Rank and Homogeneous Structures

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

April 7, 2001

The notion of constructing a homogenous structure was introduced in the 1950's by Fraïssé. It was extended by Jonsson from countable to uncountable relational structures. In 1969, Grzegorczyk asked how many theories are categorical in $\aleph_0$. The amalgamation construction was quickly used [24, 27, 21] to show there are continuum many such theories. These constructions clearly gave rise to unstable structures. As stability theory developed, the problem of classifying stable $\aleph_0$-categorical structures arose[1]. Lachlan [40] and Shelah[51] showed that every superstable $\aleph_0$-categorical theory was ω-stable. Work of Cherlin, Harrington, Lachlan, Hrushovski and Zilber showed there were only countably many ω-categorical, ω-stable structures (and much more). But, Lachlan's conjecture that there was no strictly stable $\aleph_0$-categorical structure remained open until the late 80's.

Mainly in infinitary contexts Shelah (e.g. [52]) had studied variants of the construction which strengthen amalgamation to 'free amalgamation'. The freeness of the amalgamation corresponds to a stability condition. But Hrushovski provided a concrete way of constructing such amalgams in [36]. With this method he refuted both Lachlan's conjecture and Zilber's conjecture that every strongly minimal set was 'bi-interpretable' with a discrete set, a vector space, or a field.

As we outline below these are two of a large family of variants of the amalgamation construction, which are determined by what we call here a 'rank' function on a class of models. Unexpectedly, the counterexample to the Lachlan conjecture is intimately related with the almost sure theories of random graphs. For irrational $\alpha, 0 < \alpha < 1$, let μ_n^α be the probability measure on graphs of size n given by edge probability $n^{-\alpha}$. For every first order sentence ϕ, $\lim_{n\to\infty} \mu_n^\alpha(\phi)$ is 0 or 1. The theory T^α of the almost sure sentences is

*The author thanks the organizers of the Wurzburg conference for a valuable meeting. This paper was prepared at the Mittag-Leffler institute during the Fall of 2000 and the support of the institute is greatfully acknowledged. Partially supported by NSF grant DMS-9510377.

[1]Stability theory provides a hierarchy of theories which provides a framework for dividing questions in model theory. This is described in detail in such texts as [4, 51, 42]; for an overview see the attached table???

stable and has the finite model property. It was constructed by variants on the Hrushovski construction in [12], improving the argument of [54]. More expectedly, there have been a large number of variants on the construction to construct groups, expansions of fields and various kinds of geometries with specific model theoretic properties. Our purpose here is to partially systematize this family of constructions and to provide a comprehensive list of the applications during the first 10 years of its history.

Each application of this method depends on the choice of a class $\boldsymbol{K}$ of models and a rank function δ on the members of $\boldsymbol{K}$. This rank induces a notion $\leq$ of strong substructure on some (usually all) substructures of members of $\boldsymbol{K}$. Under appropriate hypotheses a homogeneous model for the pair $(\boldsymbol{K}, \leq)$ is constructed. The rank reenters the argument in several ways to reduce the quantifer complexity of the theory of the homogeneous model and to prove the stability conditions.

As in some papers ([11]) too many adjectives were being piled on the word homogeneous, the term 'generic' (first applied in this context in [39]) is often used for the model constructed. It has long been a curiosity to me that the structure constructed in random graph case [12] is a countable first order homogeneous model which is neither saturated nor prime. In his first papers on this subject Hrushovski referred to the difficulty of determining that the model constructed was a 'first order structure'. One rendering of this metaphor is that it asks whether the structure is ω-saturated. We describe in Section 5 another explanation, which was at least implicit in [50] and clarified in [33, 48, 37, 44, 19]. The question becomes, 'Is the class of existentially closed models in $\boldsymbol{K}$ (in a suitably expanded language) an elementary class?'.

In the first section of the paper we lay out a general context for these examples. In the second, we rehearse a large number of examples showing their dependence on the choice of the class of structures and the rank. Many papers [14, 22, 59, 48, 25] have laid out the basics of the method of construction so we don't expound all of those details here. In Section 3 we discuss the definablility requirements on the rank functions and connections with algebraic geometry which arise in constructing expansions of fields. Section 4 describes how the rank produces the stability requirement. These arguments work best when the homogeneous structure is saturated; in Section 5, we examine the situation when saturation fails.

1 Setting

Let $\langle K(N), \wedge, \vee\rangle$ be a lattice of substructures of a model N. For purposes of this paper a rank is a function δ from $K(N)$ to a discrete subgroup of the reals $(\mathcal{R})$, which is defined on each N in a class $\boldsymbol{K}$. We write $\delta(A/B) = \delta(A \vee B) - \delta(B)$ to indicate the relativization of the rank. We demand *only* that δ is monotonic: if $B \subseteq A, C \subseteq N$ and $A \wedge C = B$,

$$\delta(B/A) \geq \delta(B/C).$$

This requirement can be rephrased as asserting that δ is lower semimodular:

$$\delta(A \vee B) - \delta(B) \leq \delta(A) - \delta(A \wedge B).$$

We say δ is modular if the inequality is an equality. Examples of δ include cardinality, relation size (number of instances of a relation), vector space dimension, and transcendence degree. All of these but the last are modular. The simplest example of 'relation size' is just the number of edges in a (symmetric) graph.

This notion of 'rank' is much weaker than any other of the rank notions used in stability theory. Several authors [14, 61] have used the term predimension; this seems inappropriate in the general context since the resulting combinatorial structure may not satisfy exchange and thus have nothing like a dimension.

