Some needed examples

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Theorem 1 *There are complete sentences ϕ and ψ in $L_{\omega_1,\omega}$ which consistently with ZFC are $\aleph_1$-categorical but*

1. *ϕ is not ω-stable.*
2. *ψ does not have the amalgamation property in $\aleph_0$.*

In fact, the second example also satisfies the first condition but for ease of understanding we give the simpler construction first. We rely on two old results.

Fact 2 (Baumgartner??) *It is consistent with ZFC that $2^{\aleph_0} = \aleph_2$ and any two $\aleph_1$-dense linear orders of power $\aleph_1$ which have a countable dense subset are isomorphic.*

Each example will be based on identifying two structures over a common predicate Q.

Fact 3 (Marcus) *There is a structure A in a vocabulary $\tau = \langle Q, \ldots \rangle$ such that:*

1. *Q denotes an infinite set of indiscernibles in A.*
2. *A is a minimal prime model of its first order theory.*
3. *Thus, if χ is the Scott sentence of A, χ has exactly one model.*

Let τ_1 consist of a binary relation symbol, $<$ and a 5-ary function symbol $f(x, y, u, v, z)$. Expand a model of $(\mathbb{Q}, <)$ to a τ_2' structure A_1 by naming a collection of functions which guarantee that every pair of intervals is order isomorphic. Let ϕ_1 be the Scott sentence of A_1.

Let σ_1 be the union of the vocabularies τ and τ_1, which have only the symbol Q in common. Let M consist of the countable model A of χ and a countable model of ϕ_1 which agree on Q and are otherwise disjoint. And let ϕ be the Scott sentence of M. Then ϕ is certainly $\aleph_0$ categorical and there can be no model of ϕ which properly extends Q since the reduct to τ_1 would contradict the minimality of A.

Claim 4 *In Baumgartner's model the sentence ϕ is:*

1. *$\aleph_1$ and $\aleph_0$-categorical*
2. *but not $\aleph_0$ stable.*

Proof. The 'Marcus' part of a model is ϕ is fixed up to isomorphism; the 'order' part is $\aleph_1$-categorical since every model is $\aleph_1$-dense with a countable dense subset. But there are clearly $2^{\aleph_0}$ types given by the cuts in Q. $\Box_{\textbf{??}}$ Now we turn to the more complicated example where we foil amalgamation.

The vocabulary τ_2 extends the vocabulary τ_1 by adding: a unary predicate D, and a binary relation E and another unary predicate Q. Define a τ_2-structure M satisfying the following: $<$ is a dense linear order, D and Q are disjoint dense and codense subsets; E is an equivalence relation with two classes: each class is dense and codense. Also, the set of elements in neither P nor D is dense. And for each equivalence class of E, the set of elements in that class and Q is dense and the set of elements in that class and $\neg Q$ is dense. Finally, if two points are E equivalent and satisfy the same cut in D, they are equal. Interpret the function symbol f so that for every a, b, c, d, $(\lambda x) f(a, b, c, d, x)$ preserves Q, D and the equivalence classes of E. Let ψ_1 be the Scott sentence of M.

Let σ_2 be the union of the vocabularies τ and τ_2, which have only the symbol Q in common. Let M consist of the countable model A of χ and a countable model of ψ_1 which agree on Q and are otherwise disjoint. And let ψ be the Scott sentence of M.

Claim 5 *The sentence ψ*

1. *is categorical in $\aleph_1$ and $\aleph_0$*
2. *but does not have the amalgamation property.*

Proof. The categoricity is as in Claim **??**. Moreover, ψ does not have the amalgamation property. Let M be a countable model of ψ and suppose a realizes a cut in Q not realized in M. Suppose that in M_1 and M_2 the realization of this cut are in different E classes; then M_1 and M_2 cannot be amalgamated over P. $\Box$**??**(Exercise; why is first order amalgamation possible?)