AMALGAMATION PROPERTIES AND FINITE MODELS IN L^n-THEORIES

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We study theories in the logic L^n, logic with n-variables. We write $S_n^n(T)$ for the collection L^n-types in n variables over the empty set. See [Dj 1] and [Le 2] for other notation and in particular the notion of an L^n-type in more than n-variables. Djordjević [Dj 1] deals with a stable, complete L^n-theory T in a finite relational language which has amalgamation over sets (see below). Under the assumption that $S_n^n(T)$ is finite (which is a necessary for the existence of finite models and implies T is $\aleph_0$-stable), he proves that finite models exist and that given an infinite model $M \models T$, and a finite set $A \subseteq M$, there exists a finite L^n-elementary submodel N of M containing A. Unfortunately, the L^n-theory of vector spaces over a finite field does not have amalgamation over sets in a finite relational language (or even in a functional language; see Example 10), so it is not covered by this theorem. Worse, in a later work [Dj 2], Djordjević shows that under his assumptions, all the strictly minimal sets are trivial. Eric Rosen observed that this fact can be derived from a theorem of Dugald MacPherson [Ma]. Here we examine the role of the amalgamation property in Djordjević's argument and recover several results with weaker amalgamation hypotheses. In particular, the vector space example is covered by our analysis.

Our work is built on the earlier papers of Djordjević [Dj 1] and Hyttinen [Hy 1]. They made strong amalgamation hypotheses which imply the existence of L^n-saturated models. In analogy with the work on infinitary logic of Shelah and others ([Sh:300, Sh:576, Gr]), we show that many results can be proved using such generalizations of saturated structures as 'model-homogeneous' structures.

Although, for concreteness, we formulate our work entirely in the context of L^n-theories, there are actually only a few properties which are employed, so the arguments go through in considerably more generality. We describe this situation in the last two remarks of the paper.

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As noted above, if an L^n-theory has a finite model, $S_n^n(T)$ is finite. Thus, we fix a (usually) complete L^n-theory T; $S_n^n(T)$ is finite except when we say otherwise. Countable will always mean finite or countably infinite. Our language may include function symbols but n is always at least as large as the greatest arity of any relation symbol and strictly greater than the largest arity of a function symbol. This requirement on n guarantees that any L^n-embedding is an embedding in the normal (algebraic) sense. Let $M \models T$, $B \subseteq M, \bar{a} \in M$; we write $\mathrm{tp}_{L^n}(\bar{a}/B, M) = \{\phi(\bar{x}, \bar{b}) : M \models \phi(\bar{a}, \bar{b})\}$ where $\phi(\bar{x}, \bar{y})$ is in L^n and $\mathrm{tp}_L(\bar{a}/B, M)$ for the usual first order notion.

1. Stability and ω-stability

We deal with L^n-theories T which are stable in the following sense.

Definition 1. (1) We say that $\phi(x, y) \in L^n$ has the *order property* if for some model M of T there exists $\{b_i, a_j \mid i, j < \omega\} \subseteq M$ such that $M \models \phi(a_i, b_j)$ if and only if $i < j$.
(2) We say that T is *stable* if no L^n-formula has the order property.

The next fact does not assume anything on the model M. It follows from Lemma II 2.15 (1) and Theorem I 2.10 of [Sh].

Fact 2 (Shelah). *Let Δ be a finite set of formulas, closed under permutation of variables and negation. If M is a model which does not have the order property witnessed by a formula in Δ, then for each infinite cardinal λ, the model M realizes at most λ many Δ-types over each of its subsets of cardinality λ.*

Corollary 3. *If $S_n^n(T)$ is finite and T is stable then T is $\aleph_0$-stable, namely each model realizes at most countably many L^n-types over countable sets.*

Proof. Let $\Delta = L^n$. Since $S_n^n(T)$ is finite there are at most finitely many inequivalent L^n-formulas. If T is stable then each model does not have the order property witnessed by a formula in L^n. Hence, each model realizes at most countably many L^n-types over countable subsets. $\square$

The last corollary was embedded in the proofs [Hy 1] and [Dj 1] that certain extensions of T to a complete theory are in fact ω-stable in the usual first order sense. All arguments we know for this strong conclusion require that T has some form of the amalgamation property. In contrast to [Hy 1] and [Dj 1] we make no amalgamation assumptions and obtain this weaker version of ω-stability.

As noted in [Dj 1] the requirement that $S_n^n(T)$ is finite is essential for the previous Lemma. The theory of $(Z, +, 1)$ is instructive. Every formula is equivalent to a Boolean combination of L^3 formulas. So various stability properties are the same whether speaking in L^3 or L. For example, the L^3-theory of $(Z, +, 1)$ is strictly superstable.

