Hanf numbers for Extendibility and related phenomena

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In this paper we discuss two theorems whose proofs depend on extensions of the Fraïssé method. We prove here the Hanf number for the property that every model of cardinality κ is extendible[1] of a (complete) sentence of $L_{\omega_1,\omega}$ is (modulo some mild set theoretic hypotheses that we expect to remove in a later paper) the first measurable cardinal. And we outline the description of an explicit $L_{\omega_1,\omega}$-sentence ϕ_n characterizing $\aleph_n$ for each n. We provide some context for these developments as outlined in the lectures at IPM.

The phrase 'Fraïssé construction' has taken many meanings in the over 60 years since the notion was born [Fra54] (and earlier in an unpublished thesis). There are two major streams. We focus here on variants in the original construction, which usually use the standard notion of substructure. We don't deal here directly with 'Hrushovski constructions' where a specialized notion of strong submodel varying with the case plays a central role. An annotated bibliography of developments of the Hrushovski variant until 2009 appears at [Bal].

The first variant we want to consider is the vocabulary. Fraïssé worked with a *finite, relational* vocabulary. While model theory routinely translates between functions and their graphs and there is usually little distinction between finite and countable vocabularies; in the infinite vocabulary case such extensions yield weaker but still very useful consequences. The second is a distinction in goal: the construction of *complete* sentences of $L_{\omega_1,\omega}$ (equivalently studying the *atomic* models of a complete first order theory) rather than *arbitrary* models of a first order theory. This second shift raises new questions about the cardinality of the resulting models. The second result here pins down more precisely the existence spectra for *complete* sentences of $L_{\omega_1,\omega}$. The first expresses the role of large cardinal axioms in more algebraic terms. Rephrased, it says that, consistently with the existence of a measurable cardinal, there is a nicely defined class of models that has non-extendible (maximal) models cofinally below the first measurable. The previous upper bound for such behavior was $\beth_{\omega_1}$.

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[1] We say $\boldsymbol{K}$ is *universally extendible in* κ if $M \in \boldsymbol{K}$ with $|M| = \kappa$ has a proper $\prec_{\boldsymbol{K}}$-extension in the class. Here, this means has an ∞, ω-elementary extension.

1 Hanf numbers and Spectrum functions in infinitary logic

Recent years have brought a number of investigations of the spectrum (cardinals in which the phenomenon occurs) for various phenomena and various sorts of infinitary definable classes. Some of the relevant phenomena are existence, amalgamation, joint embedding, maximal models etc. The class might be might be defined as an AEC, the models of a (complete) sentence of $L_{\omega_1,\omega}$, etc.

Hanf observed [Han60] that for any property $P(\boldsymbol{K}, \lambda)$, where $\boldsymbol{K}$ ranges over a *set* of classes of models, there is a cardinal $\kappa = H(P)$ such that κ is the least cardinal satisfying: if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. $H(P)$ is called the Hanf number of P. e.g. $P(\boldsymbol{K}, \lambda)$ might be the property that $\boldsymbol{K}$ has a model of power λ.

Morley [Mor65] showed for an arbitrary sentence of $L_{\omega_1,\omega}(\tau)$ the Hanf number for existence is $\beth_{\omega_1}$ when τ is countable (more generally, it is $\beth_{(2^{|\tau|})^+}$ [She78]); the situation for *complete* sentences is much more complicated. Knight [Kni77] found the first complete sentence characterizing ω_1 (i.e. has a model in ω_1 but no larger) by building on the construction of many non-isomorphic $\aleph_1$-like linear orderings. Hjorth found, by a procedure generalizing the Fraïssé -construction, for each $\alpha < \omega_1$, a set S_α (finite for finite α) of complete $L_{\omega_1,\omega}$-sentences[2] such that some $\phi_\alpha \in S_\alpha$ characterizes $\aleph_\alpha$. It is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. Baldwin, Koerwien, and Laskowski [BKL16] show a modification of the Laskowski-Shelah example (see [LS93, BFKL16]) gives a family of $L_{\omega_1,\omega}$-sentences ϕ_r, which characterize $\aleph_r$ for $r < \omega$. In Section 4 we sketch the new notion of n-disjoint amalgamation that plays a central role in [BKL16].

Further results by [BKS09, KLH14, BKS16], where the hypothesis are weakened to allow incomplete sentences of $L_{\omega_1,\omega}$ or even AEC are placed in context in [BB17]. Analogous results were proved earlier for *incomplete* sentences by [BKS16] who code certain bipartite graphs in way that determine specific inequalities between the cardinalities of the two parts of the graph; in this case all models have cardinality less than $\beth_{\omega_1}$.

All the exotica mentioned here and described in more detail in [BB17] occurs below $\beth_{\omega_1}$. Baldwin and Boney [BB17] have shown that the Hanf number for amalgamation is no more than the first strongly compact cardinal. This immense gap motivated the current paper. We show that for the case of univerally extendable such a gap does not exist. There is a *complete* sentences of $L_{\omega_1,\omega}$ which has a maximal model in cardinals cofinal in the first measurable (if such exists), but no larger maximal model. Is the same true of amalgamation? That is, can amalgamation eventually behave very differently than it does in small cardinalities? At the end of this paper we point to the only known example where amalgamation holds on an initial segment then fails, then holds again; then there are no larger models.

2 Disjoint Amalgamation

2.1 Classes determined by finitely generated structures

The original Fraïssé construction took place in a *finite relational* vocabulary and the resulting infinite structure was $\aleph_0$-categorical for a first order theory. We explore here several ways to construct a countable atomic model for a first order theory and thus complete sentences in $L_{\omega_1,\omega}$.

Recall (e.g. chapter 7 of [Bal09]) that the models of a complete sentence of $L_{\omega_1,\omega}(\tau)$ are the reducts to τ of the atomic (every finite sequence realizes a principal type) models of a complete first order theory in a

[2] Inductively, Hjorth shows at each α and each member ϕ of S_α one of two sentences, χ_ϕ, χ'_ϕ, works as $\phi_{\alpha+1}$ for $\aleph_{\alpha+1}$.

vocabulary τ' extending τ. We discuss classes determined by a countable set of finitely generated models. In Sections 3 and 4, we describe the examples of such classes used to prove our main results.

Definition 2.1.1. *Fix a countable vocabulary τ (possibly with function symbols). Let $(\boldsymbol{K}_0, \subseteq)$ denote a* countable *collection of* finitely generated *τ-structures and let $(\widehat{\boldsymbol{K}}, \subseteq)$ denote the abstract elementary class containing all structures M such that every* finitely generated *substructure of M is in $\boldsymbol{K}_0$.*

These classes have syntactic characterizations.

Lemma 2.1.2. *1. $\widehat{\boldsymbol{K}}$ is defined by an $L_{\omega_1,\omega}$-sentence ϕ.*

2. If $\boldsymbol{K}_0$ is closed under substructure then ϕ may be taken universal [Mal69].

3. $(\boldsymbol{K}_0, \subset)$ satisfies the axioms for AEC (except for unions under chains.)

We will see that $\subset$ can be replaced with more useful notion of strong submodel later.

Definition 2.1.3. *Fix a countable vocabulary τ (possibly with function symbols). Let $(\boldsymbol{K}_0, \leq)$ denote a countable collection of finitely generated τ-structures with $(\widehat{\boldsymbol{K}}, \leq)$ as in Definition 2.1.1.*

1. *A model $M \in \widehat{\boldsymbol{K}}$ is* rich *or* $\boldsymbol{K}_0$-homogeneous *if for all finitely generated A and B in $\boldsymbol{K}_0$ with $A \leq B$, every embedding $f : A \to M$ extends to an embedding $g : B \to M$. We denote the class of rich models in $\widehat{\boldsymbol{K}}$ as $\mathbf{R}$.*
2. *The model $M \in \widehat{\boldsymbol{K}}$ is* generic *if M is rich and M is an increasing union of a chain of finite substructures, each of which is in $\boldsymbol{K}_0$.*
3. *We let $\mathbf{R}$ denote the subclass of $\widehat{\boldsymbol{K}}$ consisting of rich models.*

Definition 2.1.4. *An AEC $(\boldsymbol{K}, \leq)$ has $(< \lambda, 2)$-disjoint amalgamation if for any $A, B, C \in \boldsymbol{K}$ with cardinality $< \lambda$ and A strongly embedded in B, C, there is a D and strong embedding of B, C into D that agree on A and such that the intersection of their ranges is their image of A.*

$\boldsymbol{K}$ has 2-amalgamation if the ranges of the embedding are allowed to intersect non-trivially.

$\boldsymbol{K}$ has the joint embedding property *(JEP) if any two models can be embedded in some larger D.*

Fraïssé 's theorem asserted that if a class of finite models in a finite relational language is closed under substructure and satisfies AP and JEP then there is a generic model whose theory is $\aleph_0$-categorical and quantifier eliminable. The following extension of Fraïssé's theorem is well-known [Hod93].

