$^{\perp}N$ AS AN ABSTRACT ELEMENTARY CLASS

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We show that the concept of an Abstract Elementary Class (AEC) provides a unifying notion for several properties of classes of modules and discuss the stability class of these AEC. An abstract elementary class consists of a class of models $\boldsymbol{K}$ and a strengthening of the notion of submodel $\prec_{\boldsymbol{K}}$ such that $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ satisfies the properties described below. Here we deal with various classes $(^{\perp}N, \prec_N)$; the precise definition of the class of modules $^{\perp}N$ is given below. A key innovation is that $A \prec_N B$ means $A \subseteq B$ and $B/A \in {}^{\perp}N$.

We define in the text the main notions used here; important background definitions and proofs from the theory of modules can be found in [EM02] and [GT06]; concepts of AEC are due to Shelah (e.g. [She87]) but collected in [Bal].

The surprising fact is that some of the basic model theoretic properties of the class $(^{\perp}N, \prec_N)$ translate directly to algebraic properties of the class $^{\perp}N$ and the module N over the ring R that have previously been studied in quite a different context (approximation theory of modules, infinite dimensional tilting theory etc.). Our main results, stated with increasing strong conditions on the ring R, are:

Theorem 0.1. (1) *Let R be any ring and N an R–module. If $(^{\perp}N, \prec_N)$ is an AEC then N is a cotorsion module. Conversely, if N is pure–injective, then the class $(^{\perp}N, \prec_N)$ is an AEC.*

(2) *Let R be a right noetherian ring and N be an R–module of injective dimension ≤ 1. Then the class $(^{\perp}N, \prec_N)$ is an AEC if and only if $^{\perp}N$ is closed under direct limits (of arbitrary homomorphisms).*

(3) *Let R be a Dedekind domain and N an R–module. The class $(^{\perp}N, \prec_N)$ is an AEC if and only if N is cotorsion.*

Since the ring of integers is a Dedekind domain, Theorem 0.1.3 exactly characterizes the abelian groups N such that $(^{\perp}N, \prec_N)$ is an AEC. We write "conversely" in Theorem 0.1 (1), although it is only a partial converse; every pure–injective module is cotorsion, but cotorsion modules are not necessarily pure–injective.

Theorem 0.2. (1) *For any module N, $(^{\perp}N, \prec_N)$ has the amalgamation property.*

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(2) *Let N be a module of injective dimension ≤ 1. Then $({}^{\perp}N, \prec_N)$ is an AEC that admits closures iff ${}^{\perp}N$ is closed under direct products iff ${}^{\perp}N$ is a cotilting class. In this case ${}^{\perp}N$ is first order axiomatizable.*

(3) *Assume that R is a Dedekind domain. Then each AEC of the form $({}^{\perp}N, \prec_N)$ admits closures. Moreover, ${}^{\perp}N$ coincides with the (first order aximatizable) class $\mathbf{K}(P)$ for a set P consisting of maximal ideals, and the closure, $\mathrm{cl}_M(X)$, coincides with the set of all $m \in M$ such that the ideal $\{r \in R : mr \in X\}$ is a finite product of elements of P.*

Conversely, for any set P of maximal ideals, there is a pure–injective module N such that $({}^{\perp}N, \prec_N)$ is an AEC with ${}^{\perp}N = \mathbf{K}(P)$.

Here, given a Dedekind domain R and its maximal ideal p, we call a module M *p–torsion–free* provided that the module R/p does not embed into M. For a set of maximal ideals P, $\mathbf{K}(P)$ is defined as the class of all modules that are p–torsion–free for all $p \in P$.

For example, if $P = \emptyset$ then ${}^{\perp}N = \mathbf{K}(P)$ is the class of all modules and $\prec_N =<_{\mathbf{K}(P)}$ is the submodule relation. If P is the set of all maximal ideals of R then ${}^{\perp}N = \mathbf{K}(P)$ is the class of all torsion–free modules and $\prec_N =<_{\mathbf{K}(P)}$ is the relation of being a pure submodule. It is easy to see that for different choices of P we get distinct classes $\mathbf{K}(P)$.

In the case of abelian groups (that is, modules over the ring of integers $\mathbb{Z}$), or more generally, modules over a principal ideal domain (p.i.d.), we can say more.

Theorem 0.3. (1) *If P is a non–empty set of prime ideals in $\mathbb{Z}$ then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is $(\aleph_0, \infty)$-tame, and if λ is an infinite cardinal, then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is stable in λ if and only if $\lambda^{\omega} = \lambda$.*

(2) *More generally, if R is a principal ideal domain, and P is a non-empty set of prime ideals in R with $|P| = \kappa$ then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is $(\mathrm{LS}(\mathbf{K}(\mathrm{P})), \infty)$-tame, and if λ is an infinite cardinal, then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is stable in λ if and only if $\lambda^{\kappa+\omega} = \lambda$.*

We discuss Theorem 0.1 in Section 1, Theorem 0.2 in Section 2, and Theorem 0.3 in Section 3.

The notion of an abstract elementary class, AEC, is usually viewed as an abstract version of 'complete first order theory' suitable for studying the class of models of a sentence in infinitary logic. However, the 'completeness' is not at all inherent in the notion. This work was stimulated by the discussion at the AIM conference on Abstract Elementary Classes in July 2006 of abelian groups as AEC (see [BCG$^+$00]). One of the goals of that conference was to identify some naturally occurring classes of abelian groups that form AEC under an appropriate notion of strong substructure. This paper provides a collection of such examples. We show certain variants of 'cotorsion pairs' of modules often satisfy the conditions for an AEC and classify when this happens. These examples provide a new type of strong extension; most previous examples are 'elementary submodel' in various logics. And as we describe in the last two sections, these examples provide further insight into the notion of Galois type. In particular, they provide examples when the notion of the Galois type of an element is readily translated into a syntactic type of a sequence.

1. When is $(^{\perp}N, \prec_N)$ an AEC?

We will begin by describing the main concepts, then prove the theorem. We recall the precise definition of an AEC since checking these axioms is the main content of this section. For background on AEC see e.g. [Gro02, Bal, She87, She99]

Definition 1.1. *A class of τ–structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an* abstract elementary class (AEC) *if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ on $\boldsymbol{K}$ are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If $M \prec_{\boldsymbol{K}} N$ then $M \subseteq N$.*
- **A2**. *$\prec_{\boldsymbol{K}}$ is a partial order on $\boldsymbol{K}$.*
- **A3**. *If $\langle A_i : i < \delta \rangle$ is a continuous $\prec_{\boldsymbol{K}}$–increasing chain:*
  (1) $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  (2) *for each $j < \delta$, $A_j \prec_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$;*
  (3) *if each $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ then $\bigcup_{i<\delta} A_i \prec_{\boldsymbol{K}} M$.*
- **A4**. *If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.*
- **A5**. *There is a Löwenheim–Skolem number $\mathrm{LS}(\boldsymbol{K})$ such that if $A \subseteq B \in \boldsymbol{K}$ there is a $A' \in \boldsymbol{K}$ with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and $|A'| \leq |A| + \mathrm{LS}(\boldsymbol{K})$.*

Here, $\langle A_i : i < \delta \rangle$ is a *continuous $\prec_{\boldsymbol{K}}$–increasing chain* provided that $A_i \in \boldsymbol{K}$ and $A_i \prec_{\boldsymbol{K}} A_{i+1}$ for all $i < \delta$, and $A_i = \bigcup_{j<i} A_j$ for all limit ordinals $i < \delta$.

If $M \prec_{\boldsymbol{K}} N$ we say that M is a *strong submodel* of N. If $f : M \mapsto N$ is 1-1 and $fM \prec_{\boldsymbol{K}} N$, we call f a *strong embedding*. Note that **A3** in toto says that $\boldsymbol{K}$ is closed under well–ordered direct limits of *strong embeddings*.

In what follows R denotes a ring (= associative ring with unit) and N a module (= unitary right R–module). When A and B are modules, $A \subseteq B$ means A is a submodule of B.

Definition 1.2. *The class $(^{\perp}N, \prec_N)$ is defined by*

$$^{\perp}N = \{A : \mathrm{Ext}^i(A, N) = 0 \textit{ for all } 1 \leq i < \omega\},$$

and

$$A \prec_N B \textit{ if and only if } A \subseteq B \textit{ and } A, B/A \in {}^{\perp}N.$$

It should be noted that the notation is not consistent in the literature. What we call $^{\perp}N$ here is often (e.g. in [GT06]) denoted by $^{\perp_\infty}N$, while $^{\perp}N$ refers to $\{A : \mathrm{Ext}^1(A, N) = 0\}$. But we use $^{\perp}N$ for the class that is most useful in our context. (For abelian groups, or more generally, for modules over right hereditary rings, the two notions coincide.)