2. Amalgamation properties

We examine here three notions of amalgamation and their associated notion of homogeneous model. In each context we obtain results on the existence of finite models of L^n-theories. Here is the weakest and most natural assumption.

Definition 4. Let T be a (not necessarily complete) L^n-theory. We say that T has the *amalgamation property over models* (AP for short), if for every triple of models M_ℓ of T, with $\ell = 0, 1, 2$, if $M_0 \prec_{L^n} M_1$ and $M_0 \prec_{L^n} M_2$, then there exist $M^* \models T$ and L^n-elementary mappings $f_\ell : M_\ell \to M^*$ which are the identity on M_0, for $\ell = 1, 2$.

If an incomplete L^n-theory T has the amalgamation property, there is a natural equivalence relation on models of T, induced by L^n-embedding, and by reducing to the equivalence classes we reduce to complete theories.

Definition 5. We say that M is $\aleph_1$ *model homogeneous* ($\aleph_1$-m-homogeneous) if for each pair of countable models $M' \prec_{L^n} M''$ and L^n-elementary mapping $f : M' \to M$, there exists an L^n-elementary mapping $g : M'' \to M$ extending f.

Many of our results depend on particular variants (AP, SAP, CAP) to be defined below of the amalgamation property. For convenience, we signal the dependence on one of these principles by writing (e.g. (AP)) at the beginning of the theorem.

Fraïssé began the study of homogeneous-universal models which has been continued and generalized by many including Jónsson, Morley-Vaught, and Shelah. The particular formulation here is close to Shelah [Sh:300]. Note that the existence of a model homogeneous model M straightforwardly implies the amalgamation property for the class of L^n-elementary submodels of M.

Fact 6. *(AP) Every model has an L^n-elementary extension which is $\aleph_1$-m-homogeneous.*

The following, which generalizes arguments of Djordjević, is the key result for transferring results from L^n to associated complete theories.

Theorem 7. *(AP) Let M and N be $\aleph_1$-m-homogeneous. Assume that for some countable $M_0 \prec_{L^n} M$, $f : M_0 \to N$ is an L^n-elementary mapping. Let $A \subseteq M_0$. Then $(M, A) \equiv (N, f(A))$.*

Proof. Write $\bar{a}$ for an enumeration of $A \subseteq M_0$ and $\bar{b}$ for $f(\bar{a})$. To show that $(M, A) \equiv (N, f(A))$, we construct a back-and-forth system I from $(M, \bar{a})$ to $(N, \bar{b})$ (see [Ho]):

Let I consist of all the countable pairs $(\bar{c}, \bar{d})$ such that there exist countable models $M' \prec_{L^n} M$ and $N' \prec_{L^n} N$ and an L^n-elementary mapping from M' onto N' sending $\bar{c}$ to $\bar{d}$ (the definition is symmetric).

Notice that $(\bar{a}, \bar{b}) \in I$ by letting N' be the range of f. Furthermore, since n is greater than the arity of the relations and greater than the arity of the function symbols of the language, if $(\bar{c}, \bar{d}) \in I$, then $\bar{c}$ and $\bar{d}$ have the same L-atomic type over the empty set.

Now let $(\bar{c}, \bar{d}) \in I$, and let $M' \prec_{L^n} M$ and $N' \prec_{L^n} N$ be countable and let g be an L^n-elementary mapping $g : M' \to N' \prec_{L^n} N$ sending $\bar{c}$ to $\bar{d}$. Let $c' \in M$ be given. Let $M'' \prec_{L^n} M$ be a countable model containing $M' \cup c$. Clearly, $M' \prec_{L^n} M''$. Then, by $\aleph_1$ model homogeneity of N, there exists $h : M'' \to N$ extending g. Let N'' be the range of h and $d' = h(c')$. Then $(\bar{c}c', \bar{d}d') \in I$. The other direction is identical, so the proof is complete. □

Note that as in [Dj 1], we could use an arbitrary infinite κ in the last result instead of $\aleph_1$. The following hypothesis holds for the remainder of the paper.

Hypothesis 8. We fix a closure operator $\mathrm{cl}(A)$ and assume this operator is uniformly locally finite: for each $m < \omega$ there exists $m' < \omega$ such that $|\mathrm{cl}(A)| \leq m'$ for each $|A| \leq m$.

There are several plausible choices for cl. Our default will be the easiest - the closure of A under the function symbols of the language. But one could use, $\mathrm{cl}(\bar{a}) = \bar{a}$, or the algebraic (definable) closure in the sense of L^n, namely the set of elements which satisfy L^n formulas with finitely many (one) solutions. (This is not the same as realizing algebraic types.) Any of these choices require an appropriate amalgamation hypothesis to guarantee that the closure is well-defined.

Now we strengthen the notion of amalgamation by allowing the amalgamation base to be a subset of the intersection.