Lemma 2.1.5. *Suppose τ is countable and $\boldsymbol{K}_0$ is a countable class of finite* or countable *τ-structures that satisfies 2- amalgamation, in particular $(\leq \aleph_0, 2)$-disjoint amalgamation, and JEP, then*

1. *A $\boldsymbol{K}_0$-generic (and so rich) τ-structure M exists.*
2. *if $\boldsymbol{K}_0$ is closed under substructure, the generic is ultra-homogeneous (every isomorphism between arbitrary finitely generated substructures extends to an automorphism).*

Proof. The Fraïssé argument works. $\square_{2.1.5}$

A key distinction from the Fraïssé situation is the distinction between $\widehat{\boldsymbol{K}}$ and $\mathbf{R}$. Fraïssé passes to the first order theory of the generic since it is $\aleph_0$-categorical in *first order logic*. In our more general situation the generic may be $\aleph_0$-categorical only in $L_{\omega_1,\omega}$. The Scott sentence of the rich model gives the $L_{\omega_1,\omega}$ sentence we study. As noted at the beginning of this section we may regard the models as reducts of atomic models of a first order theory. Thus $\widehat{\boldsymbol{K}}$ may have arbitrarily large models while $\mathbf{R}$ does not; this holds of some examples in [Hjo07, BFKL16, BKL16].

Corollary 2.1.6. *Suppose $(\boldsymbol{K}_0, \leq)$ satisfies the hypotheses of Lemma 2.1.5. Fix $\lambda \geq \aleph_0$. If $\widehat{\boldsymbol{K}}$ has $(\leq \lambda, 2)$-amalgamation and has at most countably many isomorphism types of countable structures, then every $M \in \widehat{\boldsymbol{K}}$ of power λ can be extended to a rich model $N \in \widehat{\boldsymbol{K}}$, which is also of power λ.*

Proof. Given $M \in \widehat{\boldsymbol{K}}$ of power λ, construct a continuous chain $\langle M_i : i < \lambda \rangle$ of elements of $\widehat{\boldsymbol{K}}$, each of size λ. At a given stage $i < \lambda$, focus on a specific finite substructure $A \subseteq M_i$ and a particular finite extension $B \in \widehat{\boldsymbol{K}}$ of A. If there is an embedding of B into M_i over A, $M_{i+1} = M_i$. If not, we may assume $B \cap M_i = A$. Let M_{i+1} be the disjoint amalgamation of M_i and B over A. As there are only λ-possible extensions, we can, by iterating, organize this construction so that $N = \bigcup \{M_i : i < \lambda\}$ is rich. $\square_{2.1.6}$

Crucially, in Section 3.2 the class $\hat{\boldsymbol{K}}$ under consideration will not satisfy disjoint two amalgamation even with finite models; but free members of it will.

2.2 Atomic Models of First order theories

We discuss here classes generated by finite (not finitely generated) structures. Suppose a generic τ-model M exists. When is M an atomic model of its first-order τ-theory? As remarked in Section 2 of [BKL16] this second condition has nothing to do with the choice of embeddings on the class $\boldsymbol{K}_0$, but rather with the choice of vocabulary. The following condition is needed when, for some values of n, $\boldsymbol{K}_0$ has infinitely many isomorphism types of structures of size n

We denote the class of atomic models of a complete first order theory by $\mathbf{At}$.

Definition 2.2.1. *A class $\boldsymbol{K}_0$ of finite structures in a countable vocabulary is* separable *if, for each $A \in \boldsymbol{K}_0$ and enumeration $\boldsymbol{a}$ of A, there is a quantifier-free first order formula $\phi_{\boldsymbol{a}}(\mathrm{x})$ such that:*

- *$A \models \phi_{\boldsymbol{a}}(\boldsymbol{a})$ and*
- *for all $B \in \boldsymbol{K}_0$ and all tuples b from B, $B \models \phi_A(\mathrm{b})$ if and only if b enumerates a substructure B' of B and the map $\boldsymbol{a} \mapsto \mathrm{b}$ is an isomorphism.*

In practice, we will apply the observation that if for each $A \in \boldsymbol{K}_0$ and enumeration $\boldsymbol{a}$ of A, there is a quantifier-free formula $\phi'_{\boldsymbol{a}}(\mathrm{x})$ such that there are only finitely many $B \in \boldsymbol{K}_0$ with cardinality $|A|$ that under some enumeration b satisfy $\phi'_{\boldsymbol{a}}(\mathrm{b})$, then $\boldsymbol{K}_0$ is separable.

Lemma 2.2.2. *[BKL16] Suppose τ is countable and $\boldsymbol{K}_0$ is a class of finite τ-structures that is closed under substructure, satisfies amalgamation, and JEP, then a $\boldsymbol{K}_0$-generic (and so rich) model M exists. Moreover, if $\boldsymbol{K}_0$ is separable, M is an atomic model of $Th(M)$. Further, $\mathbf{R} = \mathbf{At}$, i.e., every rich model N is an atomic model of $Th(M)$.*

Proof: Since the class $\boldsymbol{K}_0$ of finite structures is separable it has countably many isomorphism types, and thus a $\boldsymbol{K}_0$-generic M exists by the usual Fraïssé construction. To show that M is an atomic model of $Th(M)$, it suffices to show that any finite tuple $\boldsymbol{a}$ from M can be extended to a larger finite tuple b whose type is isolated by a complete formula. Coupled with the fact that M is $\boldsymbol{K}_0$-locally finite, we need only show that for any finite substructure $A \leq M$, any enumeration $\boldsymbol{a}$ of A realizes an isolated type. Since every isomorphism of finite substructures of M extends to an automorphism of M, the formula $\phi_{\boldsymbol{a}}(\mathrm{x})$ isolates $\mathrm{tp}(\boldsymbol{a})$ in M.

The final sentence follows since any two rich models are $L_{\infty,\omega}$-equivalent. $\square_{2.2.2}$

3 Hanf number for All Models Extendible

We say an abstract elementary class (the models of a complete sentence in $L_{\omega_1,\omega}$ is *universally extendible in* κ if every model of cardinality κ has a proper strong extenstion ($L_{\infty,\omega}$-elementary extension). In this section we prove the following theorem.

Theorem 3.0.3. *There is a complete sentence ϕ of $L_{\omega_1,\omega}$ that under reasonable set theoretic conditions (specified below) has arbitrarily large models. But for arbitrarily large $\lambda < \mu$, where μ is the first measurable cardinal, and unboundedly many λ if there is no measurable, ϕ has a maximal (with respect to substructure, which in this case means $\prec_{\infty,\omega}$) model with cardinality between λ and 2^λ.*

We expect to remove the set theoretic hypotheses by use of a black box as in [She1x] but that work is in progress.

If $|M|$ is at least μ, then for any $\aleph_1$-complete non-principal ultrafilter $\mathcal{D}$ on μ, $M^\mu/\mathcal{D}$ is a proper extension of M. This holds because we can find an $f \in M^\mu$ which hits each element $a \in M$ at most once. Thus the equivalence class of f cannot be that of any constant map on M (since $\mathcal{D}$ is non-principal). On the other hand, by the Łos theorem for $L_{\omega_1,\omega}$, since $\mathcal{D}$ is $\aleph_1$-complete, the ultrapower is an $L_{\omega_1,\omega}$-elementary extension of M. Thus, we have shown the Hanf number for non-maximality is at most μ:

Theorem 3.0.4. *If μ is measurable for any $\phi \in L_{\mu,\mu}$, in particular in $L_{\omega_1,\omega}$, no model of cardinality $\geq \mu$ is maximal.*

The proof of the converse (Theorem 3.0.3) fills the remainder of this section.

If we only demand the result for an arbitrary sentence of $L_{\omega_1,\omega}$ there are easy examples. An example in terms of ω-models (which is easily reinterpreted into $L_{\omega_1,\omega}$) appears in [Mag16].

3.1 Some preliminaries on Boolean Algebras

Definition 3.1.1.
1. *For $X \subseteq B$ and B a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ be the subalgebra of B generated by X.*
2. *A set Y is independent from X over an ideal I in a Boolean algebra B if and only if for any Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and any $a \in \overline{X} - I$, $p(y_0, \ldots, y_k) \wedge a \notin I$.*

Observe the following:

Observation 3.1.2.
1. *If I is the 0 ideal, (read Y is independent from X), the condition becomes: for any $a \in \overline{X} - \{0\}$, $B \models p(y_0, \ldots, y_k) \wedge a > 0$.*
2. *It is easy to check that 'Y is independent from X over I' implies the image of Y is free from the image of X in B/I.*
3. *If X is empty, the condition 'Y is independent over I' implies the image of Y is an independent subset of B/I.*
4. *Since Y is independent from X over an ideal I in a Boolean algebra B is expressed by quantifier free formulas, the condition is maintained in any B' extending B.*

For the last item, observe that condition 1) is a quantifier free statement about Y, X and I. In the construction of complete sentence I will be named by a predicate.

