Does set theoretic pluralism entail model theoretic pluralism?

II. Categoricity: in what logic?

Aberdeen

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The virtues of Categoricity

Detlefsen's first question:

(IA) Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

(IB) Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how to we justify the claim that categoricity is or would be a virtue just in the former?

Basic Definition

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T is *categorical* or *monomorphic* or *univalent* if it has exactly one model (up to isomorphism).

T is *categorical in power* κ if it has exactly one model in cardinality κ.

T is *totally categorical* if it is categorical in every infinite power.

T is *eventually categorical* if it is categorical in every sufficiently large infinite power.

A structure *M* is $\mathcal{L}$-categorical for a logic $\mathcal{L}$, if $\mathrm{Th}_{\mathcal{L}}(M) = \{\phi \in \mathcal{L}(\tau) : M \models \phi\}$ is categorical.

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For philosophers, 'categoricity' is short for 'categoricity'.

More Basic Definition

Two structures M, N for the vocabulary τ are isomorphic if there is a bijection f between their universes that preserves interpretations of the relation, function and constant symbols of τ
e.g.

$$M \models R(a_1, \ldots, a_n) \text{if and only if} N \models R(f(a_1), \ldots, f(a_n))$$

This is notion is *not* well-defined unless the vocabulary τ is specified.

Meadows' Criteria

Toby Meadows writes in 'What can a categoricity theorem tell us?'.

1. *to demonstrate that there is a unique structure which corresponds to some mathematical intuition or practice;*

   given intuition of theory show it has unique model

2. *to demonstrate that a theory picks out a unique structure;*

   given intuition of structure find axioms that characterize it

3. *to classify different types of theory.*

Axioms and Theories

Two similar but distinct results

1. The 2nd order theory of $(\mathbb{N}, +, \times, 0, 1,)$ is categorical
2. 2nd order Peano axioms of arithmetic are categorical

The choice of logic

1. How second order logic fails some of the criteria
2. How $L_{\omega_1,\omega}$ better fulfils the criteria
3. Why categoricity in power is more virtuous than categoricity

2nd order Logic

Criterion 1: Does categoricity guarantee that there is a unique structure corresponding to some theory or practice?

Is 'arithmetic' categorical?

Meadows also writes

> *there appears to be a universal belief that the language and practice of arithmetic does refer to a unique structure.*

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Yes, if 'arithmetic' means the practical rules for performing operations. No, if 'arithmetic' means the practice of mathematicians in trying to understand the properties of this structure.

The mathematical practice of ‘arithmetic’

1. Number theorists study ‘arithmetic’ in the context of ambient fields (*p*-adic, real, complex, ‘number fields’) not just the natural numbers.
2. Number theorists increasingly use methods of model theory and non-standard analysis to prove results about the natural numbers.

Nevertheless, there is a unified view that there is a unique structure (algebraically prime model) which these results are intended to describe.

Criterion 2: Demonstrating an intuition of a structure can be categorically axiomatized

Are the canonical structures as canonical as they seem? I.

Do we really have clear conception of arithmetic and geometry that specifies one model?

As Roman Kossak has pointed out,

a clear intuition or vision of the natural numbers with successor is often confused with a clear intuition of arithmetic,

the natural numbers with both addition and multiplication.

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the natural numbers with both addition and multiplication.

Few, if any, actually have the second intuition.

Criterion 2: Demonstrating a theory picks out a known unique structure

The goal of an axiomatization is to illuminate the central intuitions about the structure.

Does categoricity of an axiomatization show it fulfills this goal?

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Two similar but distinct results

1. The real linear order can be categorically axiomatized in at least 2 ways
    1. Axiomatize the naturals. Describe the the construction of the rationals from the natural numbers and then the reals as Cauchy sequences of rationals.
       As pointed out by Väänänen, this construction takes place in $V_{\omega+7}$.
    2. The reals are the unique separable complete linear order

The second axiomatization highlights the properties needed for the foundations of calculus (e.g. Spivak) as opposed to a tedious construction of the reals for the natural numbers.

Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

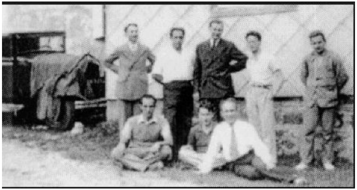
Bourbaki wrote:

> *Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a* futile logical hairsplitting not capable of fructifying any theory whatever*.*

More Bourbaki

This critical attitude can probably be accounted for by a purely historical accident.

The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, i.e. *theories which are entirely determined by their complete systems of axioms; for this reason they could not be applied to any theory except the one from which they had been abstracted (quite contrary to what we have seen, for instance, for the theory of groups).*

More Bourbaki: Bourbaki 

> *If the same had been true of all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.*

Bourbaki realizes but then forgets that the hypothesis of this last sentence is false.

They miss the distinctions between

1. axiomatization and theory
2. first and second order logic.

criterion 3: classifying theories

Following Shapiro distinguish:

1. A theory is algebraic if we expect it to have many models, and
2. non-algebraic if we expect a unique model.

Does categoricity make this distinction?

2nd order logic fails criterion 3: classifying

Theorem

1) Marek-Magidor/Ajtai *(V=L) The second order theory of a countable structure is categorical.*

2) H. Friedman *(V=L) The second order theory of a Borel structure is categorical.*

3) Solovay *(V=L) A recursively axiomatizable complete second order theory is categorical.*

4) Solovay/Ajtai *It is consistent with ZFC that there is a complete finitely axiomatizable second order theory with a finite vocabulary[a] that is not categorical.*

[a] There are trivial examples if infinitely many constants are allowed.

Moral

Fact V=L

If a second order theory is complete and easily described, (i.e.

1. recursively axiomatized
2. or has an intended model which is 'small' (countable or Borel)

then (at least in L) it is categorical.

ZFC

any countable structure that has an 'arithmetic' presentation
(each primitive relation is definable by formula of arithmetic)
is 2nd order categorical.

More unexpectedly categorical structures

Very large second order categorical structures

Many cardinals i.e. structures V_κ are characterizable in second order logic, e.g.: first inaccessible, Mahlo, Ramsey etc. the Hanf number of second order logic

Thus many theories for which non-experts have no clear intuition are 2nd order categorical.

Detlefsen's 2nd question

Question II: Given that categoricity can rarely be achieved, are there alternative conditions that are more widely achievable and that give at least a substantial part of the benefit that categoricity would?

Can completeness be shown to be such a condition?

If so, can we give a relatively precise statement and demonstration of the part of the value of categoricity that it preserves?

Completeness and categoricity

The fact that the most fundamental structures were 2nd order categorical may partially explain why it took until Gödel to clarify the distinction between syntactically complete and categorical.

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

But what if a sentence is complete?

Items 3 and 4 above show the proposition:
Every syntactically complete recursively axiomatizable second order theory is categorical.
is independent of ZFC.

$L_{\omega_1,\omega}$

Background on $L_{\omega_1,\omega}$

Defining $L_{\omega_1,\omega}$

Add to first order logic the ability to form countable conjunctions and disjunctions.

So we characterize $(N, S, 0)$ by $\forall x \bigwedge x = S^n(0)$.

Some dismiss this axiomatization as circular: the axiomatization assumes some notion the natural numbers.
Somewhat more precisely, the issue is, 'how can we grasp the notion of a single infinite recursive disjunction in the metatheory?'

While this step may trouble some, surely it is easier to grasp than 'the collection of all subsets of the natural numbers'.

$L_{\omega_1,\omega}$ and $L_{\omega,\omega}$

A translation

Given any sentence Φ of $L_{\omega_1,\omega}$ there is a countable language $L' \supseteq L$, a first-order L'-theory T, and a partial L'-type $\Delta(w)$ such that the class of models of Φ is precisely the class of L-reducts of models of T that omit $\Delta(w)$.