Notice that, since $\mathrm{Ext}^i(-, N)$ takes direct sums into direct products, the class $^{\perp}N$ is closed under arbitrary direct sums. Moreover, it is *resolving*, that is, it contains all projective modules and is closed under extensions (i.e. whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in {}^{\perp}N$ then $Y \in {}^{\perp}N$), and has the property that $X \in {}^\perp N$ whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $Y, Z \in {}^\perp N$ (see [GT06, 2.2.9]).

The notion of ${}^\perp N$ generalizes the concept of a Whitehead group: the class of all Whitehead groups is the special case for $R = N = \mathbb{Z}$. (The class $({}^\perp\mathbb{Z}, \prec_{\mathbb{Z}})$ does not form an AEC; this can be seen directly since it doesn't satisfy **A.3**(3) or by Theorem 0.1.(3), since $\mathbb{Z}$ is not a cotorsion abelian group.)

The definition of ${}^\perp N$ can be generalized to that of

$${}^\perp\mathcal{C} = \{A : \mathrm{Ext}^i(A, N) = 0 \text{ for all } 1 \leq i < \omega \text{ and all } N \in \mathcal{C}\}$$

where $\mathcal{C}$ is a class of R–modules. This is often studied in the context of *cotorsion pairs*, that is, maximal pairs of Ext–orthogonal classes; see [GT06] and [EM02]. We will state at the end of this section a generalization of Theorem 0.1 for classes of the form ${}^\perp\mathcal{C}$.

A module N is said to have *injective dimension* ≤ 1 if $\mathrm{Ext}^i(A, N) = 0$ for all $i > 1$ and all modules A. In this case of course ${}^\perp N = \{A : \mathrm{Ext}^1(A, N) = 0\}$.

A pair of classes of modules $(\mathcal{C}, \mathcal{D})$ is a *torsion pair* if $\mathcal{C} = \{M : \mathrm{Hom}(M, D) = 0 \text{ for all } D \in \mathcal{D}\}$ and $\mathcal{D} = \{N : \mathrm{Hom}(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$. There is a torsion pair of the form $(\mathcal{C}, \mathcal{D})$ if and only if $\mathcal{D}$ is a *torsion–free class of modules*, that is, $\mathcal{D}$ is closed under direct products, extensions and submodules.

If N has injective dimension ≤ 1 and ${}^\perp N = \mathrm{Cog}(C)$ where $\mathrm{Cog}(C)$ is the class of all submodules of direct products of copies of C, then N is called a *cotilting module*, and ${}^\perp N$ the *cotilting class* induced by N. Each cotilting class is a torsion–free class of modules, so there is a torsion pair of the form $(\mathcal{T}, {}^\perp N)$, called the *cotilting torsion pair* induced by N. (Again, for simplicity of notation, we deviate here from the terminology of [GT06] where the term '1–cotilting module/class' is used. We refer to [GT06, Chap. 8] for more on cotilting modules and classes.)

If all modules have injective dimension ≤ 1, then R is a *right hereditary ring*; hereditary integral domains are called *Dedekind domains*. Every Dedekind domain R is *(right) noetherian*, that is, each (right) ideal of R is finitely generated.

The notion of $\prec_N$ that we have chosen for our notion of a strong submodule arose in the module theory context in the guise of a $\mathcal{C}$–filtration. It was independently developed by the 'Abelian group group' of the AIM workshop on Abstract Elementary Classes in July 2006 [BCG+00].

Definition 1.3. *Let $\mathcal{C}$ be a class of modules. A $\mathcal{C}$–*filtration *of a module A is a continuous increasing chain $\langle A_i : i \leq \delta\rangle$ of its submodules such that $A_0 = 0$, $A = A_\delta$, and $A_{i+1}/A_i \in \mathcal{C}$ for each $i < \delta$.*

We begin the proof of Theorem 0.1 by checking that each axiom is satisfied for the class $({}^\perp N, \prec_N)$:

A1 is trivial.

A2 requires a small observation. We want to show $A \prec_N B$ and $B \prec_N C$ implies $A \prec_N C$; that is, $\mathrm{Ext}^i(C/A, N) = 0$ for all $i > 0$. We have an exact sequence:

$$0 \to B/A \to C/A \to C/B \to 0.$$

Applying $\mathrm{Hom}(-, N)$, we obtain the induced long exact sequence

$$\cdots \to \mathrm{Ext}^i(C/B, N) \to \mathrm{Ext}^i(C/A, N) \to \mathrm{Ext}^i(B/A, N) \to \ldots$$

Since the outer terms are 0, so is the middle one, for each $i > 0$.

A3 is more complicated. First, we will show that **A3**(1) and **A3**(2) always hold. The key point here is 'Eklof's Lemma' saying that each $^{\perp}N$–filtered module is actually in $^{\perp}N$:

Lemma 1.4. *Let A be a module. Suppose that $A = \bigcup_{\alpha<\mu} A_\alpha$ where $\langle A_\alpha : \alpha < \mu\rangle$ is a continuous chain of modules with $A_0 \in {}^{\perp}N$ and for all $\alpha < \mu$, $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$. Then $A \in {}^{\perp}N$.*

From this result, **A3**(1) is immediate. Lemma 1.4 follows from [EM02, XII.1.5]; another proof is in [GT06, 3.1.2].

Next, we show that **A3**(2) follows from **A3**(1). Suppose $\langle A_i : i < \alpha\rangle$ is a $\prec_N$ continuous increasing chain with union A. We must show each $A/A_i \in {}^{\perp}N$. Note that $A/A_i = \bigcup_{j>i} A_j/A_i$ so by **A3**(1) it suffices to show each $A_j/A_i \in {}^{\perp}N$ and each $A_j/A_i \prec_N A_{j+1}/A_i$. The first follows from the definition of the chain and induction using transitivity and **A3**(1) for limit stages. The second requires that

$$(A_{j+1}/A_i)/(A_j/A_i)$$

be in N. But this last is isomorphic to A_{j+1}/A_j which is in $^{\perp}N$ by hypothesis.

Condition **A3**(3), however, is quite a strong one:

Lemma 1.5. *Let R be a ring and N a module. Then* **A3***(3) holds for the class $({}^{\perp}N, \prec_N)$ if and only if $^{\perp}N$ is closed under direct limits.*

Remark 1.6. Here, we mean closure under direct limits of *homomorphisms*, not just direct limits of strong embeddings which is a well–known consequence of **A3**. One could rephrase this lemma as: $^{\perp}N$ is closed under direct limits of 'strong embeddings' ($\prec_N$–morphisms) if and only if $^{\perp}N$ is closed under direct limits of homomorphisms. We could have stated the condition as **A3** since in this context **A3.1** and **A3.2** always hold.

Proof. First, assume that $^{\perp}N$ is closed under direct limits. To verify **A3**(3), suppose $\langle A_i : i < \alpha\rangle$ is a $\prec_N$ continuous increasing chain with union A and each $A_i \prec_N B$. We must show $B/A \in {}^{\perp}N$. But B/A is the direct limit of the family of surjective homomorphisms $f_{j,i} : B/A_i \to B/A_j$, for $i \le j$.

Conversely, assume **A3**(3) holds for the class $({}^{\perp}N, \prec_N)$. By [AR94, Corollary 1.7], it suffices to prove that $^{\perp}N$ is closed under direct limits of well–ordered chains of homomorphisms. Let $\mathcal{M} = (M_\alpha, f_{\beta\alpha} : \alpha \le \beta < \sigma)$ be such a chain with $M_\alpha \in {}^{\perp}N$ for all $\alpha < \sigma$. (Though $\mathcal{M}$ is required here to be well–ordered, the maps $f_{\beta\alpha}$ are *not* assumed to be monic. Notice that [AR94] uses category-theoretic rather than algebraic terminology: in [AR94, Corollary 1.7], 'direct limits' are called 'directed colimits').

If σ is a non–limit ordinal, then $\varinjlim M_\alpha = M_{\sigma-1} \in {}^{\perp}N$ by assumption.

Assume σ is a limit ordinal. Consider the canonical presentation of $M = \varinjlim M_\alpha$

$$0 \to K \hookrightarrow A \to M \to 0$$

where $A = \bigoplus_{\alpha<\sigma} M_\alpha \in {}^\perp N$ (since ${}^\perp N$ is closed under direct sums), and K is the submodule of A generated by all elements of the form $x_{\beta\alpha} = m - f_{\beta\alpha}(m)$ where $\alpha \leq \beta < \sigma$ and $m \in M_\alpha$. Let K_γ denote the submodule of K generated only by the $x_{\beta\alpha}$'s with $\beta < \gamma$. Then $(K_\gamma : 1 \leq \gamma \leq \sigma)$ is a continuous chain of submodules of A, and $K_\sigma = K$.