Definition 9. (1) Let T be a complete L^n-theory. We say that T has the *amalgamation property over sets* (SAP for short) if for any pair of models $M, N \models T$ and countable set $\bar{a} \subseteq M \cap N$, if $\mathrm{tp}_{L^n}(\bar{a}/\emptyset, M) = \mathrm{tp}_{L^n}(\bar{a}/\emptyset, N)$, there exist a model $M^* \models T$ and L^n-elementary mappings $f : M \to M^*$ and $g : N \to M^*$ such that $f \upharpoonright \bar{a} = g \upharpoonright \bar{a}$.

(2) We say that T has the *amalgamation property for closed sets* (CAP for short) if for any pair of models $M, N \models T$ and set $\bar{a} \subseteq M \cap N$, if

$$\mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a})/\emptyset, M) = \mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a})/\emptyset, N),$$

then there exist M^* and L^n-elementary mappings $f : M \to M^*$ and $g : N \to M^*$ such that $f(\mathrm{cl}(\bar{a})) = g(\mathrm{cl}(\bar{a}))$.

Here is the structure of the Djordjević proof of the result stated in the first paragraph of this paper; he assumes SAP and $S^n_n(T)$ is finite throughout.

(1) Stability of T implies $\aleph_0$-stability of T (in L^n).
(2) There exist $(L^n, \aleph_0)$-saturated models.
(3) All $(L^n, \aleph_0)$-saturated models are elementary equivalent.
(4) This common first order theory is $\aleph_0$-stable and $\aleph_0$-categorical.
(5) The Cherlin, Harrington, Lachlan theorem (see Fact 25 below) then yields the existence of finite models.

Djordjević showed that SAP is equivalent to the existence of $\aleph_0$ saturated models (for L^n-types); we will carry out the argument using various notions of homogeneity depending on the context. Hyttinen [Hy 1] discusses a property which is closely related to SAP; essentially he replaces the class of all models of T by the class of finite models.

Example 10. Note that SAP implies CAP but not conversely; either implies AP. In particular for reasonably large n, the L^n-theory of vector spaces over a finite field F_p has CAP but not SAP. For, the property of a sequence of $n+1$ elements being n-independent but not $(n+1)$-independent is not a property of the L^n-type of the $n+1$-tuple (in either a relational or functional language). To be very specific, let $n = 3$ and $p = 2$. Consider the statement $\Sigma_{i=0}^{3} a_i = 0$. No Boolean combination of formulas each of which is a substitution instance of a 3 variable formula can express this statement. But, this general situation is expressed by the L^n-type of the p^{n+1}-tuple enumerating the subspace generated by the $(n+1)$-elements – in our case, by the L^3-type of a 16-tuple. (All the partial sums are elements of the tuple and the equations expressing this each involve only three variables.)

Moreover, the L^n-theory of $(Z/4Z)^\omega$ has CAP but does not admit quantifier elimination. (So our analysis includes cases which are covered neither by the analysis of Djordjević [Dj 1] nor by Hyttinen's [Hy 1] restriction to quantifier-free formulas.) Extending to a functional language does not help. One can amalgamate B and C over A if the closure of A in B equals the closure of A in C. But this condition is not implied by A having the same L^n-type in both B and C. Note that the countable vector space over a finite field is not L^n-homogeneous; there exist finite sequences realizing the same L^n-type which are not automorphic.

For each amalgamation property there is an appropriate variant on the notion of homogeneity.

Definition 11. (1) We say that the model N is $\aleph_1$ *set homogeneous* ($\aleph_1$-s-homogeneous) if for every countable set $A \subseteq M$ and map $f\colon A \to N$ with
$$\mathrm{tp}_{L^n}(\bar{a}/\emptyset, M) = \mathrm{tp}_{L^n}(f\bar{a}/\emptyset, N),$$
($\bar{a}$ enumerates A), there is an L^n-elementary mapping $g : M \to N$ extending f.

(2) We say that N is $\aleph_1$ *closure homogeneous* ($\aleph_1$-c-homogeneous) if for every countable model M and countable set $A \subseteq M$, and map $f\colon A \to N$ with
$$\mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a})/\emptyset, M) = \mathrm{tp}_{L^n}(\mathrm{cl}(f\bar{a})/\emptyset, N),$$
($\bar{a}$ enumerates A) there is an L^n-elementary mapping $g : M \to N$ extending f.

We develop parallel theories for each notion of amalgamation. The one corresponding to CAP gives the natural generalization of Djordjević's theorem; the one with SAP extends his result to functional languages and shows the pattern. In section 3, we describe further results on the nonexistence of finite maximal models of incomplete L^n-theories with AP.

Question 12. Is there an example of a complete L^n-theory with infinitely many finite models which is stable but does not have any of these amalgamation properties?

