1.15. (1) *Can the question of whether a class is an AEC (e.g. $(^\perp N, \prec_N)$ R a hereditary ring be independent of ZFC?*

(2) *Can the question of whether a class(e.g. Whitehead groups) is a $PC\Gamma$-class (defined as the reducts of models of say a countable theory that omitting a family of types be independent of ZFC? (Note that under $V = L$, 'Whitehead=free' and the class is easily $PC\Gamma$.*

Lemma 1.16. *For a cotorsion Abelian group N, $^{\perp}N$ is first order axiomatizable.*

Proof. Let P^N be the set of primes p such that $Ext(Z/p, N) \neq 0$. By Theorem 16.i of [ET00], $^{\perp}N$ is the class of modules is the class of modules satisfing T^N where T^N asserts that there is no p-torsion for each $p \in P^N$

Remark 1.17. *Note however, that while the class is elementary the definition of $\prec_N$ remains unusual. Even knowing the axiomatizability, we would have the definition $A \prec_N B$ iff $B/A \in T^N$.*

We have shown that if N is a cotorsion Abelian group then $(^{\perp}N, \prec_N)$ is an AEC. Now for the converse, suppose $(^{\perp}N, \prec_N)$ satisfies **A.3.3**. We show that N is cotorsion. Specifically, by criteria 2) for cotorsion it suffices to show $\mathbb{Q} \in {}^{\perp}N$. We prove the following more general lemma.

Lemma 1.18. *Suppose $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is an AEC of abelian groups where $A \prec_{\boldsymbol{K}} B$ means $A \subseteq B$ and $A, B, B/A \in \boldsymbol{K}$. Then if $\boldsymbol{K}$ contains all free abelian groups, $\mathbb{Q} \in \boldsymbol{K}$.*

Proof. So we want to write $\mathbb{Q}$ as $B/\bigcup_i A_i$ for some $B, A_i \in \boldsymbol{K}$. Fix B as $\sum Z_n$ where each $Z_n \approx \mathbb{Z}$ and choose K so that $B/K = \mathbb{Q}$. Then let

$$A_i = (\sum_{n<i} Z_n) \cap K.$$

Let B_i denote $\sum_{n<i} Z_n/A_i$. Now,

$$B/A_i = B_i \oplus (\sum_{n\geq i} Z_n)$$

(since $A_i \subseteq \sum_{n<i} Z_n$). Note that B_i is torsion-free and finitely generated so free; thus, B/A_i is free and so in $\boldsymbol{K}$. Each A_i is a submodule of a free and so in $\boldsymbol{K}$; we have $A_i \prec_{\boldsymbol{K}} B$, for each i. To check $A_i \prec_{\boldsymbol{K}} A_{i+1}$, we need $A_{i+1}/A_i \in \boldsymbol{K}$. Now A_{i+1}/A_i maps into the free B/A_i so A_{i+1}/A_i is free and in $\boldsymbol{K}$. So the A_i are an increasing $\prec_N$-chain with union K and by **A.3.3**, $B/K = \mathbb{Q} \in \boldsymbol{K}$ as required.

Remark 1.19. *There are a number of extensions that one can make.*

The ring R needed to be hereditary to guarantee axiom **A4**. *If we study $^{\perp}_{\infty}N$ instead of $^{\perp}N$, we can evade this requirement.*

$$^{\perp}_{\infty}N = \{A : Ext^i(A, N) = 0, \text{ for all } i\}.$$

To see this, note that we have $A \subset B \subset C$ with $B/A, C/B \in ^{\perp}_{\infty} N$; we must show $C/A \in ^{\perp}_{\infty} N$. We have the short exact sequence:

$$0 \to B/A \to C/A \to C/B \to 0.$$

By the long exact sequence of Ext, we get for each n.

$$Ext^n(C/A, N) \to Ext^n(B/A, N) \to Ext^{n+1}(C/B, N)$$

but the end terms are 0 by the definition of $^{\perp}_{\infty}N$ so we finish.

The other requirements behave for $^{\perp}_{\infty}N$ as they do for $^{\perp}N$???

2. Amalgamation and Stability

Having established that $(^{\perp}N, \prec_N)$ is an AEC for a number of N, we turn to establishing the model theoretic properties of the AEC. In this section we show first that for any N, $(^{\perp}N, \prec_N)$ has the amalgamation property. Then we show that for an Abelian group N, $(^{\perp}N, \prec_N)$ is stable in λ if $\lambda^{\omega} = \lambda$.