By induction on $\gamma \leq \sigma$, we prove that $K_\gamma \in {}^\perp N$ and $A/K_\gamma \in {}^\perp N$. This is clear for $K_1 = 0$. If γ is non–limit, then $K_\gamma + M_{\gamma-1} \supseteq \bigoplus_{\alpha<\gamma} M_\alpha$, hence $A = K_\gamma \oplus L_\gamma$ where L_γ denotes the direct summand of A generated by all M_α's with $\gamma - 1 \leq \alpha < \sigma$. Since $A/K_\gamma \cong L_\gamma$ is a direct sum of members of ${}^\perp N$, $A/K_\gamma \in {}^\perp N$; then $K_\gamma \in {}^\perp N$ since ${}^\perp N$ is resolving.

Let γ be a limit ordinal. By inductive premise, $K_\delta \in {}^\perp N$ and $A/K_\delta \in {}^\perp N$ for each $\delta < \gamma$. Since ${}^\perp N$ is resolving, we have also $K_{\delta+1}/K_\delta \in {}^\perp N$, so $K_\gamma = \bigcup_{\delta<\gamma} K_\delta \in {}^\perp N$ by **A3**(1), and $A/K_\gamma = A/\bigcup_{\delta<\gamma} K_\delta \in {}^\perp N$ by **A3**(3).

In particular, for $\gamma = \sigma$, we obtain $M \cong A/K_\sigma \in {}^\perp N$.

At this point, pure–injective and cotorsion modules enter the scene. In Lemma 1.8, we will discuss their role for **A3**(3) in the general setting, but we will have a complete characterization only in the particular case of Dedekind domains (see Lemma 1.9).

The *pure–injective* modules are the same as the algebraically compact modules; see [EM02, Sect. V.1] for details. For $\mathbb{Z}$–modules (abelian groups) and, more generally modules over Dedekind domains, the *cotorsion* modules can be defined as the homomorphic images of pure–injective modules. Equivalently they are the modules N satisfying $\mathrm{Ext}^1(J, N) = 0$ for every torsion–free module J, or just $\mathrm{Ext}^1(Q, N) = 0$ (where Q denotes the quotient field of the domain).

There is a structure theory for pure–injective modules over a Dedekind domain, but the class of all cotorsion modules over a Dedekind domain is quite complex. For example, any torsion module T occurs as the torsion part of a cotorsion module (namely, of $\mathrm{Ext}^1(Q/R, T)$). However the *torsion–free* cotorsion modules are fully understood: they are the pure–injective torsion–free modules, that is, they are the direct sums of copies of Q plus direct products, over all maximal ideals p, of completions of free R_p–modules where R_p denotes the localization of R at p.

A module J over a ring R is *flat* if and only if J is a direct limit of (finitely generated) projective modules. For modules over a Dedekind domain, the flat modules are precisely the torsion–free modules, because, in that case, a finitely–generated module is projective if and only if it is torsion–free.

Definition 1.7. *A module N over a ring R is* cotorsion *if* $\mathrm{Ext}^1(J, N) = 0$ *for every flat module J.*

Lemma 1.8. *Let R be a ring and N a module. If* **A3***(3) holds for the class $({}^\perp N, \prec_N)$ then N is cotorsion.*

Conversely, if N is pure–injective then $^{\perp}N$ is closed under direct limits, so by Lemma 1.5, the class $(^{\perp}N, \prec_N)$ satisfies **A3***(3).*

Proof. If **A3**(3) holds for the class $(^{\perp}N, \prec_N)$ then $^{\perp}N$ is closed under direct limits by Lemma 1.5. Since $^{\perp}N$ always contains all projective modules, it contains also all flat modules, so N is cotorsion.

For the second part, note first that it suffices to show $^{\perp}N$ is closed under homomorphic images of pure epimorphisms because the canonical map of a direct sum onto a direct limit is a pure epimorphism. As in the proof of [ET00, Lemma 9] we can show that $\{A : \mathrm{Ext}^1(A, N) = 0\}$ is closed under homomorphic images of pure epimorphisms for each pure-injective N. By [GT06, 3.2.10], for each $i > 0$ there is a pure–injective module N_i such that $\mathrm{Ext}^i(A, N) \cong \mathrm{Ext}^1(A, N_i)$ for all modules A. Thus also $^{\perp}N$ is closed under homomorphic images of pure epimorphisms, and thus under direct limits.

There is still a gap between the two parts of Lemma 1.8; we do not know exactly the rings for which the hypothesis of N cotorsion is sufficient for **A.3**(3). It is sufficient when R is a Dedekind domain:

Lemma 1.9. *Let R be a Dedekind domain and N a module. Then the following are equivalent:*

(1) *N is cotorsion;*
(2) *$^{\perp}N = {}^{\perp}PE(N)$ where $PE(N)$ denotes the pure–injective envelope of N;*
(3) *$^{\perp}N$ is closed under direct limits;*
(4) **A3***(3) holds for $(^{\perp}N, \prec_N)$.*

Proof. (1) implies (2) by [ET00, Theorem 16 (ii)(b)]. (2) implies (3) by the second part of Lemma 1.8, and (3) is equivalent to (4) by Lemma 1.5. Finally, (4) implies (1) by the first part of Lemma 1.8.

Now, we return to the open problem of characterizing **A3**(3) in the general setting. The reader who is interested mainly in the case of abelian groups, or modules over Dedekind domains, can skip this part and proceed directly to Lemma 1.14.

Lemma 1.10. *Assume that R is a right noetherian ring and the module N has injective dimension ≤ 1. Then the conditions (2), (3) and (4) of Lemma 1.9 are equivalent.*

Proof. By Lemmas 1.5 and 1.8, it suffices to prove that (3) implies (2). By (3) and the fact that N has injective dimension ≤ 1 (hence $^{\perp}N$ is closed under submodules), we have $M \in {}^{\perp}N$ iff all finitely generated submodules of M are in $^{\perp}N$. Since $PE(N)$ is elementarily equivalent to N, $PE(N)$ also has injective dimension ≤ 1 by the Baer Lemma. So also $M \in {}^{\perp}PE(N)$ iff all finitely generated submodules of M are in $^{\perp}PE(N)$. We also have that $M \in {}^{\perp}N$ iff $\mathrm{Ext}^1(M, N) = 0$ and similarly for $^{\perp}PE(N)$.

Given a finitely generated module F, the condition $\mathrm{Ext}^1(F, N) = 0$ is a first order property of N, so $^{\perp}N$ and $^{\perp}PE(N)$ contain the same finitely generated modules. Indeed, let $0 \to G \subseteq R^m \to F \to 0$ be a presentation of F where m is finite. Then $\mathrm{Ext}^1(F, N) = 0$ just says that any R–homomorphism $g : G \to N$ can be extended to an R–homomorphism $f : R^m \to N$.

Let $\{g_1, \dots, g_n\}$ be a finite R–generating subset of G, $g_j = (g_{j1}, \dots, g_{jm}) \in R^m$ for each $j \leq n$. Let $0 \to H \subseteq R^n \xrightarrow{\pi} G \to 0$ be the presentation of G with $\pi(1_j) = g_j$ ($j \leq n$). Let $\{h_1, \dots, h_p\}$ be a finite R–generating subset of H, $h_k = (h_{k1}, \dots, h_{kn}) \in R^n$ ($k \leq p$).

Then $g : G \to N$ is uniquely determined by an n–tuple $(x_1, \dots, x_n)$ of elements of N satisfying $\sum_{j \leq n} x_j h_{kj} = 0$ for each $k \leq p$. Similarly, $f : R^m \to N$ is uniquely determined by an m–tuple $(y_1, \dots, y_m)$ of elements of N. Finally, f extends g if an only if $\sum_{i \leq m} y_i g_{ji} = x_j$ for each $j \leq n$.

In general, condition (1) of Lemma 1.9 does not imply **A3**(3) for $({}^\perp N, \prec_N)$. We will see this in the next example, which is expounded from here until Lemma 1.14.

Recall that a ring R is *right artinian* if R is right noetherian and each flat module is projective. In particular, *every* module over a right artinian ring is cotorsion, so condition (1) is vacuous (satisfied for each N).

Note that right artinian rings and Dedekind domains share the property that each finitely generated module is a direct sum of a projective module and an artinian module. This is the property needed in the next lemma.