The contrast between the notion of independence above and the following is crucial for the construction here.

Definition 3.1.3. *Let X, Y be sets of elements from a Boolean algebra of sets. X is independent (free) over Y if for any infinite A that is a non-trivial finite Boolean combination of elements of X and any B which is a non-empty finite Boolean combination of elements of Y, $A \cap B$ and $A^c \cap B$ are infinite.*

Both kinds of independence will occur in the models in Section 3.2. In $\boldsymbol{K}_1$, there is a homomorphism from P_2^M into $\mathcal{P}(P_1^M)$ that does *not* translate from 'independence in the boolean algebra sense' to 'set independence'. In $\boldsymbol{K}_2$, there is an isomorphism from P_2^M into $\mathcal{P}(P_1^M)$ that correctly translates 'independence'.

We can amalgamate Boolean algebras B and A over C by the pushout/free product construction Notation 3.1.4.

Notation 3.1.4. *Let $C \subseteq A, B$ be Boolean algebras. The disjoint amalgamation $D = A \otimes_C B$ is the Boolean algebra obtained as the pushout [AB11] of A and B over C. It is characterized internally by the following condition. For $a \in A - C, b \in B - C$: $a \leq b$ in D if and only if there is a $c \in C$ with $a < c < b$ (and symmetrically). D is generated as a Boolean algebra by $A \cup B$.*

The free amalgam $A \otimes_C B$, where each of A, B have only finitely many atoms must destroy the atomicity of some elements. (If a is atom of A and $b_1, \dots b_n$ are the atoms of B, for at least one i, $A \otimes_C B \models 0 < a \wedge b_i < a$.) Thus we will have to construct a quotient algebra of the free amalgam below in order to find an amalgam which does not introduce atoms.

Notation 3.1.5. *For any Boolean algebra B, $\mathrm{At(B)}$ denotes the set of atoms of B.*

Theorem 3.1.6. *Let $A_0 \subseteq A_1, A_2$ be Boolean algebras. There is a Boolean algebra amalgamating A_1 and A_2 such that $\mathrm{At(A_3)} = \mathrm{At(A_1)} \cup \mathrm{At(A_2)}$.*

Proof. Let $\check{A}_3$ be the pushout of A_1 and A_2 over C. For each $a \in \mathrm{At(A_2)} \cup \mathrm{At(A_1)}$ we define a homomorphism g_a^i from A_i into the two element Boolean algebra $\hat{A}$ with domain $\{0, 1\}$ by: if $a, b \in A_i$ then

$$g_a^i(b) = \left\{ \begin{array}{cc} 1, & \text{for } a \leq^{A_i} b \\ 0, & \text{for } a \wedge^{A_i} b = 0 \end{array} \right\}$$

This is defined for all $b \in A_i$ because a is an atom. Extend g_a^i arbitrarily to a homomorphism from A^3 to $\hat{A}$. Let I be the ideal of A_3 generated by:

$$\{b \wedge a : b \in A_1, a \in \mathrm{At(A_2)}, \mathrm{g_a^2(b) = 0}\} \cup \{\mathrm{b \wedge a : b \in A_2, a \in At(A_1), g_a^1(b) = 0}\}.$$

Let A' denote $A_1 \cup A_2$.

First note that if $a \in \mathrm{At(A_1)}$ then for every $b \in A_2 - A_0$, either $g_a^1(b) = 0$ or $g_a^1(b^-) = 0$ so one of $a \wedge b$ or $a \wedge b^-$ is in I and $\pi(a)$ is an atom in A_3. Thus, if $a \in \mathrm{At(A_1)} \cup \mathrm{At(A_2)}$, $a \notin I$. But then if $d \in A'$ as $a \geq a$ for some such a, $d \notin A$. That is, $A' \cap I = \emptyset$.

Claim 3.1.7. *Let A^3 be $\check{A}^3/I$ by a quotient map π. π is 1-1 on $A_1 \cup A_2$.*

Now we show that if d_1, d_2 are in $A_1 \setminus A_0$, $A_2 \setminus A_0$ respectively, then $d_1 \triangle d_2$ is in A'. For this, if $d_1 \wedge d_2 = r \in A_0$, then $d_1 \triangle d_2 = (d_1 \wedge r^-) \vee (d_2 \wedge r^-)$ is a join of two elements of A' that are not in I (by 2) so cannot be in I (since I is closed down). But if $d_1 \wedge d_2 \neq i \in A_0$, there exists $c \in A_0$ with $d_1 \wedge d_2 \leq c$ and c below both d_1 and d_2. Say $d_1 \wedge d_2 \in A_1 \setminus A_0$. Now we decompose as above but using two steps to see that $d_1 \triangle d_2 \notin I$.

Thus, there is no $d_1, d_2 \in A'$ with $d_1 \triangle d_2 \in I$ as the symmetric difference is in A' and $A' \cap I = \emptyset$. So π is 1-1 on A'. $\quad \square_{3.1.7}$

No element d of $\check{A}_3 - A$ is an atom since it is a term in elements of A' and is either identically 0 or above at least two elements of $\mathrm{At}(\mathrm{A}_1) \cup \mathrm{At}(\mathrm{A}_2)$; these elements are not identified by π. Thus d is not an atom in A_3. This completes the proof of Theorem 3.1.6.

Lemma 3.1.8. *Let $B_0 \subseteq B_1 \subseteq B_2$ be Boolean algebras. Suppose I_i for $i < 3$ are sequence of ideals in the respective B_i with $I_1 \cap B_0 = I_0$ and $I_2 \cap B_1 = I_1$. If, for $i = 0, 1$, $J_i \subset B_{i+1}$ is independent from B_i over I_i in B_{i+1}, then $J = J_0 \cup J_1$ is independent from B_0 over the ideal I_2.*

Proof. Let $\mathbf{b}$ be a finite sequence of distinct elements from J. Suppose $\sigma(\mathbf{y})$ is a non-zero term in the same number of variables as the length of $\mathbf{b}$. For any $d \in B_0 - I_2$, we must show $\sigma(\mathbf{b}) \wedge d \notin I_2$.

Writing σ in disjunctive normal form it suffices to show some disjunct τ (which is just a conjunction of literals y_i and y_i^-) satisfies $\tau(\mathbf{b}) \wedge d \notin I_2$. Decompose $\tau(\mathbf{b})$ as $\tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1)$ where $\mathbf{b}_i \in J_i$. Since J_0 is independent from B_0 over I_1, $\tau_0(\mathbf{b}_0) \wedge d \notin I_1$ and clearly it is some $d_1 \in B_1$. Similarly, since J_1 is independent from B_1 over I_2, $\tau_1(\mathbf{b}_1) \wedge d_1 \notin I_2$. So $\tau(\mathbf{b}) \wedge d_1 = \tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1) \wedge d \notin I_2$ as required.

3.2 Defining the Complete Sentence

In this subsection we construct a complete $L_{\omega_1,\omega}$-sentence that we show in the next subsections has maximal models in λ for arbitrarily large λ less than the first measurable cardinal.

On a first approximation, each model consists of a two-sorted structure that consists of a Boolean algebra on P_1 and a representation of it as a field of sets on P_0 by a predicate $R(x, y)$ picking out those x that 'belong' to y. The basic idea is that an extension of a model M by adding an element a to P_0 defines an ultrafilter[3] on the Boolean algebra with domain P_1; U is the set of b such that $R^{Ma}(a, b)$. If $|M|$ is less than the first measurable μ this ultrafilter cannot be $\aleph_1$-complete. However, we can construct M such that if it has a proper (∞, ω) elementary extension, the ultrafilter is $\aleph_1$-complete. For this, we add a new predicate P_2 and functions $F_n(z)$ so that P_2 indexes countable families of elements of P_1. We add a function G_1 from P_0 into P_1 so that P_1^M includes an atomic subalgebra P_4^M corresponding to the finite Boolean combinations of the singletons of P_0^M.

This section is devoted to the construction of a countable generic structure; the details of the construction will be essential for the main argument in the next section. But we can describe the *generic* model. The unary predicate P_1^M is the domain of a Boolean algebra with an ideal P_4^M consisting of the finite joins of atoms. P_1^M/P_4^M is an atomless Boolean algebra with a basis $\{F_n^M(c) : c \in P_2^M\}$. Further, there is a set P_0^M in 1-1 correspondence with the atoms of P_1^M and a relation R such that for $b \in P_1^M$, $R(M, b)$ is a subset of P_0^M and the map $b \mapsto R(M, b)$ is an isomorphism from P_1^M into the Boolean algebra of subsets of P_0^M. A concrete example of this situation is to take P_1^M as an elementary submodel of the natural Boolean algebra on $\mathcal{P}(\omega)$, P_4^M as the distributive sublattice of finite sets, P_0^M and R^M as coding subsets of ω.