In the usual way one has

Fact 13. *If T has SAP (CAP) every model has an L^n-elementary extension which is $\aleph_1$ set ($\aleph_1$ closure) homogeneous.*

By a similar proof to Theorem 7 but identifying M and N for simplicity we have:

Theorem 14. *Let $\bar{a}$ and $\bar{b}$ be finite.*

(1) *Let N be $\aleph_1$-s-homogeneous. If*
$$\mathrm{tp}_{L^n}(\bar{a}/\emptyset, N) = \mathrm{tp}_{L^n}(\bar{b}/\emptyset, N)$$
then $\mathrm{tp}_L(\bar{a}/\emptyset, N) = \mathrm{tp}_L(\bar{b}/\emptyset, N)$.

(2) *Let N be $\aleph_1$-c-homogeneous. If*
$$\mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a})/\emptyset, N) = \mathrm{tp}_{L^n}(\mathrm{cl}(\bar{b})/\emptyset, N)$$
then $\mathrm{tp}_L(\mathrm{cl}(\bar{a})/\emptyset, N) = \mathrm{tp}_L(\mathrm{cl}(\bar{b})/\emptyset, N)$.

The previous Theorem is stated for finite sequences; the result extends immediately to sequences of arbitrary length.

Corollary 15. *If M is $\aleph_1$-s-homogeneous or $\aleph_1$-c-homogeneous then M is elementarily $\aleph_1$-homogeneous.*

Note that the truth of this conclusion for the case of $\aleph_1$-c-homogeneity depends on our assumption that on models of T, the substructure generated by a finite set is finite. It does not require that $S_n^n(T)$ be finite. More precisely, it is key to the construction that the $\mathrm{cl}(\bar{a})$ be determined by the first order type of $\bar{a}$; for this, it suffices that $\mathrm{cl}(\bar{a}) \subseteq \mathrm{acl}(\bar{a})$.

We now want to prove our generalization of Djordjević's theorem on the existence of finite models. We need to introduce two further properties.

Definition 16. An L^n-theory has the *joint embedding property*, often written JEP, if any two models of T have a common L^n-elementary extension.

Thomas exhibited in [Th] an L^n-complete theory without the joint embedding property; in fact, all models have the same finite cardinality. The following lemma provides a more concrete version of JEP.

Lemma 17. *If T has CAP, then T has JEP if and only if* $\mathrm{cl}(\emptyset)$ *has the same L^n-type in every model.*

Proof. If T has JEP, the unique type of $\mathrm{cl}(\emptyset)$ is seen by taking a common embedding of any pair of models. Conversely, by amalgamating over $\mathrm{cl}(\emptyset)$, we deduce JEP. $\square$

The following notations are then justified by (2) of Theorem 14.

Notation 18. Let T be complete theory in L^n with CAP and JEP.

(1) T^c is the first order theory of M where M is some (any) $\aleph_1$ closure homogeneous model of T.
(2) Moreover, $T^c(C)$ is the first order theory of (M, C) where M is some (any) $\aleph_1$ closure homogeneous model of T and C is $\mathrm{cl}_M(\emptyset)$.

We note the following fact about $T^c(C)$.

Lemma 19. *(CAP, JEP) Let M be a model of T with $C = \mathrm{cl}_M(\emptyset)$. The following are equivalent.*

(1) *If $p(\bar{x}_0) = \mathrm{tp}_{L(C)}(\bar{a}/\emptyset, M)$ and $q(\bar{x}) = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/\emptyset, M)$ then $T^c(C) \cup q(\bar{x}) \vdash p(\bar{x}_0)$.*
(2) *Each L^n-elementary extension of a model of $T^c(C)$ is an L-elementary extension.*
(3) *Each model of $T^c(C)$ has an elementary extension which is $\aleph_1$ model homogeneous.*

Proof. (1) implies (2) is immediate. Then (2) implies (3) follows directly from Lemma 13. Finally, suppose (3) holds. If (1) fails, there exist an ω-saturated model M of $T^c(C)$, finite sequences $\bar{a}$ and $\bar{b}$ in M such that $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/\emptyset, M) = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b})/\emptyset, M)$, but $\mathrm{tp}_{L(C)}(\bar{a}/\emptyset, M) \neq \mathrm{tp}_{L(C)}(\bar{b}/\emptyset, M)$. By (3), $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/\emptyset, M') = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b})/\emptyset, M')$, but $\mathrm{tp}_{L(C)}(\bar{a}/\emptyset, M') \neq \mathrm{tp}_{L(C)}(\bar{b}/\emptyset, M')$ for some $\aleph_1$-model homogeneous extension M' of M. This contradicts Theorem 7.

An analogous theorem holds for T^c under the hypotheses of JEP and AP. The crux is to establish any of these conditions. By assuming $S_n^n(T)$ is finite, we can do so (following the line of Djordcević.) The existence of a finite closure of the empty set is used essentially in the following argument.