Lemma 1.11. *Let be R a right noetherian ring such that each finitely generated module is a direct sum of a projective module and an artinian module. Let N be a module of injective dimension ≤ 1 such that ${}^\perp N$ is closed under direct limits. Then ${}^\perp N$ is a torsion–free class of modules.*

Proof. We modify the proof of Theorem [GT06, 8.2.17(b), p. 291] as follows:

Denote by $\mathcal{F}$ the class of all finitely generated modules in ${}^\perp N$. By assumption, if M is an arbitrary finitely generated module, then $M = F \oplus Q$ where F is projective and Q is artinian. Then $P(M) = \bigcap_{P \subseteq M, M/P \in \mathcal{F}} P$ is the smallest submodule of M such that $M/P(M) \in \mathcal{F}$ (this is because $F \in {}^\perp N$, so $P(M) \subseteq Q$ and Q is artinian, hence finitely cogenerated, and $\mathcal{F}$ is closed under submodules and finite direct products). As in [GT06, 8.2.17(b)], we see that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the category of all *finitely generated* modules where $\mathcal{T}$ denotes the class of all finitely generated modules T such that $\mathrm{Hom}_R(T, F) = 0$ for all $F \in \mathcal{F}$.

By [GT06, 4.5.2(b)], $(\varinjlim \mathcal{T}, \varinjlim \mathcal{F})$ is a torsion pair in the category of *all* modules. By assumption, ${}^\perp N$ is closed under direct limits, so ${}^\perp N \supseteq \varinjlim \mathcal{F}$. Since R is right noetherian and ${}^\perp N$ is closed under submodules, also ${}^\perp N \subseteq \varinjlim \mathcal{F}$. This proves that ${}^\perp N = \varinjlim \mathcal{F}$ is a torsion–free class of modules.

The following lemma will be helpful in constructing the example:

Lemma 1.12. *Let R be as in Lemma 1.11. Let N be a module of injective dimension ≤ 1 such that* $\mathrm{Ext}^1(N, N) = 0$ *and* **A3***(3) holds for $({}^\perp N, \prec_N)$. Then N is pure–injective.*

In particular, if R is a Dedekind domain and N is a module such that $\mathrm{Ext}^1(N, N) = 0$*, then N is cotorsion iff N is pure–injective.*

Proof. By Lemmas 1.5 and 1.11, and by [GT06, 8.2.4], $^{\perp}N$ is a cotilting class of modules, that is, there is a cotilting module C such that $^{\perp}N = {}^{\perp}C$. By [GT06, 8.1.7], each cotilting module is pure–injective. Moreover, by [GT06, 8.1.4(c)], all modules in the subclass $\mathcal{S} = \{B \in {}^{\perp}C : \mathrm{Ext}^i(A, B) = 0 \text{ for all } 1 \leq i < \omega \text{ and all } A \in {}^{\perp}C\}$ are direct summands of direct products of copies of C; in particular, they are also pure–injective. However, our assumption of $\mathrm{Ext}^1(N, N) = 0$ implies $N \in \mathcal{S}$.

If R is a Dedekind domain and N is cotorsion then **A3**(3) holds by Lemma 1.9, so N is pure–injective by the argument above.

Example 1.13. We will show that **A.3**(3) fails for $(^{\perp}N, \prec_N)$ for certain (cotorsion) modules over rings which are (right) hereditary, artinian and countable. We refer to [AHT] for more details on this example. Let R be a tame hereditary finite dimensional algebra over a field k (Note that R is countable when k is.)

If N is a finite dimensional module, then N is endofinite, hence pure–injective, so $^{\perp}N$ is closed under direct limits by Lemma 1.8, and $(^{\perp}N, \prec_N)$ satisfies **A3**(3).

By Lemma 1.12, for the desired example it suffices to take any infinite dimensional tilting module N which is not pure–injective. We claim that the (countable dimensional) Lukas tilting module N from [GT06, 5.1.5(b)] has this property. Indeed, $^{\perp}N$ is the class of all Baer modules which is not closed under direct limits (it contains neither the generic module nor the product of all indecomposable preprojective modules, see [AHT]), and Lemma 1.8 applies.

A4 is rather straightforward.

Lemma 1.14. *If R is any ring and N any module then* **A4** *holds for* $(^{\perp}N, \prec_N)$.

Proof. Suppose $A \subseteq B \subseteq C$ with $A \prec_N C$ and $B \prec_N C$. To show $A \prec_N B$, we need only show $B/A \in {}^{\perp}N$. But this is immediate from the resolving property of $^{\perp}N$ applied to the exact sequence $0 \to B/A \to C/A \to C/B \to 0$.

Verifying **A5** again relies on important concept from homological algebra. We modify the notation in [ET00].

Definition 1.15. *For any right R–module A and any cardinal κ, a (κ, N)–*refinement *of length σ of A is a continuous chain $\langle A_\alpha : \alpha < \sigma \rangle$ of submodules such that $A_0 = 0$, $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$, and $|A_{\alpha+1}/A_\alpha| \leq \kappa$ for all $\alpha < \sigma$.*

We state now a version of Theorem 6 of [ST07] (see also [GT06, 4.2.6]) where we omit some of the conclusions not needed here.

Lemma 1.16 (Hill Lemma)**.** *Let κ be a regular infinite cardinal. Suppose M admits a $\mathcal{C}$–filtration $\langle M_\alpha : \alpha \leq \sigma \rangle$, where $\mathcal{C}$ is a set of $< \kappa$–presented modules. Then there is a family $\mathcal{F}$ of submodules of $M = M_\sigma$ such that:*

1. *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*
2. *$\mathcal{F}$ is closed under arbitrary intersections and sums.*
3. *Let $N \subseteq P$ both in $\mathcal{F}$. Then P/N has a $\mathcal{C}$–filtration.*
4. *If $N \in \mathcal{F}$ and $X \subseteq M$ with $|X| < \kappa$, then there is a $P \in \mathcal{F}$ with $N \cup X \subseteq P$ and P/N is $< \kappa$ presented.*

Lemma 1.17. *Let κ be a cardinal $\geq |R| + \aleph_0$. Let N be a module.*

(1) *If every module $A \in {}^{\perp}N$ has a (κ, N)–refinement then $({}^{\perp}N, \prec_N)$ has Löwenheim–Skolem number κ.*
(2) *If $({}^{\perp}N, \prec_N)$ has Löwenheim–Skolem number κ and satisfies* **A3***(3) then every module $A \in {}^{\perp}N$ has a (κ, N)–refinement.*

Proof. Since $\kappa \geq |R| + \aleph_0$, a (κ, N)–refinement of a module M yields a $\mathcal{C}$–filtration of M where $\mathcal{C}$ is the class of all $\leq \kappa$–presented elements of ${}^{\perp}N$. We use Lemma 1.16 for κ^+. If $A \in \mathcal{F}$, then by Eklof's Lemma and (3) of Lemma 1.16, $A \prec_N M$. Write an arbitrary $X \subseteq M$ as $\bigcup_{i<\mu} X_i$ where $|X_i| \leq \kappa$ and $\mu = |X|$. Then inductively, using condition (4), we select from $\mathcal{F}$ a continuous chain of submodules of M, $\langle N_i : i \leq \mu \rangle$, so that

(1) $N_0 = 0$;
(2) $N_{i+1} \supseteq N_i \bigcup X_i$;
(3) N_{i+1}/N_i is $\leq \kappa$–presented.

Then N_μ is as required.

Conversely, assume $({}^{\perp}N, \prec_N)$ has Löwenheim–Skolem number κ. Let $A \in {}^{\perp}N$ and let X be a set of R–generators of A. A (κ, N)–refinement $\langle A_\alpha : \alpha < \sigma \rangle$ of A consisting of strong submodules of A is constructed by induction as follows:

$A_0 = 0$, and if A_α is defined and there is $x \in A \setminus A_\alpha$ then applying **A5** to A/A_α yields a submodule $A_{\alpha+1}$ containing $A_\alpha + xR$ and such that $A_{\alpha+1}/A_\alpha \in {}^{\perp}N$, $A/A_{\alpha+1} \in {}^{\perp}N$ and $|A_{\alpha+1}/A_\alpha| \leq |(A_\alpha + xR)/A_\alpha| + \kappa$. Since $(A_\alpha + xR)/A_\alpha \cong xR/(A_\alpha \cap xR)$, also $|(A_\alpha + xR)/A_\alpha| \leq \kappa$, hence $|A_{\alpha+1}/A_\alpha| \leq \kappa$. The resolving property of ${}^{\perp}N$ gives $A_{\alpha+1} \in {}^{\perp}N$.