Note that for any N, all projective and in particular all free R-modules are in $^{\perp}N$ so $(^{\perp}N, \prec_N)$ always has arbitrarily large models.

Now we show the second claim:

Lemma 2.1. $(^{\perp}N, \prec_N)$ *has the amalgamation property.*

We just check if $C \prec_N B$ and $C \prec_N A$ then the pushout D of A and B over C is in $^{\perp}N$ and $B \prec_N C$, $A \prec_N D$. Consider the short exact sequence:

$$0 \to C \to B \to B/C \to 0.$$

By the universal mapping property of pushouts we get the diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & D & \to & B/C & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & C & \to & B & \to & B/C & \to & 0 \end{array}$$

Then by the long exact sequence of Ext, we deduce:

$$Ext(B/C, N) \to Ext(D, N) \to Ext(A, N) \to 0.$$

But the first and last entries are 0, so $D \in {}^{\perp}N$. Now the commutative diagram shows $D/A \approx B/C$ so $A \prec_N D$. Performing the same construction starting with $0 \to C \to A \to A/C \to 0$, shows $B \prec_N D$ and we finish.

Question 2.2. *What about disjoint amalgamation?*

Remark 2.3 (Conclusions)**.** We can now see:

(1) $(^{\perp}N, \prec_N)$ has a monster model in usual sense of AEC, (see [Bal]).
(2) $(^{\perp}N, \prec_N)$ has EM-models and models generated by indiscernibles.

To study stability, we must define it.

Definition 2.4. (1) *Define*

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

(2) *'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N'' if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.*
(3) $\mathrm{ga}-\mathrm{S(M)}$ *denotes the collection of Galois-types over M.*
(4) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is λ-stable if for every M with $|M| = \lambda$, $\mathrm{ga}-\mathrm{S(M)} = \lambda$.*

We begin our specific study with a definition and observation from [BS]

Definition 2.5. *We say the AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$* admits closures *if for every $X \subseteq M \in \boldsymbol{K}$, there is a minimal closure of X in M. That is, the structure with universe $\bigcap\{N : X \subseteq N \prec_{\boldsymbol{K}} M\}$ is a strong submodel of M. If so, we denote it: $\mathrm{cl}_M(X)$.*

With this hypothesis we have the following check for equality of Galois types. The relevance of the second clause is that even in the absence of amalgamation it shows that equality of Galois types is determined by a basic relation, not its transitive closure. By $M_1 \restriction \mathrm{cl}_{M_1}(M_0a_1)$ we simply mean the structure M_1 induces on the minimal $\boldsymbol{K}$ substructure containing M_0a_1.

Lemma 2.6. *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admit closures.*

(1) *Suppose $M_0 \prec_{\boldsymbol{K}} M_1, M_2$ with $a_i \in M_i$ for $i = 1, 2$. Then $\mathrm{tp}(a_1/M_0, M_1) = \mathrm{tp}(a_2/M_0, M_2)$ if and only if there is an isomorphism over M_0 from $M_1 \restriction \mathrm{cl}_{M_1}(M_0a_1)$ onto $M_2 \restriction \mathrm{cl}_{M_2}(M_0a_2)$ which maps a_1 to a_2.*
(2) *(M_1, a_1, N_1) and (M_2, a_2, N_2) represent the same Galois type over M_1 iff $M_1 = M_2$ and there is an amalgam of N_1 and N_2 over M_1 where a_1 and a_2 have the same image.*

Definition 2.7. *For an Abelian group N and $X \subset B \in {}^{\perp}N$, define $\mathrm{cl}_B(x)$ as the closure of X with respect to divisibility by p for each $p \in P^N$ (see Lemma).*

We can see immediately:

Lemma 2.8. *For an Abelian group N and $X \subset B \in {}^{\perp}N$, $\mathrm{cl}_B(x)$ witnesses that $({}^{\perp}N, \prec_N)$ admits closures. Note that for any a and $M \prec_N M_1 \in {}^{\perp}N$, $\mathrm{cl}_{M_1}(Ma)$ is countably generated over M.*

Theorem 2.9. *For an Abelian group N, $({}^{\perp}N, \prec_N)$ is stable in λ if $\lambda^{\omega} = \lambda$.*

Proof. We work in the monster model $\mathbb{M}$ of $({}^{\perp}N, \prec_N)$, which exists by Lemma 2.1. Note first that the Galois type over M of any element a is determined by the isomorphism type over M (and thus the first order quantifier-free type) of the countably many generators over M of $\mathrm{cl}_{\mathbb{M}}(Ma)$. This gives the upper bound.