We build this structure as a Fraïssé-style limit of finitely generated structures; in each of these structures P_0^M and P_4^M will be finite. The classes $\boldsymbol{K}_{-1}, \boldsymbol{K}_0$ are not closed under substructure.

Definition 3.2.1. *τ is a vocabulary with unary predicates P_0, P_1, P_2, P_4, binary R, E, unary functions $'$, G_1, H_1, n-ary functions g_n, constants 0,1 and unary functions F_n, for $n < \omega$.*

Definition 3.2.2. *$M \in \boldsymbol{K}_{-1}$ is the class of structures M satisfying.*

1. *P_0^M, P_1^M, P_2^M partition M.*

[3] That is, U is an ultrafilter of the Boolean algebra with universe P_1; it is closed under meet and extension. When R is extensional, the set of $R(P_0, b)$ for $b \in U$ is an ultrafilter on $\mathcal{P}(P_0)$. We use both notations.

2. $(P_1^M, 0, 1, \wedge, \vee, <, ^-)$ *is a Boolean algebra* ($^-$ *is complement).*

3. $R \subset P_0^M \times P_1^M$ *with* $R(M,b) = \{a : R^M(a,b)\}$ *and the set of* $\{R(M,b) : b \in P_1^M\}$ *is a Boolean algebra.* $f^M : P_1^M \mapsto \mathcal{P}(P_0^M)$ *by* $f^M(b) = R(M,b)$ *is a Boolean algebra homomorphism into* $\mathbb{P}(P_0^M)$.

   Note that f *is not*[4] *in* τ*; it is simply a convenient abbreviation for the relation between the Boolean algebra* P_1^M *and the set algebra on* P_0 *by the map* $b \mapsto R(M,b)$.

4. $P_{4,n}^M = \{b \in P_1^M : |\{c \in P_{4,1}^M : c \leq b\}| = n\}$ *and* P_4^M *is the union of the* $P_{4,n}^M$.

5. *If* $b_1 \neq b_2$ *are in* P_4^M *then* $R(M,b_1) \neq R(M,b_2)$.

6. G_1^M *is a bijection from* P_0^M *onto* $P_{4,1}^M$ *such that* $R(M, G_1^M(a)) = \{a\}$. H_1^M *is defined on* $P_{4,1}^M$ *and is the inverse of* G_1^M.

7. *If* $c \in P_2^M$*, the* $F_n(c)$ *for* $n < \omega$ *are pairwise distinct.*

8. *If* $a \in P_0^M$ *and* $c \in P_2^M$ *then for every large enough* n $a \notin R(M, F_n(c))$*. Equivalently* $\bigcap_n F_n^M(c) = \emptyset$.

Note that condition 4 is not preserved under substructure.

We now define the class $\boldsymbol{K}_0$ of finitely generated members of $\boldsymbol{K}_1$; it will generate via direct limits a class $\boldsymbol{K}_1 = \hat{\boldsymbol{K}}$ and from it we will derive the rich class $\boldsymbol{K}_2 = \mathbf{R}$.

The P_1^M of models in $\boldsymbol{K}_0$ are each a direct product of a finite boolean algebra B_{n_*} and a countable atomless boolean algebra F_∞. P_0^M is in 1-1 correspondence with the atoms of B_{n_*}, P_2^M indexes families of sequences $F_m^M(c)$ of elements if P_1^M; if $m \geq n_*$, the $F_m^M(c)$ are independent. The set $\{F_m(c) : m \geq n_*, c \in P_2^M\}$ are a basis for F_∞. The $F_m^M(c)$ for $m < n_*$ are Boolean combinations of elements in B_{n_*} and the $F_n^M(c)$ with index greater than n_*; as the models are extended n_* grows all the $F_n^M(c)$ eventually have atoms below them. (See Lemma 3.2.14.) Details follow.

Definition 3.2.3. M *is in the class of structures* $\boldsymbol{K}_0$ *if* $M \in \boldsymbol{K}_{-1}$ *and there is a* witness $\langle n_*, \boldsymbol{B}, b_* \rangle$ *such that:*

1. $b_* \in P_1^M$ *is a finite union of atoms. Further, for some* k, $\bigcup_{j \leq k} P_{4,j}^M = \{c : c \leq b_*\}$ *and for all* $n > k$, $P_{4,n}^M = \emptyset$.

2. $\boldsymbol{B} = \langle B_n : n \geq n_* \rangle$ *is an increasing sequence of finite Boolean subalgebras of* P_1^M.

3. $B_{n_*} = \{c \in P_1^M : c \leq b_*\} = P_4^M$

4. $\bigcup_{n \geq n_*} B_n = P_1^M$.

5. P_2^M *is finite and not empty. Further, for each* $c \in P_2^M$ *the* $F_n^M(c)$ *for* $n < \omega$ *are independent.*

6. *The set* $\{F_m(c) : m \geq n_*, c \in P_2^M\}$ *is free over* $B_{n_*} = P_4^M$ *and* $F_m(c) \wedge b_* = 0$ *for* $m \geq n_*$.

   In detail, let $\sigma(\ldots x_c \ldots)$ *be a Boolean algebra term in the variables* x_c *which is not identically* 0*. Then, for* $n \geq n_*$ *and* $c \in P_2^M$*:*

$$\sigma(\ldots F_n(c) \ldots) > 0$$

[4] The subsets of P_0^M are *not* elements of M.

and for any non-zero $d \in B_{n_*}$ with $d \wedge b_* = 0$, (i.e. $d \in B_n - P^4_M$),

$$\sigma(\ldots F_n(c)\ldots) \wedge d > 0.$$

(Here $0 = 0^{P_1^M}$.)

7. *For every $n \geq n_*$, B_n, is generated by $B_{n_*} \cup \{F_m(c) : n > m \geq n_*, c \in P_2^M\}$. Thus P_1^M and so M is generated by $B_{n_*} \cup P_2^M$.*

8. *If $n < n_*$ and $c \in P_2^M$, $F_n^M(c) \in B_{n_*}$*

Remark 3.2.4. *Condition 3 of Definition 3.2.2 implies for any $M \in \boldsymbol{K}_{-1}$, any $a \in P_0^M$, $b \in P_1^M$, $M \models R(a,b) \vee R(a,b^-)$.*

Condition 6 implies that if $a \in P_0^M$ and $c \in P_2^M$ then for every large enough n $a \notin R(M, F_n(c))$. That is, condition 8 of Definition 3.2.2 is met in a very strong way. Equivalently $\bigcap_n F_n^M(c) = \emptyset$.

Note that if $\langle n_*, \boldsymbol{B}, b_*\rangle$ witnesses $M \in \boldsymbol{K}_0$ then for any $m \geq n_*$, so does $\langle m, \boldsymbol{B}, b_*\rangle$.

Lemma 3.2.5. *Each structure in $\boldsymbol{K}_0$ is finitely generated.*

Proof. Let $M \in \boldsymbol{K}_0$, witnessed by $\langle n_*, \boldsymbol{B}, b_*\rangle$. Then M is generated by $P_0^M \cup B_{n_*} \cup P_2^M$.

Lemma 3.2.6. *$\boldsymbol{K}_0$ is countable.*

Proof. Let M be in $\boldsymbol{K}_0$, witnessed by $\langle n_*, \boldsymbol{B}, b_*\rangle$. The isomorphism type of M is determined by the structure on P_4^M induced by the $F_n(c_i)$ and $c_i \in P_2^M$. If $m \geq n_*$, $F_m(c_i) \wedge b_* = 0$ so there is no trace on P_4^M. Since the tail, is just an atomless boolean algebra in the sense of P_1^M, it is $\aleph_0$ categorical. But there can be only countably many structures induced on the finite P_4^M by the countable set $F_n(c_i)$ through the formulas $x < F_n(c_i)$ which determine the values of R on P_4^M since only the $F_m(c_i)$ for $m < n_*$ are non-empty and P_2^M is finite. $\Box_{3.2.6}$

The following lemma shows the prototypical models in $\boldsymbol{K}_0$ in fact exhaust the class.

Lemma 3.2.7. *For any $M \in \boldsymbol{K}_0$, P_1^M has a natural decomposition as a product of an atomic and an atomless Boolean algebra*[5].