For a limit ordinal α we define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, so $A_\alpha \in {}^{\perp}N$ by **A3**(1), and $A/A_\alpha \in {}^{\perp}N$ by **A3**(3).

Remark 1.18 (Summary)**.** *We have shown that $({}^{\perp}N, \prec_N)$ is an AEC if and only if ${}^{\perp}N$ has refinements and is closed under direct limits of homorphisms.*

But the question of when refinements exist is rather complicated.

Lemma 1.19. *Each member of ${}^{\perp}N$ admits a (κ, N)–refinement under any of the following conditions.*

(1) *N is pure–injective and R is arbitrary; $\kappa = |R| + \aleph_0$.*
(2) *N is cotorsion and R is a Dedekind domain; $\kappa = |R| + \aleph_0$.*
(3) *(V=L) N is arbitrary and R is hereditary; $\kappa = \max\{|R|, |N|\} + \aleph_0$.*

Proof. These results are in [ET00]. (1) follows by Theorem 8; (3) is Theorem 14. (2) follows from (1) by Lemma 1.9.

Now, we just combine the results obtained above for a *Proof of Theorem 0.1*:
Part (1) follows by Lemmas 1.8, 1.14, 1.17(1), and 1.19(1).
Part (2) is by Lemmas 1.5, 1.10, and Part (1).
Part (3) follows by Part (2) and Lemma 1.9.

Theorem 0.1 can be generalized as follows; the proof is a straightforward generalization, where $A \prec_{\mathcal{C}} B$ means that $B/A \in {}^{\perp}\mathcal{C}$:

Theorem 1.20. (1) *Let R be a ring and $\mathcal{C}$ a class of R–modules. If every module in $\mathcal{C}$ is pure–injective, then the class $(^{\perp}\mathcal{C}, \prec_{\mathcal{C}})$ is an AEC. Conversely, if $(^{\perp}\mathcal{C}, \prec_{\mathcal{C}})$ is an AEC then every module in $\mathcal{C}$ is a cotorsion module.*

(2) *Let R be a right noetherian ring and $\mathcal{C}$ be a class of R–modules of injective dimension ≤ 1. Then the class $(^{\perp}\mathcal{C}, \prec_{\mathcal{C}})$ is an AEC if and only if $^{\perp}\mathcal{C}$ is closed under direct limits.*

(3) *If R is a Dedekind domain and $\mathcal{C}$ is a class of R–modules, the class $(^{\perp}\mathcal{C}, \prec_{\mathcal{C}})$ is an AEC if and only if every member of $\mathcal{C}$ is cotorsion.*

Here are some open questions:

Question 1.21. (1) *Can the question of whether a class is an AEC (e.g. $(^{\perp}N, \prec_N)$ for R hereditary) be independent of ZFC?*

(2) *Can the question of whether a class (e.g. Whitehead groups) is a $PC\Gamma$–class (defined as the reducts of models of say a countable theory omitting a family of types) be independent of ZFC? (Note that under $V = L$, 'Whitehead=free' and the class is easily $PC\Gamma$.)*

(3) *Characterize the cotorsion modules N over a ring R such that $(^{\perp}N, \prec_N)$ is an AEC.*

2. Amalgamation, Closures, and Stability

Recent studies in AEC ([She99, GV06, GV, Bal, BS] have focused on those AEC having additional properties such as amalgamation, tameness, and admitting closures. Having established that $(^{\perp}N, \prec_N)$ is an AEC for a number of N, we turn to establishing such model theoretic properties of the AEC.

In this section we show that for any N, $(^{\perp}N, \prec_N)$ has the amalgamation property, and if N has injective dimension ≤ 1 then $(^{\perp}N, \prec_N)$ admits closures iff $^{\perp}N$ is a cotilting class. We will also show that the latter always holds when R is a Dedekind domain, and describe the closures explicitly in this case.

Note that for any N, all projective and in particular all free R–modules are in $^{\perp}N$ so $(^{\perp}N, \prec_N)$ always has arbitrarily large models.

Now we show the first part of Theorem 0.2:

Lemma 2.1. $(^{\perp}N, \prec_N)$ *has the amalgamation property.*

Proof. We just check if $C \prec_N B$ and $C \prec_N A$ then the pushout D of A and B over C is in $^{\perp}N$ and $B \prec_N C$, $A \prec_N D$. Consider the short exact sequence:

$$0 \to C \to B \to B/C \to 0.$$

By the universal mapping property of pushouts we get the diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & D & \to & B/C & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & C & \to & B & \to & B/C & \to & 0 \end{array}$$

Then from the long exact sequence of Ext, we obtain the exact sequence

$$\mathrm{Ext}^i(B/C, N) \to \mathrm{Ext}^i(D, N) \to \mathrm{Ext}^i(A, N).$$

But the first and last entries are 0, so $D \in {}^\perp N$. Now the commutative diagram shows $D/A \cong B/C$ so $A \prec_N D$. Performing the same construction starting with $0 \to C \to A \to A/C \to 0$, shows $B \prec_N D$ and we finish.

Remark 2.2. It is easy to see that in the preceding construction, there is a copy of D over A in which A intersects the image of B in C. Thus our classes have the 'disjoint amalgamation property'.

Note that the argument for amalgamation also yields that each $({}^\perp N, \prec_N)$ has the 'joint embedding property' (any two members can be strongly embedded into some model). As with any AEC with the amalgamation property and joint embedding, we can now see:

Remark 2.3 (Conclusions)**.** If $({}^\perp N, \prec_N)$ is an AEC, then

(1) $({}^\perp N, \prec_N)$ has a monster model $\mathbb{M}$ in the usual sense of AEC, (see [Bal]) (i.e. homogeneous over strong submodels).
(2) $({}^\perp N, \prec_N)$ has EM–models and models generated by indiscernibles.

To study stability, we must define it. And for this we must introduce the notion of type that is appropriate here. We will use the monster model in our discussion of Galois types.

Definition 2.4. *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an AEC with the amalgamation property and joint embedding.*

(1) *Define*
$$(M, a, N) \cong (M, a', N')$$
if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with
$$f(a) = f'(a').$$
(2) *'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N'' if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.*
(3) *We write $\mathrm{tp}(a/M)$ for the Galois type of a over M. This can be thought of as either:*
   (a) *the equivalence class of $(M, a, \mathbb{M})$;*
   (b) *the orbit of a under $\mathrm{aut}_{\mathrm{M}}(\mathbb{M})$.*
(4) *Let $M \subseteq N \subset \mathbb{M}$ and $a \in \mathbb{M}$. The restriction of $\mathrm{tp}(a/N)$ to M, denoted $\mathrm{tp}(a/N) \restriction M$ is the orbit of a under $\mathrm{aut}_{\mathrm{M}}(\mathbb{M})$.*
(5) $\mathrm{ga-S(M)}$ *denotes the collection of Galois–types over M.*
(6) *$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is λ–stable if for every M with $|M| = \lambda$, $|\mathrm{ga-S(M)}| = \lambda$.*

The definition of restriction makes sense since our classes have a monster model. Note that the definition of Galois type in 1) and 2) makes sense without an amalgamation assumption (but 'generation' in 2) means transitive closure of the given relation). With amalgamation we get the simpler relation of 3).

The analysis of Galois types in our situation relies on a definition from [BS].

Definition 2.5. *We say the AEC* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admits closures *if for every* $X \subseteq M \in \boldsymbol{K}$*, there is a minimal closure of* X *in* M*. That is, the structure with universe* $\bigcap\{N : X \subseteq N \prec_{\boldsymbol{K}} M\}$ *is a strong submodel of* M*. If so, we denote it:* $\mathrm{cl}_M(X)$.

In our particular setting, the existence of closures translates into $^{\perp}N$ being a cotilting class. To see this, we first show that the existence of closures for modules in $^{\perp}N$ implies the existence of closures for arbitrary modules. We start by allowing the relation $\prec_N$ to hold between arbitrary modules instead of just members of $^{\perp}N$.

Definition 2.6. *If* A *and* B *are* arbitrary *modules, we will say that* $A{\prec_N}'B$ *iff* $B/A \in {}^{\perp}N$.

Notice that if $A{\prec_N}'B$ then $A \in {}^{\perp}N$ iff $B \in {}^{\perp}N$ since $^{\perp}N$ is resolving.

Lemma 2.7. *Assume that* $({}^{\perp}N, \prec_N)$ *is an AEC that admits closures. Let* X *be a subset of a module* M *and let* $Q' = \bigcap\{Q : X \subseteq Q{\prec_N}'M\}$*. Then* $Q'{\prec_N}'M$.