Lemma 20. *(CAP, JEP) Let M be an $\aleph_1$-c-homogeneous infinite model of T with $C = \mathrm{cl}_M(\emptyset)$ and suppose $S_n^n(T)$ is finite. If $\bar{a}$ is a finite sequence, $p(\bar{x}_0) = \mathrm{tp}_{L(C)}(\bar{a}/\emptyset, M)$, and $q(\bar{x}) = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/\emptyset, M)$ then $T^c(C) \cup q(\bar{x}) \vdash p(\bar{x}_0)$.*

Proof. Let $\phi(\bar{x})$ be the conjunction of the finitely many $L^n(C)$-formulas in $q(\bar{x})$. Let $\psi(\bar{x}_0)$ be in $L(C)$. If $\mathrm{cl}(\bar{b}_i) \models \phi(\bar{x})$ and $\bar{b}_i \in M$ for $i = 1, 2$, then by Theorem 14 $M \models \psi(\bar{b}_1)$ if and only if $M \models \psi(\bar{b}_2)$. Hence, either $M \models \forall \bar{x}[\phi(\bar{x}) \to \psi(\bar{x}_0)]$ or $M \models \forall \bar{x}[\phi(\bar{x}) \to \neg\psi(\bar{x}_0)]$. By completeness of $T^c(C)$, for each $\psi(\bar{x}_0) \in L(C)$, we have either $T^c(C) \vdash \forall \bar{x}[\phi(\bar{x}) \to \psi(\bar{x}_0)]$, or $T^c(C) \vdash \forall \bar{x}[\phi(\bar{x}) \to \neg\psi(\bar{x}_0)]$. But, if $\psi(\bar{x}_0) \in p$, $\phi(\bar{x}) \wedge \psi(\bar{x}_0)$ is consistent so $T^c(C) \vdash \forall \bar{x}[\phi(\bar{x}) \to \psi(\bar{x}_0)]$. This implies the conclusion. $\square$

In particular, the last result shows that for any m there are only finitely many types of closed sets of cardinality m over C (i.e. of types over the empty set in $L(C)$) realized in any $\aleph_1$-m-homogeneous structure. Since cl is uniformly locally finite this implies that for each k there can be only finitely many k types over the empty set in $L(C)$ realized in an $\aleph_1$-homogeneous model and thus in every model of $T^c(C)$. Thus:

Corollary 21. *(CAP, JEP) $T^c(C)$ is $\aleph_0$-categorical.*

Lemma 20 and Lemma 19 immediately imply the L^n-embedding in Fact 13 is actually an elementary map.

Corollary 22. *If T has SAP (CAP, JEP) and $S_n^n(T)$ is finite, every model of T^c ($T^c(C)$) has an elementary extension which is $\aleph_1$ set ($\aleph_1$ closure) homogeneous.*

We require one further assumption to show that $T^c(C)$ is $\aleph_0$-stable. This condition yields that $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/N, M') = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b})/N, M')$ implies $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a}N)/\emptyset, M') = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b}N)/\emptyset, M')$.

Hypothesis 23. (H1) For any countable models M_ℓ for $\ell = 0, 1, 2$ with $M_0 \prec_{L^n} M_\ell$ for $\ell = 1, 2$, and $\bar{a}_\ell \in M_\ell$ for $\ell = 1, 2$ if

$$\mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a}_1)/M_0, M_1) = \mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a}_2)/M_0, M_2),$$

then there exist N and L^n-elementary embeddings $f_\ell : M_\ell \to N$ such that $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ and f_ℓ is the identity over M_0 for $\ell = 1, 2$.

Note that H1 holds for the 'standard examples' of totally categorical theories: $\mathrm{Th}((Z_{p^n})^\omega)$.

Corollary 24. *(CAP, JEP) If T satisfies H1 and T is stable, then $T^c(C)$ is $\aleph_0$-stable.*

Proof. Let M be a model of $T^c(C)$. Since M is a model of T, M does not have the order property witnessed by a formula in L^n, and hence in $L^n(C)$. Let $N \prec M$ be a countable submodel. It is enough to show that M realizes only countably many $L(C)$-types over N.