To see the idea suppose $\Phi(\mathbf{x})$ is a countable conjunction of formula $\phi_i(\mathbf{x})$.
Add a new predicate symbol $R_\Phi(w)$. Let T assert for each i, $\forall w[R_\Phi(w) \to \phi_i(w)$ and let $\Delta(w)$ be the type $\{R_\Phi(w)\} \cup \{\phi_i(w) : i < \omega\}$.

Smallness

Definition

1. A τ-structure M is L^*-*small* for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many $L^*(\tau)$-types (i.e. only countably many $L^*(\tau)$-n-types for each $n < \omega$).
2. A τ-structure M is called *small* or $L_{\omega_1,\omega}$-*small* if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.

Why Smallness matters

Fact

Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.

Proof: do Scott's proof replacing 'countably many elements' by 'countably many types'

first order theories: atomic model

Definition

A formula $\phi(\mathbf{w})$, where $\lg(\mathbf{w}) = n$, is complete for T if for every formula $\psi(\mathbf{w})$, $\phi(\mathbf{w})$ decides $\psi(\mathbf{w})$ in T. I.e. $T \vdash \forall \mathbf{w}[\phi(\mathbf{w}) \rightarrow \psi(\mathbf{w})$ or $T \vdash \forall \mathbf{w}[\phi(\mathbf{w}) \rightarrow \neg\psi(\mathbf{w})$.

A model M is atomic if every finite tuple from A satisfies a complete formula.

$L_{\omega_1,\omega}$ and first order are close

If Φ is a complete $L_{\omega_1,\omega}(\tau)$-sentence, then there is a countable vocabulary $\tau' \supseteq \tau$ and a complete τ'-structure T such that the class of models of Φ is precisely the class of atomic models of T.

Conversely, given any complete theory T in a countable vocabulary τ, there is a complete sentence Φ of $L_{\omega_1,\omega}(\tau)$ whose models are precisely the atomic models of T.

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moral Studying complete sentences of $L_{\omega_1,\omega}$ is studying atomic models of first order theories.

Categoricity in $L_{\omega_1,\omega}$

Minimal Models

Definition

A model M is $L_{\omega_1,\omega}$-minimal if and only if M has no proper $L_{\omega_1,\omega}$-elementary submodel.

The model $(\mathbb{N}, S)$ is a good example of an $L_{\omega_1,\omega}$-minimal structure. The next theorem shows the mathematical importance of the *push-through construction*.

Fact: Push-through construction

Let M be structure and π a bijection of M and a set A. Then there is a structure isomorphic to M with domain A. Just transfer each relation on N to its image under π.

Minimal Models and Categoricity

Theorem

An $L_{\omega_1,\omega}$- sentence ϕ is categorical if and only if its unique countable model is $L_{\omega_1,\omega}$-minimal.

Proof. Suppose ϕ is categorical. Since $L_{\omega_1,\omega}$ satisfies the downward Löwenheim-Skolem theorem there is a countable model M of ϕ and we may as well assume that ϕ is the Scott sentence of M (since the Scott sentence is equivalent to ϕ).

Minimal Models and Categoricity: Proof cont.

Now suppose for contradiction there is a proper $L_{\omega_1,\omega}$-elementary submodel M_0 of M. As $M_0 \approx M$, by push-through M_0 has a proper elementary extension M_1.

Construct a sequence of proper elementary extensions with M_{i+1} produced from M_i and M_{i-1} as we constructed M_1 from M and M_0.

Let $M_\delta = \bigcup_{i<\delta} M_i$, when δ is a limit ordinal; it is isomorphic to M for countable δ.

Letting $M^* = \bigcup_{i<\omega_1} M_i$, M^* is an uncountable $L_{\omega_1,\omega}$-elementary extension of M. So it satisfies ϕ which thus is not categorical since it has models in two cardinals.