Let T_{-1} be a theory such that any subset X of a model N of T_{-1} is contained in a minimal submodel of N; this implies there is a natural notion of a finitely generated model. We denote this submodel $\langle X\rangle_N$, dropping the subscript N when the choice of N is evident. This condition is clearly satisfied if T_{-1} is universally axiomatized or strongly minimal and almost all of our examples fall into one of these two classes. This condition is closely connected (see [29] 6.4 and 6.5) with the requirement that T_{-1} is axiomatized by universal-existential sentences and thus that the class of its models is closed under unions of chains. Let $\overline{\boldsymbol{K}}_{-1} = \mathrm{mod}(T_{-1})$; $\boldsymbol{K}_{-1}$ is the *finitely generated* members of $\overline{\boldsymbol{K}}_{-1}$. Some of the choices for T_{-1} include: T_{-1} is a universal theory in a finite relational language; $\boldsymbol{K}_{-1}$ is the finite models of T_{-1}; T_{-1} is Acf_p; $\boldsymbol{K}_{-1}$ contains those algebraically closed fields of finite transcendence degree; more generally, T_{-1} is a strongly minimal, inductive theory with elimination of quantifiers and imaginaries and the definable multiplicity property; $\boldsymbol{K}_{-1}$ contains the models generated by finitely many independent elements.

The construction of the homogeneous model is made with respect to a notion of strong substructure. We define the most used notion now. In Application 2.2, we discuss a variant which produces simple theories.

Definition 1.1 *For $N \models T_{-1}$, $K(N)$ is the substructures of N which are in $\boldsymbol{K}_{-1}$. For $A, B \in \overline{\boldsymbol{K}}_{-1}$, we say A is a* strong substructure *of B and write $A \leq B$ if for every $B' \in \boldsymbol{K}_{-1}$ with $B' \subseteq B$, $\delta(B'/B' \cap A) \geq 0$.*

The definition easily implies that the operation of assigning to a set the smallest superset containing it that is strong in the universe satisfies all the axioms of a combinatorial pregeometry except exchange. This is a crucial exception. In most cases that we discuss there is no way to assign a single 'dimension' to a model.

Definition 1.2 *We denote by $\overline{\boldsymbol{K}}_0$ the members of $\overline{\boldsymbol{K}}_{-1}$ which have hereditarily positive rank and by $\boldsymbol{K}_0$ those which are finitely generated and have hereditarily positive rank. T_0 denotes the theory of $\overline{\boldsymbol{K}}_0$,* (In some cases, showing $\overline{\boldsymbol{K}}_0$ is an elementary class requires some effort.) Usually, it makes little difference whether one constructs homogeneous models wtih respect to $\boldsymbol{K}_0$ or $\overline{\boldsymbol{K}}_0$. We explore the sitation where it does matter in Section 5.

The following result follows easily from the monotonicity property.

Theorem 1.3 *The notion of strong substructure has the following properties.*

- **A1**. *If $M \in \overline{\boldsymbol{K}}_{-1}$ then $M \leq M$.*
- **A2**. *If $M \leq N$ then $M \subseteq N$.*
- **A3**. *If $A, B, C \in \overline{\boldsymbol{K}}_{-1}$, $A \leq B$, and $B \leq C$ then $A \leq C$.*
- **A4**. *If $A, B, C \in \overline{\boldsymbol{K}}_{-1}$, $A \leq C$, $B \leq C$ and $A \subseteq B$ then $A \leq B$.*

Since $\leq$ is imposed by δ, the following property holds, which is more special than the general case, e.g. [39].

- **A5**. *If $A, B, C \in K(N)$, $A \leq C$, $B \subseteq C$, then $A \cap B \leq B$.*

  We restrict to $\overline{\boldsymbol{K}}_0$ precisely to obtain:
- **A6**. *$\emptyset \in \overline{\boldsymbol{K}}_0$ and $\emptyset \leq A$ for all $A \in \overline{\boldsymbol{K}}_0$.*

Definition 1.4 *The pair $(\boldsymbol{K}, \leq)$ has the* amalgamation property *if for $N, M \in \boldsymbol{K}$ with $A \leq M, N$, there exists $N_1 \in \boldsymbol{K}$ and $M, N \leq N_1$.*

We have not required $A \in \boldsymbol{K}$. When applied to theories T_{-1} which are universally axiomatized, clearly $A \in \overline{\boldsymbol{K}}_{-1}$. for other T_{-1}, allowing $A \notin \overline{\boldsymbol{K}}_{-1}$ strengthens the amalgamation hypothesis. In the presence of **A6**, the amalgamation property for $(\overline{\boldsymbol{K}}_{-1}, \leq)$, implies $\overline{\boldsymbol{K}}_{-1}$ has both the amalgamation and the joint embedding (any two members of $\overline{\boldsymbol{K}}_{-1}$ have a common strong extension) properties.

Definition 1.5 *The model M is* $(\boldsymbol{K}, \leq)$-homogeneous *(or generic) if $A \leq M, A \leq B \in \boldsymbol{K}$ implies there exists $B' \leq M$ such that $B \cong_A B'$.*

Theorem 1.6 *If a class $(\boldsymbol{K}, \leq)$ has the amalgamation property and the joint embedding property then there is a $(\boldsymbol{K}, \leq)$-homogeneous structure.*

If $(\boldsymbol{K}, \leq)$ has only countably many countable members then there is a countable $(\boldsymbol{K}, \leq)$-homogeneous structure. If $(\boldsymbol{K}, \leq)$ is closed under unions of chains (as in all the examples considered here) there is a $(\boldsymbol{K}, \leq)$-homogeneous structure in $\boldsymbol{K}$.

If $(\boldsymbol{K}_{-1}, \leq)$ has the amalgamation property, both $(\boldsymbol{K}_0, \leq)$ and $(\overline{\boldsymbol{K}}_0, \leq)$ have the amalgamation and joint embedding properties and thus a homogeneous model. We let M denote the $\boldsymbol{K}_0$-homogenous model and M_1 denote the $\overline{\boldsymbol{K}}_0$-homogeneous model. We distinguish several notions that might be termed 'locally finite' in this discussion.