Suppose that M realizes uncountably many $L(C)$-types over N. By Corollary 22, we can choose M', an $\aleph_1$-c-homogeneous elementary extension of M. Then, M' realizes uncountably many $L(C)$-types over N. Corollary 3 implies that only countably many $L^n(C)$-types over N are realized in M'. Hence, by the pigeonhole principle, there exist $\bar{a}, \bar{b}$ such that $\mathrm{tp}_{L(C)}(\bar{a}/N, M') \neq \mathrm{tp}_{L(C)}(\bar{b}/N, M')$ but $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a})/N, M') =$

$\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b})/N, M')$. By finite character, there exists a finite $\bar{d} \in N$, such that $\mathrm{tp}_{L(C)}(\bar{a}\bar{d}/\emptyset, M') \neq \mathrm{tp}_{L(C)}(\bar{b}\bar{d}/\emptyset, M')$ but $\mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{a}\bar{d})/\emptyset, M') = \mathrm{tp}_{L^n(C)}(\mathrm{cl}(\bar{b}\bar{d})/\emptyset, M')$. Invoking H1, Hypothesis 23, this contradicts Theorem 14 (2). $\square$

The next fact is proved in [CHL]:

Fact 25 (Cherlin, Harrington, Lachlan)**.** *If $Th_L(M)$ is $\aleph_0$-categorical and $\aleph_0$-stable, A is a finite subset of M and ψ is a sentence such that $M \models \psi$, then there exists a finite M' with $A \subseteq M' \subset M$ such that $M' \models \psi$.*

Theorem 26. *Let T be a complete stable L^n-theory. Assume that T has CAP, JEP, H1 and $S^n_n(T)$ is finite. Then T has arbitrarily large finite models.*

Proof. Let M be an $\aleph_1$-c-homogeneous model and $C = \mathrm{cl}_M(\emptyset)$. Construct $Th_{L(C)}(M) = T^c(C)$ as above. Then $T^c(C)$ is $\aleph_0$-categorical and $\aleph_0$-stable. By Fact 25, for each $k < \omega$ there is a finite model of T with cardinality at least k. $\square$

This argument allows an improvement of Djordjević's theorem where we allow functional languages. There are no hidden assumptions. In particular, the L^3-theory of infinitely many 3 cycles (in a language with one unary function symbol) is now covered by the theorem.

Corollary 27. *Let T be a complete stable L^n-theory and with $S^n_n(T)$ finite. Suppose that T has SAP. Then, there exist finite models and each finite model has arbitrarily large finite L^n-elementary extensions.*

Proof. First, SAP implies CAP and JEP when we take $\mathrm{cl}(\bar{a}) = \bar{a}$ SAP also implies H1. So we apply the last result. $\square$

3. Forbidding maximal finite models

Although much of the development here parallels Section 2, the final result is quite different. We show that under reasonable amalgamation conditions one cannot have the phenomena of Thomas [Th, Th1] - an L^n-theory with finitely many finite models and also infinite models which extend them. For the remainder of the paper, we drop the assumption that T is a *complete* theory in L^n. We do assume that T has AP (amalgamation over models). If we take the disjunction of a stable complete L^n sentence with only finitely many L^n types, AP and JEP and a sentence with a single finite model, we will obtain a theory which meets the hypotheses of this section.

In our context, we will fix a particular finite model A and work with $T(A)$, the L^n-theory of A.

Eric Rosen has pointed out a general construction for counterexamples. Form a two-sorted model with a 'well-behaved' example (e.g. one of the totally categorical examples considered in the earlier sections) in one sort and a 'pathological' example (e.g. the Thomas examples with a few finite models and also infinite models [Th1]) in the other sort. Since Djordjević has shown [Dj 1] that the Thomas example fails SAP, the resulting theory will fail SAP, has infinite models, and fail to have maximal finite models. Is there an example of a complete theory in L^n which has, say AP, and infinite models but also maximal finite models?

Notice that if a finite model A has arbitrarily large finite L^n-elementary extensions, then by AP, A has an infinite L^n-elementary extension (see the proof of Corollary 35). We set out to prove the converse of this fact.

Hypothesis 28. From now until Corollary 35, we assume that there exists a finite model of T with infinite L^n-elementary extensions.

Let A be this finite model. By our amalgamation hypothesis, there exist $\aleph_1$-m-homogeneous L^n-elementary extensions of A.

Corollary 29. *(AP,H1) Let M, N be $\aleph_1$-m-homogeneous models. Suppose that M_0 is countable and $M_0 \prec_{L^n} M, N$. Suppose that*

$$\mathrm{tp}_{L^n}(\mathrm{cl}(\bar{a})/M_0, M) = \mathrm{tp}_{L^n}(\mathrm{cl}(\bar{b})/M_0, N),$$

for $\bar{a} \in M$ and $\bar{b} \in N$ finite. Then $(M, M_0\bar{a}) \equiv (N, M_0\bar{b})$.