What structures are $L_{\omega_1,\omega}$-Categorical?

Fact.

A structure M is $L_{\omega_1,\omega}$-minimal if and only if there is a first order theory T such that M is an atomic minimal model to T. That is,

1. M is atomic model of T
2. There is no $N \subsetneq M$ such that $M \models T$.

Criterion 1: Demonstrating an intuition of a structure can be categorically axiomatized

Theorem

Each of the following is categorical in $L_{\omega_1,\omega}$.

1. Arithmetic $(\mathbb{N}, +, \times, 0, 1)$
2. Euclidean Geometry
3. the complete ordered field $(\Re, +, \times, 0, 1)$
4. geometry over the complete ordered field $(\Re, +, \times, 0, 1)$

Proof. Every element of $(\mathbb{N}, +, \times, 0, 1)$ is definable from the constants.

Are the canonical structures as canonical as they seem? II.

Basic Question

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But what about Euclidean geometry?

Equally clearly no.

Interlude: Euclidean Geometry

By the theory of Euclidean geometry we mean the geometrical results in Euclid's Elements.

Hilbert gave a first order axiomatization (EG) of Euclidean geometry.

1. clarified the 'betweenness' notion.
2. added a congruence axiom
3. (implicitly) added circle-circle intersection

These axioms allow the bi-interpretation of Euclidean geometry with a Euclidean ordered field (every positive element has a square root).

Non-first order axioms

1. Euclid implicity relied on the Euclidean axiom.
2. Hilbert used Archimedes and completeness axioms to study relations between axioms and to axiomatize real geometry.

These are distinct categorical axiomatizations of Euclidean Geometry

The varieties of Euclidean geometry

These are distinct categorical axiomatizations of (variants) of Euclidean Geometry

1. Euclidean geometry
2. Descartes-Tarski geometry – no trancendentals required
3. real geometry

Criterion 1: Euclidean Geometry can be categorically axiomatized

The sentence ϕ_{EG} just says that the geometry is over the minimal Euclidean field. Each element of that field is first order definable over the empty set so the sentence ϕ_{EG} has only one model and it is *countable*.

Criterion 1: Real Geometry can be categorically axiomatized

the real numbers

The reals are slightly more complicated. In addition to the first order axioms of real closed fields, one must say that the set of rationals is dense in the model. We can define the set of rationals in $L_{\omega_1,\omega}$ since each rational number is first order definable and there are only countably many of them.

Then, just say there is a rational number between any two members of the field. Finally one must say each cut in the rationals is realized. This requires $2^{\aleph_0}$-sentences, so it is only a theory not a single sentence that is categorical. Further, that theory has no countable models.

Real geometry

The categoricity of real geometry follows from the binterpretability of (appropriate) fields and their geometries.

Criterion 2 Demonstrating a theory picks out a known unique structure

If a countable structure M is minimal, then it is straight-forward to define an $L_{\omega_1,\omega}$-sentence characterizing M.
Write the first order diagram in constants $a_i : i < \omega$ and add the sentence

$$(\forall x) \bigvee x = a_i.$$

But this is not a very informative axiomatization.

As Button and Walsh have pointed out adding * to first order Peano is a more insightful axiomatization.

$L_{\omega_1,\omega}$ satisfies criterion 3: classifying

Theorem

1. A (necessarily complete) $L_{\omega_1,\omega}$-sentence ϕ is categorical if and only if its unique countable model is $L_{\omega_1,\omega}$-minimal.
   Proof. Expand vocabulary by adding constants c for elements of the minimal models form the diagram $\mathcal{D}(M)$. ϕ_M is:
$$\bigwedge_{\phi(\mathbf{c}\in\mathcal{D}(M))} \wedge \forall x \bigvee_{c\in M} x = c.$$
2. An $L_{\omega_1,\omega}$-theory T is categorical if and only if the model is given by the realization of $\kappa \leq 2^{\aleph_0}$-types over a countable minimal structure. ???? what about iterating?