Proof. Let $M \in \boldsymbol{K}_0$, witnessed by $\langle n_*, \boldsymbol{B}, b_*\rangle$. Then the atomic part is the collection of elements of P_1^M that are $\leq b_*$. And the independent generation by the $F_n^M(c_i)$ for $n \geq n_*$ and $c_i \in P_2^M$ shows the quotient is atomless. $\Box_{3.2.7}$

Definition 3.2.8. *The class $\boldsymbol{K}_1 = \hat{\boldsymbol{K}}$ is the collection of all direct limits of models in $\boldsymbol{K}_0$.*

Lemma 3.2.9. *There is a minimal model M_0 of $\boldsymbol{K}_0$, and so of $\boldsymbol{K}_1$, that can be embedded in any model.*

Proof. Let $P_0^{M_0}$ be empty; $P_2^{M_0} = \{c\}$; let the $F_n^{M_0}(c)$ be independent generators of $P_1^{M_0} - P_4^M$; P_4^M is empty. $\Box_{3.2.9}$

Thus, in the minimal model $P_1^{M_0}$ is the direct product of the 2-element Boolean algebra with an atomless Boolean algebra. More generally, if $M \in \boldsymbol{K}_0$, P_1^M is the direct product of the finite ideal P_4^M with an atomless Boolean algebra.

[5]The atomic component need not have a maximal element (in $\mathbf{R}$).

Lemma 3.2.10. *If $M_0 \subseteq M_1$ are both in $\boldsymbol{K}_0$, witnessed by $\langle n^i_*, \boldsymbol{B}^i, b^i_*\rangle$, then for sufficiently large n, $\boldsymbol{B}^0_n = \boldsymbol{B}^1_n \cap P_1^{M_1}$.*

Proof. Since the B^1_n exhaust $P_1^{M_1}$ and $B^0_{n_*}$ is finite, for all sufficiently large n, and since $F_r^{M_1}(c) = F_r^{M_0}(c)$, B^1_n contains the $F_r^{M_0}(c)$ and thus B^0_n. But if some $b \in B^0_n$ but not in $B^1_c \cap P_1^{M_0}$, then for some k, $b \in B^0_{k+1} - B^0_k$. But then B^0_{k+1} is not generated by $B^0_{n^*}$ along with the $F_r^{M_0}(c)$ for $r < k$. $\square_{3.2.10}$

Lemma 3.2.11. $(\boldsymbol{K}_0, \subseteq)$ *has the disjoint amalgamation property.*

Proof: Suppose M^0 is extended by M^1 and M^2. Let $\mathcal{B}^i$ be the Boolean algebra with domain $P_1^{M_i}$. Lemma 3.1.6 gives an amalgamation $\mathcal{B}^3$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ over $\mathcal{B}_0$ with the same atoms, where $\mathcal{B}^3 = \check{\mathcal{B}}^3/I$. ($\check{\mathcal{B}}^3$ is thus usual free amalgam and I is the ideal in Lemma 3.1.6.) We define a structure $M_3 \in \boldsymbol{K}_0$ with $\mathcal{B}^3$ as the Boolean algebra $P_1^{M_3}$. We take $P_0^{M_1} \cup P_0^{M_2}$ as $P_0^{M_3}$. Let $P_2^{M_3} = P_1^{M_1} \cup P_2^{N_2}$ and interpret the F_n as they are in the substructures M_1 and M_2. The interpretation of G_1 is immediate. For $a \in P_1^{M_1} - P_1^{M_0}$, $b \in P_1^{M_2} - P_1^{M_0}$, let $R^{M_3}(a, b)$ if there is a $c \in M_0$ with $a < c < b$ (in M_3). Extend R to all of $P_1^{M_3}$ to make it a homomorphism. This gives us an a model in $\boldsymbol{K}_{-1}$.

Let $M_0 \subseteq M_1, M_2 \in \boldsymbol{K}_0$. By taking n_* as the maximum of n^i_* for $i < 3$, we can assume this is witnessed by tuples $\langle n_*, \boldsymbol{B}^i, b^i\rangle$. We must find an $\langle n_*, B^3_{n_*}, b^3\rangle$ witnessing $M_3 \in \boldsymbol{K}_0$, i.e. is finitely generated. Rechoosing n_* by Lemma 3.2.10 we can assume for all $n \geq n_*$, $B^1_n \cap P_1^{M_0} = B^0_n = B^2_n \cap P_1^{M_0}$.

Set B^3_n as $\langle B^1_n \cup B^2_n\rangle^{M_3}$ for each n; use $\langle n_*, \boldsymbol{B}^3, b^1 \vee b^2\rangle$ as the witness. We must verify that $\{F_n^{M_3}(c) : n \geq n_*, c \in P_2^{\check{M}_3}\}$ is independent from $B^3_{n_*}$ over $P_4^{M_3}$. We first work in $\check{M}^3$. Note that for $i = 1, 2$, $\{F_n^{M_i}(c) : n \geq n_*, c \in P_2^{M_i}\}$ is independent from B^i_n over $P_4^{M_i}$ in $P_1^{M_i}$. By free amalgamation, we have $Y = \{F_n^{\check{M}_3}(c) : n \geq n_*, c \in P_2^{\check{M}_3}\}$ is independent from $\check{B}^3_{n_*} = \langle B^1_{n_*} \cup B^2_{n_*}\rangle^{\check{M}_3}$ over $P_4^{\check{M}_3}$ in $P_1^{\check{M}_3}$. We want to remove the checks.

So, we have in $\check{M}^3$ for any Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and any $a \in (\check{B}^3_n - P_4^{\check{M}_3})$, and any $y_0, \ldots, y_k \in Y$, $p(y_0, \ldots, y_k) \wedge a \notin P_4^{\check{M}_3}$. Since I is contained in the ideal generated by $P_4^{\check{M}_3}$ in $P_1^{\check{M}_3}$, $\pi(\sigma(y_0, \ldots, y_k) \wedge a \notin P_4^{M_3}$ and so we finish. $\square_{3.2.11}$

Since $\boldsymbol{K}_0$ has joint embedding, amalgamation and only countably many models, we have:

Corollary 3.2.12. *There is a countable generic model M for $\boldsymbol{K}_0$. We denote its Scott sentence by ϕ_M.*

Definition 3.2.13. *We say a model M in $\boldsymbol{K}_1$ is rich if for any $N_1, N_2 \in \boldsymbol{K}_0$ with $N_1 \subseteq N_2$ and $N_1 \subseteq M$, there is an embedding of N_2 into M over N_1. We denote the class of rich models in $\boldsymbol{K}_1$ by $\boldsymbol{K}_2 = \mathbf{R}$.*

Here are some of the properties that distinguish the theory of $\boldsymbol{K}_2$ from that of $\boldsymbol{K}_1$.

Lemma 3.2.14. *If M is the generic model then*

1. *if $b_1 \neq b_2 \in P_1^M - P_4^M$ then $R(b_1, M) \neq R(b_2, M)$, i.e. the map f from Definition 3.2.2.1.c is injective.*
2. *The sets $R(M, F_n^M(c)$ with $F_n^M(c) \notin P_4^M$ and $c \in P_2(M)$ are independent in the algebra of sets on P_0^M.*
3. *If $b \in P_1^M - P_4^M$, $R^M(b, M)$ is infinite and b is not an atom. So P_1^M/P_4^M is an atomless boolean algebra, hence free.*

Proof. Fix a finitely generated model M_0 containing b_1, b_2; there is a finitely generated extension M_1 in $\boldsymbol{K}_0$ by adding $a \in P_1^{M_1} - P_1^{M_0}$ with $R^{M_1}(a, b_1) \wedge \neg R^{M_1}(a., b_2)$. The other statements hold by similar arguments. $\square_{3.2.14}$

By the usual argument for rich models (given amalgamation with free over finite intersection), it is easy to construct a back-and-forth-showing:

Lemma 3.2.15. *The class $\boldsymbol{K}_2$ is the collection of models of the Scott sentence of the rich model constructed in Lemma 3.2.12.*

Proof. If $N \in \boldsymbol{K}_2$ or $N \models \phi_M$, N is back and forth equivalent to M, $\square_{3.2.15}$

Lemma 3.2.16. *There is an expansion τ^* of τ and a complete first order theory quantifier eliminable T such that $\mathbf{R}$ is the collection of τ-reducts of the atomic models of T.*

Proof. Add a predicate for each quantifier free type of a finite sequence of variables realized in the generic model M. Let T be the theory of the resulting structures.

This completes our description of the class $\boldsymbol{K}_2$ of rich models and its Scott sentence. We now introduce a notion of free-extension which will be crucial for the main construction.

Definition 3.2.17. *We say M_2 is free over M_1 and write $M_1 \subseteq_{fr} M_2$ if*

(a) There is an $\boldsymbol{I}$ with $\boldsymbol{I} \subset (P_1^{M_2} - P_1^{M_2}) \cup P_4^{M_2}$ such that i) $\boldsymbol{I} \cup P_1^{M_1} \cup P_4^{M_2}$ generates $P_1^{M_2}$ and ii) $\boldsymbol{I}$ is independent from $P_1^{M_1}$ over $P_4^{M_2}$ in $P_1^{M_2}$.