Proof. Using H1, we can find an $\aleph_1$-m-homogeneous model M^* and L^n-elementary embeddings $f : M \to M^*$ and $g : N \to M^*$ which agree on M_0 such that $f(\bar{a}) = g(\bar{b})$. By Theorem 7, we have $(M^*, f(M_0)f(\bar{a})) \equiv (M, M_0\bar{a})$ and $(M^*, g(M_0)g(\bar{b})) \equiv (M, M_0\bar{b})$. Hence, $(M, M_0\bar{a}) \equiv (N, M_0\bar{b})$ by choice of f and g. □

Notation 30. By Theorem 7, if M and N are $\aleph_1$-m-homogeneous models of T and $A \prec_{L^n} M, N$, then $(M, A) \equiv (N, A)$. We can therefore unambiguously denote by $T^c(A)$ the $Th_L(M, A))$ for any $\aleph_1$-m-homogeneous model M with $A \prec_{L^n} M$. Then $T^c(A)$ is a first order theory in the finite language $L(A)$ expanding L with finitely many constants for the elements of A. Denote by $D(T^c(A))$ the set of all complete $L(A)$-types over the empty set consistent with $T^c(A)$. Notice that since A has an infinite L^n-elementary extension M, then $T' = Th_{L^n}(M)$ is a complete L^n-theory with finite models, hence there are only finitely many L^n-types over the empty set consistent with T' and finitely many inequivalent L^n-formulas over each finite set of parameters modulo T', and hence modulo $T^c(A)$.

The next two results are proved analogously to Lemma 20, and Corollaries 21 and 22.

Lemma 31. *(AP,H1) Let M be an $\aleph_1$-m-homogeneous infinite L^n-elementary extension of A. If $p(\bar{x}_0) = \mathrm{tp}_{L(A)}(\bar{a}/A, M)$ and $q(\bar{x}) = \mathrm{tp}_{L^n(A)}(\mathrm{cl}(\bar{a})/A, M)$ then $T^c \cup q(\bar{x}) \vdash p(\bar{x}_0)$.*

Corollary 32. *(AP,H1) $T^c(A)$ is $\aleph_0$-categorical.*

Corollary 33. *(AP,H1) If $M \models T^c(A)$ and N is an $\aleph_1$-m-homogeneous L^n-elementary extension of M, then $M \prec_{L(A)} N$. In particular, every model of $T^c(A)$ has an elementary extension which is $\aleph_1$-m-homogeneous.*

As in Corollary 24, we have:

Corollary 34. *(AP,H1) $T^c(A)$ is $\aleph_0$-stable.*

Now we can obtain the following characterization of when a finite model of a possibly incomplete L^n-theory has arbitrarily large finite L^n elementary extensions. We need the hypothesis H1 only in the direction: 2 implies 1.

Corollary 35. *Let T be a stable L^n-theory. Assume that T has AP and T satisfies H1. The following conditions for a finite model $A \models T$ are equivalent:*

(1) *There exist arbitrarily large finite L^n-elementary extensions of A;*
(2) *There exists an infinite L^n-elementary extension of A.*

Proof. (1) implies (2): Let $\{N_i \mid i < \omega\}$ be an enumeration of all finite L^n-elementary extensions of A. By AP, we construct an increasing and continuous increasing sequence of models M_i such that $M_0 = A$ and M_{i+1} is an amalgam of M_i and N_i. Then, by assumption for each $m < \omega$ there exists $i < \omega$ such that $\|M_i\| \geq m$. Let $M = \bigcup_{i<\omega} M_i$. It is an infinite L^n-elementary extension of A.

We prove (2) implies (1). Let A be given with an infinite L^n-elementary extension. Then there exists an infinite $\aleph_1$-m-homogeneous L^n-elementary extension. Construct $Th_{L(A)}(M) = T^c(A)$ as above. Then $T^c(A)$ is $\aleph_0$-categorical and $\aleph_0$-stable. Let $k < \omega$. By Fact 25 applied to $T^c(A)$ there exists a finite L^n-elementary extension of A with at least k elements. $\square$

4. Conclusions

We began with the observation that Djordjević's theorem did not cover an important case; we have recovered that case and more. Is our result best possible? We can formalize this question as follows. Hrushovski [Hru] showed every totally categorial theory is axiomatized by a single finite sentence plus axioms asserting the universe is infinite. He generalized this result to show that every ω-stable ω-categorical theory is axiomatized by a single sentence and finite family of 'axiom schemes of infinity'. We have shown that every stable theory in L^n with CAP, JEP and $S^n(T)$ finite is naturally associated with a theory of the second sort. Eric Rosen asked whether this map is onto.

Question 36. Is every totally categorical (ω-stable ω-categorical) theory axiomatized by an L^n sentence which has CAP plus axioms asserting the universe is infinite (suitable axiom schema of infinity)?

In effect, this question asks what kind of L^n-homogeneity conditions are satisfied by the countable model of theories satisfying the specified conditions.