Proof. Fix an epimorphism $\pi : F \to M$ where F is a free module. For each submodule Q of M, denote by $\mu_Q : Q \to M$ the inclusion, and consider the pullback of π and μ_Q:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_Q & \xrightarrow{\nu_Q} & F & \longrightarrow & M/Q & \longrightarrow & 0 \\ & & \big\downarrow & & \pi\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Q & \xrightarrow{\mu_Q} & M & \longrightarrow & M/Q & \longrightarrow & 0 \end{array}$$

Notice that P_Q is just the set of all pairs $(m, f) \in M \oplus F$ where $m \in Q$, $\pi(f) = m$, and ν_Q is the (injective) restriction to P_Q of the projection $M \oplus F \to F$. Moreover, if $Q{\prec_N}'M$ then $F, M/Q \in {}^{\perp}N$, so $P_Q \in {}^{\perp}N$ since $^{\perp}N$ is resolving, and $\nu_Q(P_Q){\prec_N}F$.

Let $\mathcal{Q} = \{Q : X \subseteq Q{\prec_N}'M\}$. Notice that $\nu_{Q'}(P_{Q'}) = \bigcap\{\nu_Q(P_Q) : Q \in \mathcal{Q}\}$: the inclusion $\subseteq$ is obvious, and if $f \in \bigcap\{\nu_Q(P_Q) : Q \in \mathcal{Q}\}$ then $\pi(f) \in Q'$ so $f \in \nu_{Q'}(P_{Q'})$.

Since $({}^{\perp}N, \prec_N)$ admits closures, $\nu_{Q'}(P_{Q'})$ is a strong submodule of F. But this implies $M/Q' \in {}^{\perp}N$, that is, $Q'{\prec_N}'M$.

Lemma 2.8. *Let* R *be a ring, and* N *be a module of injective dimension* ≤ 1*. Then the following conditions are equivalent:*

(1) $({}^{\perp}N, \prec_N)$ *is an AEC that admits closures;*
(2) $^{\perp}N$ *is closed under direct products;*
(3) $^{\perp}N$ *is a cotilting class.*

These conditions imply that $^{\perp}N$ *is first order axiomatizable and moreover, the closure,* $\mathrm{cl}_M(X)$*, is determined by* $\mathrm{cl}_M(X)/X$ *being the* $\mathcal{T}$*–torsion part of* M/X *where* $(\mathcal{T}, {}^{\perp}N)$ *is the cotilting torsion pair.*

Proof. The implication (1) $\to$ (2) holds for any module N:

Assume that (1) holds and there are a cardinal κ and modules $M_\alpha \in {}^\perp N$ ($\alpha < \kappa$) such that $P = \prod_{\alpha<\kappa} M_\alpha \notin {}^\perp N$. We take κ the smallest possible; notice that κ is infinite since ${}^\perp N$ is closed under direct sums. For each $\alpha < \kappa$, let $P_\alpha = \{p \in P : p_\beta = 0 \text{ for all } \beta < \alpha\}$. Then $P/P_\alpha \cong \prod_{\beta<\alpha} M_\beta \in {}^\perp N$ by the minimality of κ, so $P_\alpha \prec_N P$. Lemma 2.7 implies that $\bigcap_{\alpha<\kappa} P_\alpha \prec_N{}' P$. But $\bigcap_{\alpha<\kappa} P_\alpha = 0$, so $P/(\bigcap_{\alpha<\kappa} P_\alpha) \cong P \in {}^\perp N$, a contradiction.

(2) implies (3) by [GT06, 8.1.10].

Assume (3). Since any cotilting class is of the form ${}^\perp N$ for a cotilting module N, and N is pure–injective by [GT06, 8.1.7], $({}^\perp N, \prec_N)$ is an AEC by Theorem 0.1(1).

Let $X \subseteq M \in {}^\perp N$. Let $Q' = \bigcap\{Q : X \subseteq Q \prec_N M\}$. Then the map $\varphi : M/Q' \to \prod_{X\subseteq Q\prec_N M} M/Q$ assigning to each $m + Q'$ the sequence $(m+Q)_{X\subseteq Q\prec_N M}$ is monic. Since ${}^\perp N$ is a cotilting class, it is closed under direct products and submodules, so $\prod_{X\subseteq Q\prec_N M} M/Q \in {}^\perp N$, and $M/Q' \in {}^\perp N$. As ${}^\perp N$ is resolving, we conclude that Q' is a strong submodule of M, so $({}^\perp N, \prec_N)$ admits closures.

Finally, any cotilting class is 'definable' (that is, closed under direct products, direct limits and pure submodules), so it is axiomatizable (by axioms saying that certain of the Baur-Garavaglia-Monk invariants are equal to 1), cf. [CB98, 2.3] and [Pr88, p.34].

Moreover, since $(\mathcal{T}, {}^\perp N)$ is a torsion pair, we have $M/Y \cong (M/X)/(Y/X) \in {}^\perp N$ where Y/X is the $\mathcal{T}$–torsion part of M/X, and if $X \subseteq P \prec_N M$ then $Y \subseteq P$ since otherwise $(Y/X)/(P/X \cap Y/X) \cong (P/X + Y/X)/(P/X) \cong (P+Y)/P$ is isomorphic to a non-zero $\mathcal{T}$–torsion submodule of $M/P \in {}^\perp N$. This proves that $\mathrm{cl}_M(X)/X = Y/X$.

Our main applications are to abelian groups, or modules over Dedekind domains, so we note that in that case the AEC $({}^\perp N, \prec_N)$ always admits closures, that is, ${}^\perp N$ is a cotilting class. Moreover, we provide an explicit construction of the closures.

Recall that for a set of maximal ideals P, $\mathbf{K}(P)$ is defined as the class of all modules that are p–torsion–free for all $p \in P$.

Lemma 2.9. *Let R be a Dedekind domain. If $({}^\perp N, \prec_N)$ is an AEC, then ${}^\perp N$ coincides with the (first order aximatizable) class $\mathbf{K}(P)$ for a set P of maximal ideals of R and $({}^\perp N, \prec_N)$ admits closures. The closure $\mathrm{cl}_M(X)$ coincides with the submodule of M consisting of all $m \in M$ such that the ideal $\{r \in R : mr \in X\}$ is a finite product of elements of P.*

Conversely, for any set of maximal ideals P, there is a pure–injective module N such that $({}^\perp N, \prec_N)$ is an AEC such that ${}^\perp N = \mathbf{K}(P)$.

Proof. For the first claim, let P be the set of all maximal ideals p of R such that $R/p \notin {}^\perp N$. Then ${}^\perp N = \mathbf{K}(P)$ by [ET00, Theorem 16(i)]. For the second claim, it suffices, by Lemma 2.8, to show that $\mathbf{K}(P)$ is closed under direct products; but this is obvious from the definition of $\mathbf{K}(P)$.

By Lemma 2.8, $\mathrm{cl}_M(X)/X$ coincides with the $\mathcal{T}$–torsion part of M/X where $(\mathcal{T}, {}^{\perp}N)$ is the cotilting torsion pair. By [GT06, 8.2.11], this torsion pair is hereditary in the sense of [St75, VI.§3], that is, there is a Gabriel topology (filter) $\mathfrak{F}$ of ideals of R such that for each module N, $N \in \mathcal{T}$ iff each element from N is annihilated by an element of $\mathfrak{F}$. Since each non–zero ideal of R is uniquely a finite product of prime ideals (see [Ma94, Theorem 11.6]), and ${}^{\perp}N = \mathbf{K}(P)$, we infer that $\mathfrak{F}$ coincides with the set of all ideals of R which are finite products of elements of P (here we include R as the 'empty' case of such product). In particular, the $\mathcal{T}$–torsion part of M/X consists of all elements $m \in M$ such that $\{r \in R : mr \in X\}$ is a finite product of prime ideals from P.

Conversely, for any set of maximal ideals P, $\mathbf{K}(P) = {}^{\perp}\prod_{p\in P} \hat{R}_p$ where $\hat{R}_p$ denotes the completion of R_p, and R_p is the localization of R at p (see [ET00, Theorem 16(ii)(a)]); $\hat{R}_p$ is a complete torsion–free module, hence $\hat{R}_p$ is pure injective.

Finally, we prove that $\mathbf{K}(P)$ is first order axiomatizable for any set of maximal ideals P. For each $p \in P$, let $\{a_{p,1}, \dots, a_{p,k_p}\}$ be a finite R–generating subset of p. The first–order theory T_P which asserts that the simple module R/p does not embed into N for all $p \in P$ just consists of the implications $(x.a_{p,1} = \cdots = x.a_{p,k_p} = 0) \to x = 0$ where p runs over all maximal ideals in P.