Now we want a lower bound on the number of Galois types. For this, introduce a further concept.

Definition 2.10. (1) *Fix a family Δ of first order formulas such that if $\phi(\mathbf{x}) \in \Delta$ and $\boldsymbol{a} \in M \prec_{\boldsymbol{K}} N$ then $M \models \phi(\boldsymbol{a})$ if and only if $N \models \phi(\boldsymbol{a})$.*
(2) *The τ-type $p \in S_\Delta(M)$ is* acceptable *if there is an $N' \in \boldsymbol{K}$ with $N \prec_{\boldsymbol{K}} N'$ and p is realized in N'.*
(3) *We denote the set of acceptable 1-types over M by $S^a(M)$.*

For example, if $A \prec_{\boldsymbol{K}} B$ implies A is pure in p the following lemma holds where Δ is the existential formulas. (Positive existential should suffice.)

Lemma 2.11. *Let $N \in \boldsymbol{K}$, $p \in S_\Delta(N)$ is acceptable if and only if there is an $N' \in \boldsymbol{K}$ with $N \prec_{\boldsymbol{K}} N'$ and p is realized in N'.*

Proof. By clause 2b) of the presentation theorem, if there are such N, N', they can be expanded to models witnessing the definition of acceptable. And the converse is 2a) in the presentation theorem. $\square_{2.11}$

Corollary 2.12. *For any M, $|S^a(M)| \leq \mathrm{ga-S(M)}$.*

The stability section needs some work. The idea is that a) there is an upper bound of λ^ω on the number of Galois types since they are determined by the quantifier free type of a countable. b) this bound will be realized because there are models which have λ^ω distinct syntactic types over them that are realized in ${}^\perp N$. (This was clearer to me when 'pure' was in the definition of $\prec_N$ so that the existential formulas had to be preserved. I have left a start of a proof below. But it is probably confused both by the question of purity and my algebraic problems with whether there have to be many submodules of infinite index.

Lemma 2.13. *Let Δ be the collection of existential formulas.*
(1) *If $r(x) \in S_\Delta(M)$ is consistent and for each $m \in M$, r does not assert $p|(x-m)$, then $r(x)$ is acceptable.*
(2) *If there is $M \in {}^\perp N$ such that there are infinitely many $q \notin P^N$ such that $M \neq qM$ then $({}^\perp N, \prec_N)$ is not Galois-stable in any μ with $\omega \leq \mu \leq 2^{\aleph_0}$.*
(3) *If there is $M \in {}^\perp N$ such that there are infinitely many $q \notin P^N$ such that M/qM is infinite then $({}^\perp N, \prec_N)$ is not Galois-stable in λ unless $\lambda^\omega = \lambda$*
.

Proof. For the first, by compactness let a realize p along with the existential diagram of M in $M_1 \models T^N$. The second is easier than the third so we do only the third. By compactness choose $M \models N_p$ so that for each of indicated q, $|M/qM| = \lambda$. Now each type given by assigning for each such q a coset of M/qM to x determines a distinct acceptable type over M. By Corollary 2.12, we finish.

Remark 2.14. *The stability spectrum problem for arbitrary AEC has not been solved. There are explicit results for tame AEC in* [GV06] *and* [BKV00]. *See also for example* [She99].

Question 2.15. (1) *Does the condition that $({}^\perp N, \prec_N)$ is stable in all cardinals (or all cardinals beyond the continuum) provide any further algebraic conditions on N?*

(2) *Are the implications in 2) and 3) of Lemma 2.13 reversible?*

3. Further Directions

Question 3.1. (1) *What are the tameness properties of* $({}^{\perp}N, \prec_N)$*?*
(2) *Is* $({}^{\perp}N, \prec_N)$ *finitary in the sense of Hyttinnen-Kesala* [HK]

Question 3.2. (1) *When, if ever is* $({}^{\perp}N, \prec_N)$ *an excellent class. Note that for a cotorsion Abelian group this question is clearly specified. I mean excellence in the sense of Shelah* [She83a, She83b]. *For other classes one have to define an appropriate notion of independence to formulate excellence.*
(2) *For which* R *and* N *is* $({}^{\perp}N, \prec_N)$ *axiomatizable (in infinitary logic)? One might expect to use* $L_{\kappa,\omega}$ *if the ring had cardinality* κ*.*
(3) *What can we say about the number of models in various cardinalities of* $({}^{\perp}N, \prec_N)$*?*

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