Under the SAP, we can extend the analysis of Lemma 19 and shed some light on a question of Djordjević. He showed, assuming SAP and $S^n_n(T)$ finite that every (L^n, κ)-saturated model of T^c is κ-saturated and asked whether the requirement that $S^n_n(T)$ be finite is essential. This question remains open but we have:

Lemma 37. *Let T be a complete L^n-theory with SAP and canonical completion T^c. The following are equivalent.*

1. $p(x) \cup T^c \models q(x)$*, where* $p(x)$ *is the restriction of* $q(x)$ *to* L^n *and* $q(x)$ *is a type over the empty set consistent with* T^c*.*
2. *Every L^n-elementary extension of a model of T^c is an L-elementary extension.*
3. *Every model of T^c has an elementary extension which is $(L^n, \aleph_0)$-saturated.*
4. *Every $\aleph_0$-saturated model of T^c is L^n-saturated.*
5. *Every $(L^n, \aleph_0)$-saturated model is $\aleph_0$-saturated.*

Proof. The cycle of (1) implies (2) implies (3) is proved as in Lemma 19. And a slight variant on the proof of (3) implies (1) there yields (5) implies (1) here. For (3) implies (4), let M be an $\aleph_0$-saturated model of T^c. Then there exists an elementary extension M' which is $(L^n, \aleph_0)$-saturated. Let $p(x)$ be an L^n-type over a finite subset A of M which is consistent with $Th_{L^n}(M, A) = Th_{L^n}(M', A)$. Then $p(x)$ is realized in M', so it is consistent with $Th_L(M', A) = Th_L(M, A)$, hence realized in M by $\aleph_0$-saturation of M. By (4), some and hence all $(L^n, \aleph_0)$-saturated models of T^c are weakly saturated. But by Theorem 7, an $(L^n, \aleph_0)$-saturated model is ω-homogeneous (in the usual first order sense). Thus, it is $\aleph_0$-saturated, giving (5) and completing the cycle.

Note that (1) and thus any of the above conditions is equivalent to: Every $\psi(x) \in L$ is equivalent (modulo T^c) to a boolean combination of L^n formulas. We can rephrase Djordjevic's question as:

Question 38. Is T is a complete L^n theory with SAP, must the equivalent conditions of Lemma 37 hold?

Finally, there is a natural extension (asked by Lessmann) of our analysis in Section 2.

Question 39. Suppose T is a complete theory in L^n with $S_n^n(T)$ finite, JEP and AP. Must T have a finite model?

Remark 40. For ease of notation, we always worked with the class of *all* models of an L^n-theory. This restriction is not essential. Suppose $\mathcal{K}$ is a class of L^n-elementary equivalent models, which is closed under isomorphism and unions of L^n-elementary chains; for example the class of models of a complete L^n-theory omitting a prescribed set of (first order) types. All the amalgamation properties we described can be defined for $\mathcal{K}$. Similarly, the appropriate 'homogeneous' models will exist in $\mathcal{K}$; for example SAP will give (L^n, κ)-homogeneous models, rather than (L^n, κ)-saturated ones. Finally, we can define stability for $\mathcal{K}$, by asking that no model of $\mathcal{K}$ has the order property witnessed by a formula in L^n. This is equivalent to assuming that the appropriate sufficiently homogeneous models do not have the order property witnessed by a formula in L^n. The reader can easily check that all the proofs work in this more general setting, provided one rephrases the conclusions in terms of L^n-elementary finite submodels of the appropriately homogeneous model. For example, any $(L^n, \aleph_1)$-homogeneous model which does not have the order property witnessed by a formula in L^n, and which realizes only finitely many L^n-types over the empty set has arbitrarily large finite L^n-elementary submodels. Similar generalizations but under stronger stability assumptions are considered in [Dj 2].

We finish with a comment about how to extend the results of this last section to other sublogics of first order logic or for incomplete first order axiomatizations.

Remark 41. As we remarked in the introduction, only a few features of L^n have been exploited in this paper. Let L^* be a fragment of first order logic and T a set of sentences in L^*. We can define L^*-types with parameters by substituting L^* for L^n in the definition of types we gave. We can also define $M \prec_{L^*} N$ in the natural way, as well as L^*-elementary mappings: those which preserve L^*-formulas. This leads to the corresponding definitions of SAP, CAP, and AP. All the results of this last section hold for L^*, with the obvious change in the definition of $H1$ provided that the following four conditions are satisfied: (0) L^* is closed under negation and permutations of variables. (1) For each $m < \omega$, the set of L^*-types in m variables over the empty set is finite. (2) The set of inequivalent L^*-formulas is finite. (3) If $f : M \to N$ is an L^*-elementary mapping then, for each $\bar{a} \in M$, we have $\mathrm{atp}(\bar{a}, \emptyset, M) = \mathrm{atp}(f(\bar{a})/\emptyset, N)$, namely atomic types are preserved under L^*-elementary mappings. The first fact is true for L^n. The next two are true for a complete L^n-theory if there exists a finite model. The last fact is true if n is greater than the arity of any function or relation in the language.

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