Definition 1.7

1. *The class* $\overline{\boldsymbol{K}}_{-1}$ *is* locally finite *if for every* $A \in \overline{\boldsymbol{K}}_{-1}$ *and every finite* $A_0 \subset A$ *there is a finite* $A_1 \in \boldsymbol{K}$ *with* $A_0 \subseteq A_1 \subset A$. A_1 *is finitely generated over* $A_0 \in \overline{\boldsymbol{K}}_{-1}$, *if* $A_1 = A_0 \vee B$ *where* B *is in* $\boldsymbol{K}_0$.

2. *The class* $(\overline{\boldsymbol{K}}_0, \leq)$ *is* locally closed *if for every* $A \in \overline{\boldsymbol{K}}_0$ *and every* $A_0 \subset A$ *there is an* $A_1 \in \overline{\boldsymbol{K}}_{-1}$ *with* $A_0 \subseteq A_1 \leq A$ *and* A_1 *is finitely generated over* A_0.

With this vocabulary we now describe some of the main applications of the method. Later we will discuss in more detail the techniques to establish quantifier reduction and stability conditions.

2 Combining rank functions

This section lists many applications of the method. Each application depends on the choice of class $\boldsymbol{K}$ and a rank function on members of $\boldsymbol{K}$. Many of the ranks are obtained by standard combinations of ones that are already known. If δ_1, δ_2 are ranks defined on a class $\boldsymbol{K}$, so are

$$\delta = \alpha\delta_1 + \beta\delta_2$$

for any *positive* reals α, β and

$$\delta = \alpha\delta_1 - \beta\delta_2$$

for any *positive* reals α, β if δ_2 is modular!

With this observation, most of the examples of this construction can be seen as built up from the examples given at the beginning. We describe a collection of examples which all use the relation:

$$\delta = \alpha\delta_1 - \beta\delta_2$$

In many cases it is more convenient to define the rank function of finite sequequences from models of T_{-1} rather than on substructures; we expand on this point in Section 3.

Applications 2.1 *Many of the examples are obtained with* δ_1 *the cardinality of a finite relational structure and* δ_2 *the number of tuples which satisfy a fixed list of symmetric relations on the structure. We refer to these as* ab initio *examples.*

1. *ab initio* finite relational language: δ_1 is cardinality, δ_2 is 'relation size'.

   (a) $\alpha = \beta = 1$. This is the dimension function for the first application of the method: Hrushovski's new strongly minimal set [36].

i. The class $\boldsymbol{K}_0^\mu$ depends on a function bounding multiplicity as described in Section 4 and yields [36] a strongly minimal set. If the μ-function is relaxed to allow even one infinite value, the rank is infinite [10]. There are continuum many different theories of this sort depending on the choice of μ.

ii. Working with the class of all structures $\boldsymbol{K}_0$ with hereditarily non-negative rank yields a theory of rank ω [25]. There are countably many classes which satisfy a certain 'δ-invariance' condition; they are classified in [1, 2].

iii. It is straightforward that Hrushovski's example does not admit elimination of imaginaries but Verbovskiy [58] provides a variant which does.

iv. Hedman [26] has shown that none of the almost strongly minimal theories which can be constructed in this way can be axiomatized with finitely many variables.

(b) $\alpha = 2$, $\beta = 1$. Baldwin [5] varied the method to construct almost strongly minimal projective planes which have no infinite definable groups of automorphisms. In [6] he showed these planes had the least possible structure in the sense of the Lenz-Barlotti classification.

(c) $\alpha = n-1$, $\beta = n-2$. Debonis and Nesin (for odd n) [41] and Tent [57] (uniformly for all n) constructed almost strongly minimal generalized n-gons. The automorphism groups of Tent's structures were highly transitive even though they were not Moufang. Thus she showed that the analog of the Feit-Higman theorem [23] did not hold for finite Morley rank n-gons.

(d) $\alpha = 1$, β irrational;

i. $\boldsymbol{K}_0^1$: Hrushovski [34] constructed a strictly stable $\aleph_0$-categorical theory thereby refuting Lachlan's conjecture.

ii. Herwig [28] varied the construction by allowing an infinite language to find a theory with infinite p-weight. This paper also contains the best published exposition of Hrushovski's $\aleph_0$-categorical stable theory.

iii. $\boldsymbol{K}_0^2$. Baldwin and Shi [14] modified the second Hrushovski construction only by changing the class of finite structures. The resulting theory is uneventful model theoretically: another non-small strictly stable theory. But, it turned out [12] that this is the almost sure theory of random graphs with edge probability $n^{-\beta}$ which had originally been constructed by [54]. Baldwin and Shelah [13] show this theory has the dimensional order property but not the finite cover property.

iv. Baldwin [3] (see also Shelah [53]) has generalized this argument to show a 0-1-law for expansions of successor by graphs with edge probablility $n^{-\beta}$.

(e) $\alpha = 1$, β rational; This gives rise [14] to a class of ω-stable theories. There is no longer a connection with random graphs.

The other cases involve more 'algebraic' dimensions as ingredients for δ and we treat these as separate cases.

2. *Baudisch groups*:

   (a) δ_1, δ_2 are the vector space dimension of a vector space E and an associated subspace of $\bigwedge^2 E$. $\delta = \delta_1 - \delta_2$. Baudisch [15] constructs a nilpotent $\aleph_1$-categorical group which does not interpret a field.

   (b) In [16], Baudisch analyzes some obstructions to extending Hrushovski's construction of a strictly stable structure to find a strictly stable $\aleph_0$-categorical group.