First order logic

Detlefsen's first question:

(IA) Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?
(IB) Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how to we justify the claim that categoricity is or would be a virtue just in the former?

What is virtue?

From the second standpoint it is better to take theories as closed under logical consequence.

What is virtue?

I take ‘ a virtuous property’ to be one which has significant mathematical consequences for a theory or its models. Thus, a better property of theories has more mathematical consequences for the theory.

Changing the question

We argued categoricity of a second order theory does not, by itself, shed any mathematical light on the categorical structure.

But categoricity in power for first order and infinitary logic yields significant structural information about models of theory.

This kind of structural analysis leads to a fruitful classification theory for complete first order theories.
Indeed, fewer models usually indicates a better structure theory for models of the theory.

Our Argument

1. Categoricity in power implies strong structural properties of each categorical structure.
2. These structural properties can be generalized to all models of certain (syntactically described) complete first order theories.

GEOMETRIES

Definition. A pregeometry is a set G together with a 'dependence' relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
A4. $cl(cl(X)) = cl(X)$
If points are closed the structure is called a geometry.

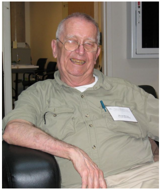
STRONGLY MINIMAL

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.
A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.
The complex field is strongly minimal.

$\aleph_1$-categorical theories

Morley  **Lachlan**  **Zilber** 

Strongly minimal set are the building blocks of structures whose first order theories are categorical in uncountable power.

$\aleph_1$-categorical theories

Theorem (Morley/ Baldwin-Lachlan/Zilber) TFAE

1. T is categorical in one uncountable cardinal.
2. T is categorical in all uncountable cardinals.
3. T is ω-stable and has no two cardinal models.
4. Each model of T is prime over a strongly minimal set.
5. Each model of T can be decomposed by finite 'ladders'. Classical groups are first order definable in non-trivial categorical theories.

Item 3) implies categoricity in power is absolute.

Any theory satisfying these properties has either one or $\aleph_0$ models of cardinality $\aleph_0$.

Zilber's Thesis

Zilber's Thesis

Fundamental structures are canonical

Fundamental mathematical structures can be characterized in an appropriate logic.

Conversely, characterizable structures are 'fundamental'.

The relevant notion of 'characterize' is categoricity in power

Specifying the thesis I

Find an axiomatization for $\mathrm{Th}(\mathcal{C}, +, \cdot, \exp)$.

Specifying the thesis II

Zilber Conjecture

Every strongly minimal first order theory is

1. disintegrated
2. group-like
3. field-like

Specifying the thesis III

Cherlin-Zilber Conjecture

Every simple ω-stable group is an algebraic group over an algebraically closed field.

This led to:
Is there an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group?

Challenge

Hrushovski Construction I

There is a strongly minimal set which is not locally modular and not field-like. (Hrushovski)

Hrushovski Construction II

There is an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group. (Baudisch, Hils, Martin-Pizarro, Wagner)

Response

Response

Strengthen the hypotheses:
Extend first order to more powerful ”Logics”.

1. $L_{\omega_1,\omega}(Q)$
2. Zariski Structures

What is $L_{\omega_1,\omega}(Q)$?

Response

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

It is Godelian:

The integers are defined as $\{a : e^a = 1\}$.
The first order theory is undecidable and 'wild'.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x) e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi.$$

Consequences of Zilber's thesis

1. There is a sentence of $L_{\omega_1,\omega}(Q)$ that is categorical in power and defines an 'exponentiation' on $\mathcal{C}$.
2. The study of this structure has stimulated new conjectures in number theory.
3. It is open whether its model in the continuum is complex exponentiation.
4. This topic is entirely done in ZFC.

What is our standpoint?