Parts (2) and (3) of Theorem 0.2 now follow immediately from Lemmas 2.8 and 2.9, respectively.

Remark 2.10. (i) The class ${}^{\perp}N = \mathbf{K}(P)$ above is elementary, but the unusual definition of $\prec_N$ yields some new phenomena: we require that $A \prec_N B$ iff $B/A \models T_P$. This distinction was reflected for example in Lemma 2.7. While, by definition of cotilting, if N is cotilting any submodule A of a member B of ${}^{\perp}N$ is in ${}^{\perp}N$, $A \prec_N B$ may fail. In [BS] an AEC $\boldsymbol{K}$ is called *model complete* if $A \subset B$ and $A, B \in \boldsymbol{K}$ implies $A \prec_{\boldsymbol{K}} B$. Here we have natural examples where this condition fails: just consider a non–pure subgroup A of a torsion–free group B.

(ii) The equivalence (2) $\leftrightarrow$ (3) in Lemma 2.8 extends to any module N of injective dimension $n < \omega$ if we replace 'cotilting class' by 'n–cotilting class' in the sense of [GT06, Chap. 8].

(iii) That $({}^{\perp}N, \prec_N)$ admits closures whenever $({}^{\perp}N, \prec_N)$ is an AEC holds for any right noetherian ring whose finitely generated modules decompose into direct sums of projective and artinian modules, and any module N of injective dimension ≤ 1.

Notice that there exist rings R and AEC's of the form $({}^{\perp}N, \prec_N)$ that do not admit closures:

Example 2.11. Consider any ring R which is right perfect (that is, all flat modules are projective), but not left perfect (see [AF92, p.322]). Then R is not left coherent (that is, there exists a finitely generated left ideal which is not finitely presented, see [Pr88, 14.23]). Let N be the direct sum of a representative set of all simple modules. By [GT06, 4.1.8], ${}^{\perp}N$ is the class of all projective modules (and $\prec_N$ is the relation of being a direct summand). Since ${}^{\perp}N = {}^{\perp}\mathcal{C}$ where $\mathcal{C}$ is the class of all pure–injective modules (see [GT06, 2.2.3]), Theorem 1.20(1) implies that $({}^{\perp}N, \prec_N)$ is an AEC.

But R is not left coherent, so the class of all flat (= projective) modules is not closed under direct products by a classical result of Chase [AF92, 19.20]. Hence $({}^{\perp}N, \prec_N)$ does not admit closures by (the implication (1) → (2) of) Lemma 2.8.

We conclude this section by describing some general properties of Galois types. In Section 3, we will use these concepts to study the stability of $({}^{\perp}N, \prec_N)$ for abelian groups N.

If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admits closures then we have the following check (from [BS]) for equality of Galois types. The relevance of the second clause is that even in the absence of amalgamation it shows that equality of Galois types is determined by the basic relation of Definition 2.4, rather than its transitive closure. By $M_1 \restriction \mathrm{cl}_{M_1}(M_0a_1)$ we simply mean the structure M_1 induces on the minimal $\boldsymbol{K}$ substructure containing M_0a_1.

Lemma 2.12. *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ admit closures.*

(1) *Suppose $M_0 \prec_{\boldsymbol{K}} M_1, M_2$ with $a_i \in M_i$ for $i = 1, 2$. Then $\mathrm{tp}(a_1/M_0, M_1) = \mathrm{tp}(a_2/M_0, M_2)$ if and only if there is an isomorphism over M_0 from $M_1 \restriction \mathrm{cl}_{M_1}(M_0a_1)$ onto $M_2 \restriction \mathrm{cl}_{M_2}(M_0a_2)$ which maps a_1 to a_2.*

(2) *(M_1, a_1, N_1) and (M_2, a_2, N_2) represent the same Galois type over M_1 iff $M_1 = M_2$ and there is an amalgam of N_1 and N_2 over M_1 where a_1 and a_2 have the same image.*

The syntactic type of an element a over a set A is the set of formulas $\phi(x)$ in some logic with parameters from A that are satisfied by a. Such a type describes the relation of a to A in the sense of the logic under consideration. In saturated models of a first order theory the type of a over A determines the orbit of a under automorphisms of the monster model fixing A. The Galois type of a over A is only defined when A is a model M. But then, it corresponds precisely to the orbit of a under automorphisms of the monster model fixing M. Syntactic types have certain natural locality properties.

locality: Any increasing chain of types has at most one upper bound;
tameness: two distinct types differ on a finite set;
compactness: an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general. We will show that when N is an abelian group, Galois types of elements correspond to syntactic types of countable sequence. This means that these locality properties do hold. Since the translation rather than the specific property is the crucial issue here, we give the technical definition of only one of the properties.

Definition 2.13. (1) *We say $\boldsymbol{K}$ is (χ, μ)–*tame *if for any $N \in \boldsymbol{K}$ with $|N| = \mu$ if $p, q, \in \mathrm{ga{-}S(N)}$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.*

(2) *$\boldsymbol{K}$ is (χ, ∞)–*tame *if it is $\boldsymbol{K}$ is (χ, μ)–*tame *for all μ.*

3. The case of abelian groups

The main goal of this section is the proof of Theorem 0.3. First we note that closures in the sense of Definition 2.5 take a simpler form for abelian groups:

By Lemma 2.9, $\mathrm{cl}_B(X)$ is just the closure of X with respect to divisibility by p for each $p \in P$, that is, $\mathrm{cl}_B(X)$ is the set of all $y \in B$ such that for some m which is a product of powers of primes in P, $my \in X$. Note that for any $A <_{\mathbf{K}(P)} B \in \mathbf{K}(P)$ and $b \in B$, $\mathrm{cl}_B(A, b)$ is countably generated over A.

This has corollaries concerning tameness and stability.

Corollary 3.1. *If N is an abelian group, the Galois type of a over $M \in {}^{\perp}N$ is determined by the quantifier-free type of a countable sequence associated with a.*

This implies $({}^{\perp}N, \prec_N)$ is $(\aleph_0, \infty)$–tame and stable in λ if $\lambda^{\omega} = \lambda$.

Proof. Let $A \prec_N B$ and $a \in B$. Then $\mathrm{cl}_B(Aa)$ is generated by some countable set $\mathbf{c}$. So by Lemma 2.12, the Galois type of a over B is determined by the quantifier-free first order type of $\mathbf{c}$ over B. This means that if a and a' have different Galois types over B, the associated sequences $\mathbf{c}, \mathbf{c}'$ have different quantifier-free types over B and thus over a countable submodel B_0 of B; this is tameness. And the number of Galois types over a model of cardinality λ is at most the number of quantifier-free types of ω–sequences. Since over any module of cardinality λ there are only λ quantifier free types of finite sequences, there are λ^{ω} types of ω–sequences.

The translation to quantifier-free types also gives full compactness and locality for Galois types in this context; see the chapter, Locality and Tameness, in [Bal] for precise definitions.

Now we can prove Theorem 0.3(1), that is, if P is a non–empty set of primes of $\mathbb{Z}$ and λ is an infinite cardinal, then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is stable in λ if and only if $\lambda^{\omega} = \lambda$. We just proved that $\lambda^{\omega} = \lambda$ is a sufficient condition for stability. Fix a prime p in P. For a given infinite cardinal λ, fix an enumeration $\{f_i : i \in \lambda^{\omega}\}$ of λ^{ω} different one-one functions from ω to λ. Let $S' = \omega \times \lambda$, and let A be the free abelian group with basis $S = \{e_s : s \in S'\}$. For each $i \in \lambda^{\omega}$, let B_i be F_i/K_i where F_i is the free group on $S \cup \{b_i\} \cup \{z_{n,i} : n \in \omega\}$ and K_i is the subgroup generated by

$$w_{n,i} = p^{n+1} z_{n,i} - b_i - \sum_{\ell=0}^{n} p^{\ell} e_{\ell, f_i(\ell)}.$$

It is easy to check that this generating set is a basis. We show that the map taking $a \in A$ to $a + K_i$ embeds A as a subgroup of B_i; indeed, suppose that $a = \sum_{s \in S'} c_s e_s$ and $a + K_i = 0$, that is,

$$\sum_{s \in S'} c_s e_s = \sum_{n \in \omega} d_n w_{n,i}$$

or

$$\sum_{s\in S'} c_s e_s = \sum_{n\in\omega} d_n(p^{n+1} z_{n,i} - b_i - \sum_{\ell=0}^{n} p^\ell e_{\ell,f_i(\ell)})$$

where $c_s, d_n \in \mathbb{Z}$. This is an equation in F_i, and since $S\cup\{b_i\}\cup\{z_{n,i} : n \in \omega\}$ is a basis of F_i, we can equate coefficients of basis elements on each side. In particular, the coefficient of $z_{n,i}$ on the right is $d_n p^{n+1}$ and on the left is 0; thus $d_n = 0$. But then the right–hand side is zero, i.e. $a = 0$.