3. *fusions*: δ_1, δ_2 are Morley rank on two strongly minimal sets which share the same universe. Let,

$$\delta(\mathbf{x}) = \delta_1(\mathbf{x}) + \delta_2(\mathbf{x}) - \lg(\mathbf{x}).$$

   The resulting amalgam is a strongly minimal expansion of both of the original structures [36]. Holland [31, 30] clarifies this construction and in [32] proves that these theories (as well as the Hrushovski strongly minimal set) are model complete.

4. *enriched fields* $\alpha = 2$, $\beta = 1$, Work in $L = (+, 0, -, *, 1, B)$, the language of fields. N is an algebraically closed field and $K(N)$ is its algebraically closed subfields. Let $\delta_1 = d_f$ be transcendence degree and let δ_2 vary as indicated in the subcases. In each case, $(\overline{\boldsymbol{K}}_0, \leq)$ has the amalgamation property. (Take linearly disjoint copies of the fields and paint all possible new points in the join white.) These constructions were motivated by the search for first any (expansion of a) field with finite Morley rank and then for so-called bad fields: fields with 'new' definable subgroups of the multiplicative group.

   (a) *bicolored fields*: $\delta_2(\boldsymbol{a}) = |B \cap \boldsymbol{a}|$. Poizat [46] constructs a bicolored field of rank $\omega \times 2$ working with the class $\boldsymbol{K}_0$; Baldwin and Holland [9] find a rank 2 field by showing (for appropriate choice of μ) the $(\boldsymbol{K}_0^\mu, \leq)$-generic structure is ω-saturated. A corrected version of their proof is available on line and will appear in their paper [7], which considers the slightly tricky technicalities which arise for $\alpha > 2$).

   (b) *additive bad fields*: δ_2 additive linear dimension. Poizat [47] has completed the infinite rank case.

   (c) *multiplicative bad fields*: δ_2 multiplicative linear dimension. Again, Poizat [47] has completed the infinite rank case.

5. *A nondefinability result:* [20]: Using a dimension on algebraically closed fields which assigns to a sequence of ordered pairs twice the transcendence degree of its union minus its length, the authors show there is no first order formula $\phi(R)$ which holds of the relation R on an algebraically closed field just if R is Zariski closed.

6. Ikeda [38] has used variants of this method to settle a question raised in [18]. He shows that there are structures of every finite dimension which are minimal (every definable subset is finite or cofinite) but not strongly minimal (i.e. in some elementary extension there is an infinite/coinfinite definable set.)

7. Sudoplatov [56, 55] has used variants of this method to show there exists an ω-stable *group trigonometry* on a projective plane. Group trigonometries and some generalizations were devised by Sudoplatov to study combinatorial geometries arising in model theory.

All the *ab initio* examples are locally finite in the sense of Definition 1.7. When the parameter α is a rational number or in the $\aleph_0$-categorical example, the class is locally closed. But the strictly stable example of Baldwin-Shi is not locally closed. The Baudisch group is also locally finite.

Applications 2.2 *Simple theories:* Hrushovski introduced a variant on the notion of strong substructure which allows the construction of strictly simple theories. The key is to make the inequality in the definition of strong substructure strict. For $A, B \in \overline{\boldsymbol{K}}_{-1}$, we say A is a $*$-*strong substructure* of B and write $A \leq^* B$ if for every $B' \in \boldsymbol{K}_{-1}$ with $B' \subseteq B$, $\delta(B'/B' \cap A) > 0$. With this notion Hrushovski constructed an $\aleph_0$-categorical strictly simple theory where forking is not locally modular. This argument was sketched in [34], treated more expansively in [33] and has a full exposition in [48]. Pourmahdian [48], following Pillay [44] also develops the appropriate notions of simplicity for the class of existentially closed models even if it is not an elementary class.

Applications 2.3 *Pseudoexponentiation:* Zilber [61] defines a rank on two sorted structures (D, ex, R) where D is a field of characteristic 0, R is a field of characteristic p and ex is a homomorphism of the additive group of D onto the multiplicative group of R. He defines a 'rank' or 'predimension'

$$\delta(X) = d_f(X) + d_f(ex(X)) - d_{vs}(X)$$

and investigates, for example, connections with the Schanuel conjecture.

Pillay and Baudisch ([17, 45, 43] have investigated more intensively the geometric conditions (such as CM-triviality) which arise in these constructions and also studied higher dimensional analogues.

3 Definability Conditions on δ

The Hrushovski construction produces a countable model. We want to draw conclusions about all models of the theory of this model. For this, we need to impose several conditions on

δ. The first of the following just establishes notation, the second reaffirms our commitment to monotonicity; the third and fourth specify definability requirements on δ. We earlier described δ as a function on models. In order to develop the definability constraints we must consider a variant where δ is defined on finite sequences. In the *ab initio* situation these definability requirements are so obvious they go unnoticed. They become increasingly more difficult to fulfill for various kinds of enriched fields.