Standpoint so far

Meta theory: ZFC
Choose logic:

1. first order: little entanglement
2. $L_{\omega_1,\omega}$ or $L_{\omega_1,\omega}(Q)$: deep entanglement
3. 2nd order: What's the difference?

Alternative Metatheories

1. pluralistic interpretation of ZFC
2. subsystems of second order arithmetic
3. 2nd order logic

A multisorted vocabulary for ‘2nd order’ arithmetic

Fix a two sorted language $(N, P, 0, s, +, \times, <, \epsilon)$:

0 names an element of N, $+, \times$ are binary functions on N, $<$ is binary relation on N.

ϵ is binary relation on $N \times P$.

Without loss we assume a pairing function on N.

$X, Y, \ldots$ range over P; $x, y, \ldots$ range over N.

Subsystems of second order arithmetic

Definition

ACA_0 satisfies the axioms:

1. first order *PA*. (restricted to *N*)
2. (extensionality) $\forall X, Y[\forall x(x \in X \leftrightarrow x \in Y]) \rightarrow X = Y]$.
3. recursive comprehension. For $\phi(x, Y, y)$ with no second order quantifiers and Δ^0_1,

$$\forall Y \forall y \exists X \forall x(x \in X \leftrightarrow \phi(x, Y, y)).$$

4. 2nd order induction: $\forall X[(0 \in X \wedge \forall x(x \in X \rightarrow sx \in X)) \rightarrow \forall y(y \in X)]$.

Definition:

ACA_0 replaces RCA with ACA arithmetic comprehension. For $\phi(x, Y, y)$ with no second order quantifiers,

$$\forall Y \forall y \exists X \forall x(x \in X \leftrightarrow \phi(x, Y, y)).$$

What is used to prove $L_{\omega_1,\omega}$-categoricity?

Definition

A structure with universe $\mathbb{N}$ is recursive if each basic predicate is interpreted by a recursive set.
It is dcl-recursive if it minimal, recursive and for some finite sequence $\boldsymbol{a}$, $M = \mathrm{dcl}(\boldsymbol{a})$.

The previous condition can be weakened substantially.

Theorem

Provably in RCA_0, if M is a countable dcl-recursive minimal of a theory T, there is an $L_{\omega_1,\omega}$-sentence ϕ_M:

1. $M \models \phi_M$.
2. Every countable model of ϕ_M is isomorphic to M.

Possible objections

RCA_0 can't prove anything about uncountable models.

There are statements that are false as stated (for cardinality reasons), yet provable in RCA_0, for example that any two dense linear orderings without endpoints are isomorphic.

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There are statements that are false as stated (for cardinality reasons), yet provable in RCA_0, for example that any two dense linear orderings without endpoints are isomorphic.

The categorial structure depends on the model of RCA_0.

Indeed, if we change from a well-founded model M of RCA_0 to an ill-founded model N, the isomorphism type of models $\mathcal{A}^N$ and $\mathcal{A}^M$ may differ.

But, this is a problem for stronger meta-theories as well.

Dependence on model in *ZFC*

Fact

ω is absolute between *transitive* models of *ZFC*.

But the transitive is essential. If I take a non-principal ultrapower $N = M^{\omega}/D$ of a transitive model M (So $\omega^M = \omega^V$)), ω^N is uncountable in V.

Dependence on model in *ZFC*: ω

Fact

The real order is defined up to isomophism in ZFC by the assertion $(A, <)$ is dense, separable and Dedekind complete without endpoints.

But in a model of $ZFC + CH$, that order has cardinality $\aleph_1$.
And in a model failing *CH* it doesn't.
So the categoricity is only relative to the choice of the model of ZFC.

Internal Categoricity I

Definition (Väänänen)

The sentence ϕ in a vocabulary τ is internally categorial if for any model (M, G) satisfying the comprehension axiom and any $A, A', R, R' \epsilon G$ such that $(A, R) \approx (A', R')$, there is an $f \in G$ such that $f : (A, R) \approx (A', R')$.