(b) There is a function H from $P_2^{M_2} \setminus P_2^{M_1}$ to $\mathbb{N}$ such that the $F_n(c)$ for $n \geq H(c)$ are distinct and

$$\{F_n^M(c) : c \in P_2^{M_2} \setminus P_2^{M_1} \text{ and } n \geq H(c)\} \subset \boldsymbol{I}.$$

1. *M is free if it is free over the empty model i.e., P_1^M has a free basis over P_4^M.*

2. *M is weakly free over N if M contains a free extension M' of N.*

Remark 3.2.18. *Note that a free model in $\boldsymbol{K}_1$, like those in $\boldsymbol{K}_0$, has a free ideal picked out by P_4^M, and P_1^M / P_4^M is an atomless boolean algebra.*

We need only speak of the generation of P_1^M and P_2^M since for in $M \in \boldsymbol{K}_1$ elements of P_0^M are generated from P_1 by G_1. Here are some basic facts about free extensions.

Lemma 3.2.19. *1. If $M_1 \subseteq_{fr} M_2$ by $\boldsymbol{I}_1$ and $M_2 \subseteq_{fr} M_3$ by $\boldsymbol{I}_2$ then $M_1 \subseteq_{fr} M_3$ by $\boldsymbol{I}_1 \cup \boldsymbol{I}_2$. Thus, $\subseteq_{fr}$ is a partial order.*

2. More generally if M_α with $\alpha < \delta$ is continuous $\subseteq_{fr}$ increasing then $M = \bigcup M_\alpha$ satisfies $M_\alpha \subseteq_{fr} M$ witnessed by $\bigcup_{\alpha < \beta < \delta} I_\beta$.

Proof. For 1) the requirements on the I_i follow directly Lemma 3.1.8 (taking the ideals as $P_4^{M_2}$ and $P_4^{M_3}$) and the given generating conditions. And clearly the $F_n(c)$ are contained in $I_1 \cup I_2$ and continue to be independent. Part 2 follows by induction. $\square_{3.2.19}$

Note that each model in $\boldsymbol{K}_0$ is free over the emptyset.

In order to construct a sequence as appears in Lemma 3.2.19, we need to construct free extensions.

Lemma 3.2.20. *Suppose $M_1 \in \widehat{\boldsymbol{K}} = \boldsymbol{K}_1$ is free and $N_1 \subset M_1$. Let $N_1 \subset N_2$ with both in $\boldsymbol{K}_0$.*

Then there are an M_2 and an f such that:

1. *$M_2 \in \boldsymbol{K}_1$, $M_1 \subseteq_{fr} M_2$ and so M_2 is free. Similarly $N_2 \subseteq_{fr} M_2$.*

2. *f maps N_2 into M_2 over N_1. Moreover, the image of N_2 is free in M_2.*

Proof. We construct an amalgam M_2 of M_1 and N_2 over N_1. Fix n_1 such that for $n \geq n_1$, $\{F_n^{M_1}(c) : n(c) \geq n_1, c \in P_2^{M_1}\} \cap \{F_n^{N_2}(c) : n(c) \geq n_1, c \in P_2^{N_2}\}$ and $F_n^{N_2}(c)$ for $c \in P_2^{M_1}$ and $c \in P_2^{N_2}$, respectively are in disjoint boolean algebras, $P_1^{M_1} - P_1^{N_1}$ and $P_1^{N_2} - P_1^{N_1}$ and $P_1^{M_1}, P_1^{N_2}$ are generated by I_1, I_2 free from N_1 over $P_4^{N_1}$. respectively. Let $P_1^{M_2}$ be amalgam of $P_1^{M_1}$ and $P_1^{N_2}$ over $P_2^{N_1}$ with no new atoms as constructed in Theorem 3.1.6. So $P_4^{M_2} = P_4^{M_1} \cup P_4^{N_2}$. The argument for Lemma 3.2.11 shows the image of $I_1 \cup I_2$ is independent from $P_1^{N_1}$ over $P_4^{M_2}$ and $I_1 \cup I_2$ generates $P_1^{M_2}$. It follows that M_1 and N_2 are freely embedded in M_2. $\square_{3.2.20}$

Now we extend the range of the amalgamation from from $\boldsymbol{K}_0$ to free models in $\boldsymbol{K}_1$. The argument is essentially the same

Corollary 3.2.21. *Let M_1 be free in $\boldsymbol{K}_1$.*

1. *There exists an M_2 which is a free extension of M_1.*

2. *We can choose $M_2 \in \boldsymbol{K}_2$.*

Proof. For 1) embed the minimal model M_0 (Lemma 3.2.9) into M_1; let N_1 be an extension of M_0 by adding one more d to $P_2^{M_0}$ to form $P_2^{N_1}$ and setting $R(N_1, F_n(d)) = \emptyset$. Amalgamate M_1 and N_1 over M_0 by Lemma 3.2.20. Note the result M_2 is a free extension of M_1.

For part 2) iterate Lemma 3.2.20 as in Corollary 2.1.6 to obtain a rich model; note that freeness is preserved in each stage. $\square_{3.2.21}$

The crucial distinction from Corollary 2.1.6 is that here we extend only 'free models' in $\boldsymbol{K}_1$ to $\mathbf{R}$. $\square_{3.2.21}$

Proceeding inductively we get:

Corollary 3.2.22. *For every μ there is a free $M \in \boldsymbol{K}_1$, $(\boldsymbol{K}_2)$ of cardinality μ.*

In the next section, we show under appropriate set theoretic hypotheses on λ, that we can build a chain of free extensions such that the union is not free and in fact is maximal. For this, we will need one further refinement.

We now easily get a free extension of a free model in $\boldsymbol{K}_1$ by a model in $\boldsymbol{K}_1$. We needed the amalgamation to prove the generic exists and for 2) of each corollary– getting structures in $\boldsymbol{K}_2$. Note that step 1 does not extend P_0, (but M_1 is assumed free).

Corollary 3.2.23. *Let M_1 be free in $\boldsymbol{K}_1$. Suppose $A \subset P^{M_1}$.*

1. *There exists an M_2 which is a free extension of M_1 and a $b_0 \in P_1^{M_2}$ such that b is free from $P_1^{M_1}$ over $P_4^{M_2}$ and $R(M_2, b_0) = A$.*

2. *We can choose $M_2 \in \boldsymbol{K}_2$.*

Proof. If some $b \in P_1^{M_1}$ satisfies $R(M_1, b) = A$, there is nothing to do. If not, extend $P_1^{M_1}$ by taking the direct product with a new copy of the free algebra on countably many generators. Add one new c to $P_2^{M_1}$ to get $P_2^{M_2}$. Let $b_n = F_n(c)$ list the generators of the extension. Let $R(M_2, b_0) = A$ and $R(M_2, b_i) = \emptyset$ for $i > 0$. Extend R to a Boolean homomorphism (which is trivial off $\langle P_1^{M_1} \cup b_0\rangle$ by the obvious interpretation on finite boolean combinations. Since M_2 is free, we can extend it to a model in $\boldsymbol{K}_2$ as in Corollary 3.2.21. $\square_{3.2.23}$

3.3 Constructing maximal models in an extension of ZFC

We show that in each cardinal below a measurable cardinal, assuming a mild set theoretic hypothesis described below, $\mathbf{R}$ has maximal models. We begin by defining a pair of set theoretic notions and some specific notions of maximal model.

Definition 3.3.1 ($\diamond_S$)**.** *Given a cardinal κ and a stationary set $S \subseteq \kappa$, $\diamond_S$ is the statement that there is a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that*

1. *each $A_\alpha \subseteq \alpha$*
2. *for every $A \subseteq \kappa, \{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in κ*

Definition 3.3.2 (S reflects)**.** *Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. If $\alpha < \kappa$ has uncountable cofinality, S reflects at α if $S \cap \alpha$ is stationary in α . S reflects if it reflects at some $\alpha < \kappa$.*

Definition 3.3.3.

1. *A model $M \in \boldsymbol{K}_2 = \mathbf{R}$ is* P_0-maximal *(for $\boldsymbol{K}_1$) if $M \subseteq N$ and $N \in \boldsymbol{K}_2 = \boldsymbol{K}_1$ implies $P_0^M = P_0^N$.*
2. *A model $M \in \boldsymbol{K}_2 = \mathbf{R}$ is* maximal *(for $\mathbf{R}$) if $M \subseteq N$ and $N \in \boldsymbol{K}_2 = \mathbf{R}$ implies $M = N$.*

Let $S^\lambda_{\aleph_0}$ denote the stationary set $\{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0, \delta \text{ is divisible by } |\delta|\}$.