Definition 3.1 *A class $\boldsymbol{K}$ has δ-formulas over parameters if the following conditions hold. ($\boldsymbol{K}$ has δ-formulas over the empty set if they hold when $\mathbf{b} \in \mathrm{acl}(\emptyset)$.)*

1. *If* $\mathrm{Diag}(\boldsymbol{a}) = \mathrm{Diag}(\mathbf{b})$ *then* $\delta(\boldsymbol{a}) = \delta(\mathbf{b})$.
2. *(monotonicity)* $\delta(\boldsymbol{a}/\mathbf{b}) \geq \delta(\boldsymbol{a}/\mathbf{bc})$ *whenever* $\mathbf{c} \cap \boldsymbol{a} = \mathbf{c} \cap \mathbf{b}$.
3. *(definability) For any integer k and $\boldsymbol{a}, \mathbf{b} \subseteq C \in \boldsymbol{K}$ with $\delta(\boldsymbol{a}/\mathbf{b}) = k$, there is a* **$\delta$-formula for $\boldsymbol{a}$ over $\mathbf{b}$**, *$\varphi(\overline{x}; \overline{y}) \in \mathrm{Diag}(\boldsymbol{a}; \mathbf{b})$, such that the following hold for any $\boldsymbol{a}', \mathbf{b}' \subseteq B \in \boldsymbol{K}$: if $B \models \varphi(\boldsymbol{a}'; \mathbf{b}')$, then $\delta(\boldsymbol{a}'/\mathbf{b}') \leq k$.*
4. *(q.f. type determined) For any $N \in \boldsymbol{K}$ and $\boldsymbol{a}, \boldsymbol{a}', \mathbf{b} \subseteq N$, if $N \models \phi(\boldsymbol{a}; \mathbf{b}) \wedge \phi(\boldsymbol{a}'; \mathbf{b})$ and*
$$\delta(\boldsymbol{a}/\mathbf{b}) = \delta(\boldsymbol{a}'/\mathbf{b}) = k$$
and
$$\mathbf{b}\boldsymbol{a} \leq \langle \mathbf{b}\boldsymbol{a} \rangle,$$
$$\mathbf{b}\boldsymbol{a}' \leq \langle \mathbf{b}\boldsymbol{a}' \rangle,$$
then
$$diag(\mathbf{b}\boldsymbol{a}) = diag(\mathbf{b}\boldsymbol{a}').$$

Definition 3.2 *Let $\boldsymbol{a}, \mathbf{b} \in N \in \boldsymbol{K}$. $\boldsymbol{a}$ is a* **minimal intrinsic extension** *of* $\mathbf{b}$ *if $\delta(\boldsymbol{a}/\mathbf{b}) < 0$ but for every $\boldsymbol{a}'$ properly contained in $\boldsymbol{a}$, $\delta(\boldsymbol{a}'/\mathbf{b}) \geq 0$.*

Now suppose the $\boldsymbol{K}$ in Definition 3.1 is one of the $\overline{\boldsymbol{K}}_{-1}$ described above. Note that if δ-formulas can be found over the empty set, then there is a first order theory T_0 axiomatizing $\overline{\boldsymbol{K}}_0$; just assert that the intrinsic closure of the empty set is empty. Such δ-formulas are found easily in the *ab initio* case; Hrushovski's construction for fusions in [35] extends painlessly to bicolored fields enriched only by a predicate. When the predicate is required to define a group very delicate questions arise. A lemma of Zilber [62] (with ideas from Hrushovski) provides a partial solution to this difficulty for the case of multiplicative bad fields.

Definition 3.3 *Let T_W be the minimal torus containing a variety W. Let U be an irreducible subvariety of an irreducible variety W. V is an* atypical component *of $W \cap T_V$ if*
$$d_f(W) - d_f(V) < d_f(T_W) - d_f(T_V).$$

Lemma 3.4 (Zilber) *Fix a variety W_1 defined over $\overline{Q}$ by the equations $\psi(\mathbf{x}, \mathbf{y})$. There is a finite set $T_1, \ldots T_n$ of proper subtori of T_W such that: for any $\mathbf{b}$, and $W = W(\mathbf{b}) = \{\boldsymbol{a} :\models \psi(\boldsymbol{a}, \mathbf{b})\}$ if V is an irreducible subvariety of W and V is an atypical component of $W \cap T_V$ with then for some $i < n$ and some $\mathbf{f}$, $V \subseteq T\mathbf{f}$.*

The following corollary (and a proof of the lemma incorporating ideas of Marker) can be found both in [8] and [47].

Corollary 3.5 *Let T_{-1} be as in Application 2.1 4c). For any sequence $\boldsymbol{a}$ with $\delta(\boldsymbol{a}/\emptyset) > -\infty$, there is a formula $\phi(\mathbf{x})$ which is a δ-formula for $\boldsymbol{a}$ over $\emptyset$.*

In order to find a finite rank expansion of the complex numbers, with a predicate for a subgroup of the multiplicative group either an improvement or a clever application of Lemma 3.4 is required. This project only seems really possible in characteristic 0 both because the proof of Lemma 3.4 uses that hypothesis heavily and because Wagner [60] shows a bad field of finite characteristic is extremely unlikely.

4 Stability and Finite Rank

The main purpose of marrying ranks to the Fraïssé-construction is to control the stability class and more particularly the Morley Rank of the resulting structure. How is this control exercised? In one approach, the map $\delta(a/X)$ is made monotone in the second argument by replacing δ with $d_M(a/X) = \inf\{\delta(a/Y) : X \subseteq Y \subseteq_{fin} M\}$ and it is shown that a notion of independence built on 'a is independent from A over B if $d_M(a/AB) = d_M(a/B)$' (with another condition) satisfies most of the axioms of forking. This works for forking with respect to formulas of low quantifier complexity.

There are several strategies to prove stability or (ω-stability). First get the result for formulas with low quantifier complexity by combinatorial means: counting types [46], or following the unpublished Hrushovski paper using d-independence [14] or by checking the order property [37]. If T^* (the theory of the generic model) has been shown model complete, the 'stability' result follows immediately. If not, specific technical arguments can be given [14].

Hrushovski [33] suggested the following device to simplify the description of the situation.

Definition 4.1 *Form the language L^+ by adding a relation symbol $R_{AB}(\mathbf{x})$ for each pair (A, B) where B is a minimal intrinsic extension of A. For any of our theories, T^0, T^0_{nat} is the L^+-theory extending T^0 which asserts:*

$$[\exists \mathbf{y} \Delta_{AB}(\mathbf{x}, \mathbf{y})] \leftrightarrow R_{AB}(\mathbf{x}).$$

We denote the natural expansion of an L-structure N to L^+ by N^+ and the collection of expansions of models in a class $\boldsymbol{K}$ by $\boldsymbol{K}^+$.