We also prove that A embeds as a pure subgroup of B_i, which implies that it embeds as a $\mathbf{K}(P)$-submodel, since B_i is torsion-free. Aiming at a contradiction, suppose that q is a prime and for some $a \in A$, $a + K_i$ is divisible by q in B_i but not in A. Say $a = \sum_{s\in S'} c_s e_s$ and

$$\sum_{s\in S'} c_s e_s = qy + \sum_{n\in\omega} d_n(p^{n+1} z_{n,i} - b_i - \sum_{\ell=0}^{n} p^\ell e_{\ell,f_i(\ell)})$$

where $y \in F_i$ and some c_s is not divisible by q. Comparing the coefficient of b_i on both sides, we conclude that $\sum_{n\in\omega} d_n \equiv 0$ (mod q); but then the coefficient of e_s on the right is also congruent to 0 mod q, a contradiction.

Also, in $B_i/\langle A, b_i\rangle$ the coset of $z_{n,i}$ has order p^{n+1}, so $z_{n,i}$ belongs to $cl_{B_i}(A, b_i)$ for all $n \in \omega$.

We claim that for all $i \neq j$, there is no isomorphism $\varphi : cl_{B_i}(A, b_i) \to cl_{B_j}(A, b_j)$ which is the identity on A and takes b_i to b_j. This will suffice to prove Theorem 0.3(1) in view of Lemma 2.12(1). Suppose, to the contrary that ϕ is such an isomorphism. Let n be minimal such that $f_i(n) \neq f_j(n)$. By construction, in $cl_{B_i}(A, b_i) \subseteq B_i$ we have

$$p^{n+1} z_{n,i} = b_i + \sum_{\ell=0}^{n} p^\ell e_{\ell,f_i(\ell)}$$

so applying φ we get

$$p^{n+1}\varphi(z_{n,i}) = b_j + \sum_{\ell=0}^{n} p^\ell e_{\ell,f_i(\ell)}$$

in B_j. But by construction, in B_j we have

$$p^{n+1} z_{n,j} = b_j + \sum_{\ell=0}^{n} p^\ell e_{\ell,f_j(\ell)}.$$

Subtracting the last two equations, and taking into account the minimality of n, we get that p^{n+1} divides

$$p^n(e_{n,f_i(n)} - e_{n,f_j(n)})$$

in B_j. But this is a contradiction of the purity of $A + K_j$ in B_j, so the proof of 0.3(1) is complete.

Now we take up the proof of 0.3(2). Suppose that R is a p.i.d. and P is a non-empty set of primes of cardinality κ (finite or infinite). Let $\kappa' = \kappa + \omega = \max\{\kappa, \omega\}$. Fix an infinite cardinal λ. The argument in 3.1 generalizes to show that in $\mathbf{K}(P)$, the Galois type of an element is determined by the quantifier-free type of a sequence of length κ', and hence, as in 0.3(1), $\lambda^{\kappa'} = \lambda$ is a sufficient condition for stability at λ. If two Galois types are different, the associated quantifier free types must disagree (actually on finite set) and therefore the Galois types have different restrictions to an $M \in \mathbf{K}(P)$ with $|M| = \mathrm{LS}(\mathbf{K}(\mathrm{P}))$, so $(\mathbf{K}(P), \prec_{\mathbf{K}(P)})$ is $(\mathrm{LS}(\mathbf{K}(\mathrm{P})), \infty)$-tame If $\kappa' = \omega$ (i.e., κ is at most countable), the argument in part (1) shows that $\lambda^\omega = \lambda$ is also necessary for stability. So we consider the case when $\kappa = \kappa'$ is an infinite (even uncountable) cardinal. Fix an enumeration $\{p_\nu : \nu < \kappa\}$ of the elements of P, and fix an enumeration $\{f_i : i \in \lambda^\kappa\}$ of λ^κ different one-one functions from κ to λ. Let A be the free abelian group with basis $S = \{e_\alpha : \alpha \in \lambda\}$. For each $i \in \lambda^\kappa$, let B_i be F_i/K_i where F_i is the free group on $S \cup \{b_i\} \cup \{z_{\nu,i}\}$ and K_i is the subgroup generated by

(1) $$\{p_\nu z_{\nu,i} - b_i - e_{f_i(\nu)} : \nu \in \kappa\}.$$

Once again, it is easy to check that this generating set is a basis and that the map taking $a \in A$ to $a + K_i$ embeds A as a pure subgroup of B_i. Also, in $B_i/\langle A, b_i\rangle$ the coset of $z_{\nu,i}$ has order p_ν, so $z_{\nu,i}$ belongs to $cl_{B_i}(A, b_i)$ for all $\nu \in \kappa$.

Aiming at a contradiction, suppose that for some $i \neq j$, there is an isomorphism $\varphi : cl_{B_i}(A, b_i) \to cl_{Bj}(A, b_j)$ which is the identity on A and takes b_i to b_j. Fix $\nu \in \kappa$ such that $f_i(\nu) \neq f_j(\nu)$. By construction, in $cl_{B_i}(A, b_i) \subseteq B_i$ we have

$$p_\nu z_{\nu,i} = b_i + e_{f_i(\nu)}$$

so applying φ we get

$$p_\nu \varphi(z_{\nu,i}) = b_j + e_{f_i(\nu)}$$

in B_j. But by construction, in B_j we have

$$p_\nu z_{\nu,j} = b_j + e_{f_j(\nu)}.$$

Subtracting the last two equations, we get that p_ν divides

$$e_{f_i(\nu)} - e_{f_j(\nu)}$$

in B_j. But this contradicts the fact that A is a pure subgroup of B_j and completes the proof of 0.3(2).

We finish with a more specific comment about abelian groups, recall that if $P \neq \emptyset$, then $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is stable only in λ with $\lambda^\omega = \lambda$. In contrast [BCG+00] shows that the class of all abelian groups under subgroups is stable in all infinite cardinalities. And this is exactly the case $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ when P is empty. So in this one case, $(\mathbf{K}(P), <_{\mathbf{K}(P)})$ is stable in all cardinalities.

One is naturally led to ask whether analogs to properties of the first order stability classification hold in this situation. Recall that a countable first order theory that is not superstable has 2^λ models in every uncountable power. The analogous question for AECs is open; indeed, the choice of the appropriate version of superstable in this context is open. But it is easy to see for an abelian group N that each $^\perp N$ has 2^λ models in each cardinal as each class contains a non–superstable group whose theory thus has 2^λ models of cardinality λ and $^\perp N$ is closed under elementary equivalence since it is first order axiomatizable.

Remark 3.2. The stability spectrum problem for arbitrary AECs has not been solved. There are explicit results for tame AECs in [GV06] and [BKV00]. See also for example [She99].

Question 3.3. (1) *For which rings R does each AEC of the form $({}^{\perp}N, \prec_N)$ admit closures? (This holds for all Dedekind domains and all right artinian right hereditary rings, but not for all rings by Lemma 2.9, Remark 2.10(iii) and Example 2.11.)*
(2) *What is the stability spectrum of $({}^{\perp}N, \prec_N)$ for other rings R and modules N? Does the condition that $({}^{\perp}N, \prec_N)$ is stable in all cardinals (or all cardinals beyond the continuum) provide any further algebraic conditions on N?*

4. Further Directions

These examples have provided illustrations for a number of notions of notions of AEC; in particular, the notion of closure is clarified by providing classes where it does and does not hold. This family of examples also provides one explanation of how tameness can be obtained: the Galois type of an element is determined by the syntactic type of an associated short sequence. But tameness can hold for deeper reasons [BS, BK]; it would be interesting to explore when this simple explanation works. Here are some more specific questions.

Question 4.1. (1) *For modules over rings other than Z, what are the tameness properties of $({}^{\perp}N, \prec_N)$?*
(2) *Does $({}^{\perp}N, \prec_N)$ have finite character in the sense of Hyttinnen-Kesala* [HK]*?*
(3) *For which R and N is $({}^{\perp}N, \prec_N)$ axiomatizable (in infinitary logic)? One might expect to use $L_{\kappa,\omega}$ if the ring had cardinality κ.*
(4) *What can we say about the number of models in various cardinalities of $({}^{\perp}N, \prec_N)$ (for general N and R)?*

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