Theorem 3.3.4. *Fix $\boldsymbol{K}_0, \boldsymbol{K}_1 = \hat{K}$, and $\boldsymbol{K}_2 = \mathbf{R}$ as in Definitions 3.2.3, 3.2.8 and 3.2.13. There is a P_0-maximal (for $\boldsymbol{K}_1$) model $M \in \mathbf{R}$ of card λ if there is no measurable cardinal ρ with $\rho \leq \lambda$, $\lambda = \lambda^{<\lambda}$, and there is an $S \subseteq S^\lambda_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds.*

Under $V = L$, the hypotheses are clearly consistent and imply there are arbitrarily large maximal models of $\mathbf{R}$ in L. When a measurable cardinal exists, the consistency of the conditions can be established by forcing; see the article by Cummings in the Handbook of Set Theory [Cum08] or by considering the inner model of a measurable $L[D]$ where is D is a normal ultrafilter on μ.

The argument for Theorem 3.3.4 will have three parts. First, we describe the requirements a construction of a model; then we carry out the construction. Finally, we show the model is constructed is P_0-maximal if its cardinality is below the first measurable.

Construction 3.3.5 (Requirements)**.** Let $\langle U_\alpha : \alpha < \lambda\rangle$ list $[\lambda]^{<\lambda}$ so that each subset is enumerated λ times and $U_\alpha \subseteq \alpha$. Without loss, each $\alpha \in S$ is a limit ordinal and is divided by $|\alpha|$. Let $\overline{A}^* = \langle A^*_\delta : \delta \in S\rangle$ be a $\diamond_S$-sequence.

We will choose M_α for $\alpha < \lambda$ by induction to satisfy the following conditions. (Since the universe of M is a subset of λ, its elements are ordinals so we may talk about their order although the order relation is not in τ.)

1. $M_\alpha \in \boldsymbol{K}_1$ has universe an ordinal between α and λ and M_0 is empty. Each $M_\alpha \in \boldsymbol{K}_1$
2. $\langle M_\beta : \beta < \alpha\rangle$ is $\subseteq$- continuous.
3. If $\beta \in \alpha - S$ then M_α is free over M_β *and* $M_\alpha \in \boldsymbol{K}_2 = \mathbf{R}$.

4. If $\alpha = \beta + 2$ and $U_\beta \subseteq P_0^{M_\beta}$ then there is a $b_\beta \in P_1^{M_\alpha}$ such that $R(M_\alpha, b_\beta) \cap M_{\beta+1} = U_\beta$ and in the Boolean algebra $P_1^{M_\alpha}$, $\{b_\beta\}$ is free from $P_1^{M_{\beta+1}}$ over $P_4^{M_\alpha}$.

5. If $\delta \in S$ and $\alpha = \delta + 1$ then a) implies b), where:

   (a) there is an increasing sequence $\overline{\gamma} = \langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$, where the $\gamma_{\delta,n}$ are increasing with n and not in S satisfying:

      i. $\gamma_{\delta,n} < \gamma_{\delta,n+1} < \delta$ with $\sup_n \gamma_{\delta,n} = \delta$;

      ii. $b_{\delta,n} \in P_1^{M_{\gamma_{\delta,n+1}}} \cap A^*_\delta$ and so $b_{\delta,n} \in P_1^{M_\delta}$;

      iii. $\{b_{\delta,n} : n < \omega\}$ is independent over $P_1^{M_{\gamma_n}} \cup P_4^{M_\delta}$;

      iv. if $a \in P_0^{M_\delta}$ then for all but finitely many n, $\neg R(a, b_{\delta,n})$.

   (b) For some $\overline{\gamma} = \langle \gamma_{\delta,n}, b^\delta_n : n < \omega \rangle$, there is a $c_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(c_\delta) = b_{\delta,n}$.

Note that each M_i for $i < \lambda$ is free; but, as we will see, M_λ may not be. We now carry out the inductive construction.

Note that if we know that $\delta \in S$, we can guarantee the M_δ is free by choosing a sequences of successor ordinals γ_n with limit δ. By induction each M_{γ_n} is free so by Lemma 3.2.19.3 so is M_δ.

Construction 3.3.6. *Details*

Case 1: $\alpha = 0$. Let M_0 be the prime model from Lemma 3.2.9. The generic can be taken as M_1.
Case 2: $\alpha = \beta + 1$ and $\beta \notin S$. If β is a limit we only have to choose, by Lemma 3.2.21, M_α to be a free extension of M_β in $\mathbf{R}$. If β is a successor, there is an additional difficulty if $U_\beta \in P_0^{M_\beta}$; we must choose b_β to satisfy condition 4) and with $M_{\alpha+2} \in \boldsymbol{K}_2$. For this, apply Corollary 3.2.23.2.
Case 3: $\alpha = \delta$, a limit ordinal that is not in S. Set $M_\delta = \bigcup_{\gamma<\delta} M_\gamma$. We must prove that if $\beta \in \delta \setminus S$ then M_δ is free over M_β. Since S does not reflect there exists an increasing continuous sequence $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ of ordinals less than δ, which are not in S and with $\alpha_0 = \beta$. By the induction hypothesis, for each $i < j < \mathrm{cf}(\delta)$, M_{α_j} is free over M_{α_i}. And clearly then M_δ is free over M_β as required.
Case 4a: $\alpha = \delta + 1$, $\delta \in S$, and clause 5a fails. This is just as in case 2.
Case 4b: $\alpha = \delta + 1$, $\delta \in S$, but clause 5a holds.

So, suppose $\langle M_\beta, b_\beta \rangle$ for $\beta < \delta$ have been defined. If there exists $\overline{\gamma}$ as in condition 5a) we must construct $M_{\delta+1}$ and c_δ to satisfy condition 5b). Such a choice is immediate from the following Claim 3.3.7. To see hypothesis A) of Claim 3.3.7 holds, recall that δ is divisible by $|\delta|$ so we can choose the γ_n so that $\gamma_{n+1} \geq \gamma_n + \omega$ and so $P_2^{M_{n+1}} - P_2^{M_n}$ is infinite. Hypotheses 2) and 3) of Claim 3.3.7 are conditions iii) and iv) of 5a) from Construction 3.3.5.

Claim 3.3.7. *Suppose that for $n < \omega$, $M_n \subset_{\mathrm{fr}} M_{n+1}$ are in $\widehat{\boldsymbol{K}}$. If Condition A) holds then so does condition B).*

A

1. $P_2^{M_{n+1}} - P_2^{M_n}$ *is infinite*
   or at least there are no finite $Y \subset P_1^{M_{n+1}}$ and $X \subset P_2^{M_{n+1}}$ so that $P_1^{M_{n+1}}$ is generated as a Boolean algebra by $P_1^{M_n} \cup Y \cup \{F_m^{M_{n+1}}(c) : c \in X, m \in \omega\}$
2. *there is a $b_n \in P_1^{M_{n+1}}$ so that $\{b_n\}$ is free over $P_1^{M_n}$.*
3. *if $a \in P_1^{M_i}$, then for all but finitely many $n \geq i$, $a \notin R(M_{n+1}, b_n)$.*

B) then there is a pair (M, c)

1. *$M = \bigcup M_n \cup \{c\}$, $c \in P_2^M$, c is not in any M_n,*
2. *$M_n \subset_{\rm fr} M$ for each n,*
3. *$F_n^M(c) = b_n$.*

Proof. For simplicity write N_n for the M_{γ_n} in Construction 3.3.5.5a). The difficulty is that while we know each N_{n+1} is free over N_n, witnessed by some I_n, we don't know $b_n^\delta \in I_n$. Clearly, it suffices to find I'_n which also witnesses $N_n \subset_{\rm fr} N_{n+1}$ and $b_n \in I'_n$ as the description of M in b) determines the rest. To find I'_n, we first find (X, J_n) such that:

1. $X \subseteq P_1^{N_n}$ is finite.
2. $J_n \subset I_n$ is countable.
3. If $c \in P_2^{N_{n+1}} - P_2^{N_n}$ then for sufficiently large m, $F_m^{N_{n+1}}(c) \notin J_n$.
4. $b_n \in BA(X \cup \bigcup J_n)$, the Boolean algebra generated by $X \cup \bigcup J_n$.

Now we construct such an (X, J_n).

First step: Note that b_n is in a subalgebra generated by a finite subset X of $P_1^{N_n}$ and a finite subset J'_n of I_n. Now, by hypothesis A.1) of Claim 3.3.7, we find countably infinite J''_n contained in $I_n - J'_n$ such that for each $c \in P_2^{N_{n+1}} - P_2^{N_n}$ all but finitely many of the $F^{N_{n+1}}(c)$ are not in J''_n. Set $J_n = J'_n \cup J''_n$.

Second step: Let $\langle b_k^n : k < \omega\rangle$ list J_n without repetitions. Now choose $d_i = d_i^n$ for $i < \omega$ by induction, with $d_0 = b_n$, so that

1. $d_\ell \in BA(X \cup J_n)$
2. $\langle d_j : j \leq \ell\rangle$ is independent from X over $P_1^{M_n}$
3. If $\ell = k + 1$, $b_k^n \in BA(X \cup \langle d_j : j \leq \ell\rangle)$.