Baldwin and Holland [9] provide a necessary and sufficient condition for the homogeneous structure to be ω-saturated. We need a little vocabulary.

Definition 4.2 *For any* $\mathbf{b} \in A \in \boldsymbol{K}$, *let* $I^*(\mathbf{y})$ *be a collection of formulas so that if* $B \models I^*(\mathbf{b}')$, *then* $\mathbf{b}' \leq B$ *and* $\langle \mathbf{b} \rangle \cong \langle \mathbf{b}' \rangle$ *(I.e. the* L^+*-diagram of* $\mathbf{b}$.*)*

Definition 4.3 $(\boldsymbol{K}, \delta)$ admits strong separation of quantifiers *if for any* $\mathbf{b} \leq \boldsymbol{a}\mathbf{b} \leq \langle \boldsymbol{a}\mathbf{b} \rangle \in \boldsymbol{K}$ *with* $\boldsymbol{a}$ *minimal strong over* $\langle \mathbf{b} \rangle$, *the following holds: For any formula* $\tau(\mathbf{x};\mathbf{y})$ *in* $I^*(\boldsymbol{a}, \mathbf{b})$, *there is* $\sigma(\mathbf{y}) \in I^*(\mathbf{b})$ *such that whenever* $\mathbf{b}' \subseteq C \in \boldsymbol{K}$ *and* $C \models \sigma(\mathbf{b}')$, *there is* $D \in \boldsymbol{K}$ *with* $C \leq D$ *and* $\boldsymbol{a}' \in D$ *such that*

$$D \models \tau(\boldsymbol{a}'; \mathbf{b}').$$

Theorem 4.4 ([9]) $(\overline{\boldsymbol{K}}_0, \delta)$ *admits strong separation of quantifiers iff the generic is* ω*-saturated.*

With each of the dimension functions for the ω-stable examples above two classes of theories can be constructed. If one amalgamates over all structures with hereditarily nonnegative dimension, the theory has infinite Morley rank. To investigate the finite rank case requires a different choice of the amalgamation class. In order to describe it we need other notions.

Definition 4.5 $\boldsymbol{a}$ *is* primitive *over* $\mathbf{b}$ *if*

1. $\delta(\boldsymbol{a}/\mathbf{b}) = 0$
2. $\delta(\boldsymbol{a}/\mathbf{b}') < 0$ *if* $\mathbf{b} \subset \mathbf{b}' \subset \boldsymbol{a}\mathbf{b}$.

Suppose $A/B \in \boldsymbol{K}_0$ is primitive, let M be $(\boldsymbol{K}_0, \leq)$-homogeneous and let $\chi_M(A/B)$ denote the number of copies of A over B in M. There are three possibilities.

- $\delta(A/B) < 0$ implies $\chi_M(A/B)$ is finite.
- $\delta(A/B) > 0$ implies $\chi_M(A/B)$ is infinite.
- $\delta(A/B) = 0$ implies $\chi_M(A/B)$ is undetermined.

A key point for the completeness of the theories T^α (the random graph case) is that when α is irrational, the third case cannot occur. For rational α (e.g. $\alpha = \beta = 1$ in the original strongly minimal construction), in the full class $\boldsymbol{K}_0$, the primitive structures also occur infinitely often and the theory has infinite rank. To guarantee that the theory has finite rank we make the following restriction; we remove the ambivalence by changing 'undetermined' in the third case to finite. But to axiomatize this restriction requires great care.

A *primitive code* is a sequence of parameters $\boldsymbol{c}$ which completely describes a primitive pair. It includes a formula specifying the quantifier free type of $\boldsymbol{a}/\mathbf{b}$. For any function μ from codes to natural numbers we define $\mathbf{K}^{\mu}$ to be the members of $\overline{\boldsymbol{K}}_0$ which have less than $\mu(\mathbf{c})$ independent realizations over the same canonical parameter of each code $\mathbf{c}$.

In all the examples so far considered it is fairly straightforward to transfer amalgamation from $\boldsymbol{K}_0$ to $\boldsymbol{K}_0^{\mu}$ provided only that $\mu(A/B)$ is sufficiently large ([36, 5, 46]). In order that $\boldsymbol{K}_0^{\mu}$ admit strong separation of variables, a stronger condition on μ is required: it suffices that μ be finite-to-one [30, 35, 9]. In fact, this condition is essential:

Theorem 4.6 *[9] For $\boldsymbol{K}_0$, the class of bicolored fields, $(\boldsymbol{K}_0, \delta)$ and if μ is finite-to-one the class of bicolored fields, $(\boldsymbol{K}_0^{\mu}, \delta)$ admits strong separation of quantifiers.*

Theorem 4.7 *[9] For appropriate μ which is not finite to one,*

1. *The generic is not saturated.*

2. *In fact, T^{μ} is not ω-stable, nor even small.*

5 Infinitary Logic and Finite Diagrams

There are two examples which are particularly intriguing. For irrational α, the Baldwin-Shi homogenous model is not saturated; indeed the theory has no countable saturated model. Again, Theorem 4.7 shows that if μ is not finite-to-one the theory T^{μ} of a bicolored field may not have a countable saturated model. Infinitary logic and Shelah's notion [49] of finite diagram are useful for explaining this situation. In fact, the reason is different for the two cases.

The *finite diagram* of a model M, $D = D(M)$ is the collection of finite types over the empty set realized in M. A D-model is a structure N with $D(N) \subseteq D$. The class of D-models is $L_{\omega_1,\omega}$ definable but may not be an elementary class. Note that 'N is existentially closed' (for $\boldsymbol{K}$) is a property of $D(N)$.

Consider any class $\overline{\boldsymbol{K}}_0$ with associated theories T_0 in L and T^0_{nat} in L^+ (see Definition 4.1). Let M be generic for $\boldsymbol{K}_0$; we denote its theory by T^*. In all our examples, $(T^0_{nat})_\forall$ is a universal theory with amalgamation and joint embedding. In particular, in the language of [33], it is a Robinson theory. Let $Ex(T^0_{nat})$ denote the class of existentially closed (for $\overline{\boldsymbol{K}}_0$) models of $(T^0_{nat})_\forall$. Since T^0_{nat} is a Robinson theory, on $Ex(T^0_{nat})$, every formula is equivalent to a (possibly infinite) Boolean combination of quantifier-free formulas. So if $Ex(T^0_{nat})$ is first order, it admits quantifier elimination in L^+. Since the added relation symbols represented existential formulas this means that in L, the theory of the generic is *nearly model complete*: every formula is a Boolean combination of existentials.

Consider three class of models of T^*: all models of T^*, $Ex(T^0_{nat})$ and $Mod(D)$ where D is the finite diagram of M^+:

$$Mod(D) \subseteq Ex(T^0_{nat}) \subseteq Mod(T^*).$$

The first containment holds as a model N is existentially complete if it omits (incomplete) types of the form $\{\neg\theta(\mathbf{y}) : (\forall \mathbf{x}\mathbf{y})[\phi(\mathbf{x},\mathbf{y}) \rightarrow \theta(\mathbf{y})\} \cup \{\neg(\exists \mathbf{x})\phi(\mathbf{x},\mathbf{y})\}$, where θ and ϕ are quantifier-free in L^+. Thus, any model of a complete theory whose diagram is contained in the diagram of an existentially closed model is existentially closed.

In general, $D(M^+)$ and $Ex(T^0_{nat})$ may not be first order. Consider T^μ as in Lemma 4.7 and T^α, the almost sure theory of Application 2.1 1.d.iii.

The first order theory T^μ of Lemma 4.7 is not ω-stable nor even small. But it is easy to see from condition 4) on the definition of δ-formulas (Definition 3.1) that $Ex(T^\mu_{nat})$ is ω-stable as a finite diagram and so ω-stable as a first order order theory if it is axiomatiable. This implies $Ex(T^\mu_{nat}) \neq Mod(T^\mu)$.

On the other hand taking T^* as T^α, $Mod(D) \neq Ex(T^0_{nat})$ while $Ex(T^0_{nat}) = Mod(T^*)$. The equality follows from the proof in [12] that T^α is nearly model complete; alternatively, Hyttinen proved it directly [37]. The inequality then follows since D is countable while $Mod(T^*)$ has continuum many types over the empty set.

As M^+ is existentially closed, if M^+ is ω-saturated then T^* is model complete (in L^+!). Thus the three classes coincide if the generic is ω-saturated. This happens for bicolored fields if μ is finite-to-one. If not, we can either investigate the first order theory or the infinitary classes, which are well behaved, using the technology of [19]. Several interesting questions arise from such considerations. For which μ (beyond the finite-to-one), if any, is T^μ stable or even simple? Let M_1 be the homogeneous model for $\overline{\boldsymbol{K}}_0$. Where does $D(M_1)$ fit into the picture above? Can $Ex(T^\mu_{nat})$ always (in the contexts discussed in this paper) be realized as the class of models of $D(N)$ for some N?

References

[1] R.D. Aref'ev. $K_{(H,|A|-e(A),\leq)}$-homogeneous-universal graphs. In B.S. Baizhanov M.B. Aidarkhanov, editor, *Proceedings of Informatics and Control Problems Institute, Almaty*, pages 27–40. 1995. In Russian.

[2] R.D. Aref'ev, J.T. Baldwin, and M. Mazzucco. δ-invariant amalgamation classes. *Journal of Symbolic Logic*, 64:1743–1750, 1999.

[3] J.T. Baldwin. Random expansions of geometries. submitted.

[4] J.T. Baldwin. *Fundamentals of Stability Theory.* Springer-Verlag, 1988.

[5] J.T. Baldwin. An almost strongly minimal non-desarguesian projective plane. *Transactions of the American Mathematical Society*, 342:695–711, 1994.

[6] J.T. Baldwin. Some projective planes of Lenz Barlotti class I. *Proceedings of the A.M.S.*, 123:251–256, 1995.

[7] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Infinite rank. preprint, 199x.

[8] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Bad fields. in progress, 200?

[9] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Rank 2 fields. *Journal of Symbolic Logic*, 65:371–391, 2000.

[10] J.T. Baldwin and M. Itai. K-generic projective planes have Morley rank two or infinity. *Mathematical Logic Quarterly*, 40:143–152, 1994.

[11] J.T. Baldwin and S. Shelah. Abstract classes with few models have 'homogeneous-universal' models. *Journal of Symbolic Logic*, 60:246–266, 1995.

[12] J.T. Baldwin and S. Shelah. Randomness and semigenericity. *Transactions of the American Mathematical Society*, 349:1359–1376, 1997.

[13] J.T. Baldwin and S. Shelah. DOP and FCP in generic structures. *Journal of Symbolic Logic*, 63:427–439, 1998.

[14] J.T. Baldwin and Niandong Shi. Stable generic structures. *Annals of Pure and Applied Logic*, 79:1–35, 1996.

[15] A. Baudisch. A new uncountably categorical group. *Transactions of the American Mathematical Society*, 348:889–940, 1995.

[16] A. Baudisch. Closures in $\aleph_0$-categorical bilinear maps. *Journal of Symbolic Logic*, 65:914–922, 2000.

[17] A. Baudisch and Pillay Anand. A free pseudospace. *Journal of Symbolic Logic*, 65:443–460, 2000.

[18] O. Belegradek. On minimal structures. *J. Symbolic Logic*, 63:421–425, 1998.

[19] S. Buechler and O. Lessmann. Simple homogeneous models. preprint, 200?

[20] O. Chapuis, E. Hrushovski, P. Koiran, and B. Poizat. La limite des théories de courbes génériques. preprint, 2000.

[21] A. Ehrenfeucht. There are continuum ω_0-categorical theories. *Bulletin de l'Acadamie Polonaise des sciences math., astr., et phys.*, XX:425–427, 1972.

[22] D. Evans. $\aleph_0$-categorical structures with a predimension. preprint.

[23] W. Feit and G. Higman. The nonexistence of certain generalized polygons. *Journal of Algebra*, 1:114–131, 1964.

[24] W. Glassmire. There are $2^{\aleph_0}$ countably categorical theories. *Bulletin de l'Acadamie Polonaise des sciences math., astr., et phys.*, XIX:185–190, 1971.

[25] J. Goode. Hrushovski's Geometries. In Helmut Wolter Bernd Dahn, editor, *Proceedings of 7th Easter Conference on Model Theory*, pages 106–118, 1989.

[26] S. Hedman. Finitary axiomatizations and local modularity of strongly minimal theories. preprint, 2000.

[27] W. Henson. Countable homogeneous relational structures and $\aleph_0$-categorical theories. *The Journal of Symbolic Logic*, 37:494–500, 1972.

[28] B. Herwig. Weight ω in stable theories with few types. *J. Symbolic Logic*, 60:353–373, 1995.

[29] W. Hodges. *Model Theory.* Cambridge University Press, 1993.

[30] K. Holland. An introduction to the fusion of strongly minimal sets: The geometry of fusions. *Archive for Mathematical Logic*, 6:395–413, 1995.

[31] K. Holland. Strongly minimal fusions of vector spaces. *Annals of Pure and Applied Logic*, 83:1–22, 1997.

[32] K. Holland. Model completeness of the new strongly minimal sets. *J. Symbolic Logic*, 64:946–962, 1999.

[33] E. Hrushovski. Simplicity and the Lascar group. preprint.

[34] E. Hrushovski. A stable $\aleph_0$-categorical pseudoplane. preprint, 1988.

[35] E. Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel Journal of Mathematics*, 79:129–151, 1992.

[36] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[37] T. Hyttinen. Canonical finite diagrams and quantifier elimination. preprint.

[38] K. Ikeda. On minimal structures. preprint.

[39] D.W. Kueker and C. Laskowski. On generic structures. *Notre Dame Journal of Formal Logic*, 33:175–183, 1992.

[40] A.H. Lachlan. Two conjectures regarding the stability of ω-categorical theories. *Fundamenta Mathematicae*, 81:133–145, 1974.

[41] A. Nesin M. J. De Bonis. There are $2^{\aleph_0}$ many almost strongly minimal generalized n-gons that do not interpret an infinite group. *J. Symbolic Logic*, 63:485–508, 1998.

[42] A. Pillay. *An introduction to stability theory.* Clarendon Press, Oxford, 1983.

[43] A. Pillay. The geometry of forking and groups of finite Morley rank. *Journal of Symbolic Logic*, 60:1251–1259, 1995.

[44] A. Pillay. Forking in the category of existentially closed structures. preprint, 1999.

[45] A. Pillay. the geometry of forking and cm-triviality. *Journal of Symbolic Logic*, 65:474–480, 2000.

[46] Bruno Poizat. Le carré de l'egalité. *The Journal of Symbolic Logic*, 64:1339–1356, 1999.

[47] Bruno Poizat. L'egalité au cube. preprint, 1999.

[48] M. Pourmahdian. *Simple Generic Theories.* PhD thesis, Oxford University, 2000.

[49] S. Shelah. Finite diagrams stable in power. *Annals of Mathematical Logic*, 2:69–118, 1970.

[50] S. Shelah. The lazy model-theoretician's guide to stability. *Logique et Analyse*, 18:241–308, 1975.

[51] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models.* North-Holland, 1978.

[52] S. Shelah. Universal classes: Part 1. In J. Baldwin, editor, *Classification Theory, Chicago 1985*, pages 264–419. Springer-Verlag, 1987. Springer Lecture Notes 1292.

[53] S. Shelah. 0-1 laws. preprint 550, 199?

[54] S. Shelah and J. Spencer. Zero-one laws for sparse random graphs. *Journal of A.M.S.*, 1:97–115, 1988.

[55] Sudoplatov S.V. On hypergraphs of minimal prime models. In *International Conference devoted to the 90 years of birthday of A.I.Maltsev. Novosibirsk: IDMI,*, pages 112–113, 1999.

[56] Sudoplatov S.V. On type identifications in trigonometrical theories. In *International Conference devoted to the 90 years of birthday of A.I.Maltsev. Novosibirsk: IDMI*, pages 111–112, 1999.

[57] Katrin Tent. Very homogeneous generalized n-gons of finite morley rank. *J.London Math. Soc. (2)*, 62:1–15, 2000.

[58] V. Verbovskiy. On the elimination of imaginaries for the strongly minimal sets of Hrushovski. preprint.

[59] F. Wagner. Relational structures and dimensions. In *Automorphisms of first order structures*, pages 153–181. Clarendon Press, Oxford, 1994.

[60] F. Wagner. Fields of finite morley rank. preprint, 200x.

[61] B.I. Zilber. Fields with pseudoexponentiation. preprint, 2000.

[62] B.I. Zilber. Intersecting varieties with tori. preprint, 2000.