This suffices, as if we succeed, I'_n can be taken as $(I_n - J_n) \cup \langle d_\ell : \ell < \omega\rangle$ and we choose a new c and define M as required by setting $F_n^M(c) = b_n$ for each c. But how do we choose the d_i? Let $d_0 = b_n$. Clearly this choice satisfies the first three conditions and the fourth is vacuous for 0.

Now suppose $\ell = k + 1$ and $d_0, \ldots d_k$ have been chosen. Suppose $\langle e_{\ell,j} : j < j_\ell\rangle$ lists the atoms of $BA(X \cup \langle d_j : j \leq k\rangle)$. Let $i(k)$ be minimal such that $b_{i(k)}^n \notin BA(X \cup \{d_0, \ldots d_k\})$, equivalently $b_{i(k)}^n \notin BA(P_1^{N_n}) \cup P_4^{N_{n+1}} \cup \{d_0, \ldots d_k\}$.

Let $u_\ell^0 = \{j < j_\ell : e_{\ell,j} \wedge b_k^n \in \{0, e_{\ell,j}\}$ and $u_\ell^1 = \{j < j_\ell : j \notin u_\ell^0\}$. Then choose, $m(\ell)$ such that $d_0, \ldots d_k \in BA(X \cup \{b_0^n \ldots b_{m(\ell)-1}^n\}$ and $m(\ell) > k$. Now the required d_ℓ is

$$\bigvee\{b_{m(\ell)}^n \wedge e_{\ell,j} : j \in u_\ell^0\} \vee \bigvee\{b_{i(k)}^n \wedge e_{\ell,j} : j \in u_\ell^1\}.$$

$\square_{3.3.7}$

This completes the construction. We use the following observation in proving Claim 3.3.12.

Fact 3.3.8. *There is a closed unbounded set C_1 such that if $\delta \in C_1$, for every sequence $\overline{\gamma} \in M_\delta^\omega$ satisfying condition 5a), there is a $c_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(c_\delta) = b_{\delta,n}$.*

Claim 3.3.9. *The structure $M = \bigcup_{i<\lambda} M_i \in \mathbf{R}$.*

Proof. Since we required the extension to be in $\boldsymbol{K}_2 = \mathbf{R}$ in requirement 3 of Construction 3.3.5, for cofinally many i, $M_i \in \mathbf{R}$. By Lemma 3.2.14, all are ∞, ω equivalent. Hence $M \in \mathbf{R}$. $\Box_{3.3.9}$

Construction 3.3.10. *Verification that the construction suffices*

Now we now show that M is P_0-maximal for $\boldsymbol{K}_1 = \hat{K}$. Suppose for contradiction there exists N in $\boldsymbol{K}_1$ extending M such that $P_0^N \supsetneq P_0^M$. Choose $a^* \in P_0^N - P_0^M$. Let

$$A = \{b \in P_1^M : R^N(a^*, b)\}.$$

Then, by Remark 3.2.4, for every $a \in P_0^N$, in particular a^* and every $b \in P_1^N$ (and so every $b \in P_1^M$) either $R^N(a, b)$ or $R^N(a^*, b^-)$. Thus, A is an ultrafilter on P_1^M. If A is non-principal, it is generated by some atom $b_0 \in P_1^M$. Then b_0 must be in $P_{4,1}$ and so $\neg R^N(a^*, b_0)$, contrary to hypothesis. We do not know whether this ultrafilter is $\aleph_1$-complete; we will show it induces an $\aleph_1$-complete ultrafilter on $\mathcal{P}(P_0^{M_{\alpha^*}})$ for some $\alpha^* < \lambda$.

We now deduce a contradiction from the properties of A. Recall that the A^*_δ are a diamond sequence fixed in requirement 3.3.5. Note

$$S_A = \{\delta \in S : M_\delta \text{ has universe } \delta \;\&\; A^*_\delta = A \cap \delta\}$$

is a stationary subset of λ.

Recall that in the construction, we chose b_α for $\alpha < \lambda$ which satisfied requirement 4 of Construction 3.3.5.

Notation 3.3.11. *Note $C = \{\delta < \lambda : \delta \text{ limit } \;\&\; \alpha < \lambda \rightarrow b_\alpha < \delta\}$ is a club of λ.*

There are two cases. We will show the first is impossible and the second implies λ is measurable contrary to hypothesis.

Case i): For every $\alpha < \lambda$ there is a $b_\alpha \in P_1^M \cap A$ such that $R(M, b_\alpha)$ is disjoint from α and $\{b_\alpha\}$ is independent over $P_1^{M_\alpha} \cup P_4^M$.

Choose $\delta^* \in S_A \cap C \cap C_1$. Since δ^* has cofinality ω we can choose a sequence γ_n^δ such that each is successor (so not in S), and as we are in case i) with $b_{\gamma_n^{\delta^*}} < \gamma_{n+1}^{\delta^*}$. But this is forbidden by the following Claim 3.3.12. This contradiction completes the proof of case i).

Claim 3.3.12. *There can be no sequence $\langle \gamma_n^{\delta^*} : n < \omega \rangle$ with limit δ^* so that $\gamma_n^\delta \notin S$ and $b_{\gamma_n^{\delta^*}} < \gamma_{n+1}^{\delta^*}$.*

Proof. If such a sequence exists, as $\delta^* \in C_1$, Fact 3.3.8 implies there is a $c^*_\delta \in M_{\delta^*+1}$ such that for each n, $F_n^{M_{\delta^*+1}}(c^*_\delta) = b_{\gamma_n^{\delta^*}}$. Since $N \in \widehat{\boldsymbol{K}}$, by clause 8 of Definition 3.2.2, $N \models \neg(\exists x) \bigwedge_n R(x, F_n(c^*_\delta))$. This contradicts that we chose $b_{\gamma_n^{\delta^*}} \in A$, so by the definition of A, for each $n < \omega$, $R^N(a, b_{\gamma_n^{\delta^*}})$ holds. $\Box_{3.3.12}$

case ii) For some α^*, there is no such b_{α^*}. That is, if $b \in P_1^M$ is independent from $P_1^{M_\alpha}$ and $R(M, b)$ is disjoint from α^* then $\neg R(a^*, b)$.

Let $\langle v_\gamma : \gamma < \lambda \rangle$ list $\mathcal{P}(P_0^{M_{\alpha^*}})$ with each element appearing λ times in the list. We now choose inductively by requirement 4 of Goal 3.3.5 a subsequence of the $b_\gamma \in P_1^M$ and M_γ. For local intelligibility (and at the risk of global confusion) we use indices b_γ rather than b_{α_γ} that would keep track of the subsequence fact, as it does not matter.

Choose inductively $b_\gamma \in P_1^M$ such that $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}} = v_\gamma$ and moreover $R(M, b_\gamma) \cap P_0^{M_{\gamma+1}} = v_\gamma$ and $\langle b_\beta : \beta \leq \alpha\rangle$ is independent over $P_1^{M_{\alpha^*}} \cup P_4^M$ in the Boolean algebra P_1^M. Moreover $\{b_\beta\}$ is independent over $P_1^{M_\beta} \cup P_4^M$.

We claim that if $\gamma_1 < \gamma_2 \wedge v_{\gamma_1} = v_{\gamma_2}$ then $R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2})$ holds. Let $b' = b_{\gamma_1} \triangle b_{\gamma_2}$. Then $R(M, b') \cap P_0^{M_{\alpha^*}} = \emptyset$ so by the case choice, $\neg R(a^*, b')$. But $\neg R(a^*, b')$ implies

$$R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2}),$$

as required.

Continuing the proof of case ii) we define an ultrafilter $\mathcal{D}$ on $\mathcal{P}(P_0^{M_{\alpha^*}})$ by $v \in \mathcal{D}$ if for some (and hence any) b_γ with $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}} = v$, $R^N(a^*, b_\gamma)$.

> We guarantee that this is an ultrafilter as each $u \subset P_0^{M_{\alpha^*}}$ is $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}}$ by step 4 of the construction.

But there is no $\aleph_1$-complete ultrafilter on $\mathcal{P}(P_0^{M_{\alpha^*}})$ since λ is not measurable. So there are $\langle v_n \subseteq P_0^{M_{\alpha^*}} : n < \omega\rangle$, each in $\mathcal{D}$, that are decreasing and intersect in $\emptyset$.

Claim 3.3.13. *For any $b \in P_1^M$, if $v = R(M, b) \cap P_0^{M_{\alpha^*}}$ and $v \in \mathcal{D}$ then $N \models R(a^*, b)$.*

We can choose a β large enough so that $\alpha_0 < \beta <, b \in P_1^{M_\beta}$ and there is a $\beta_1 > \beta$ such that $v_{\beta_1} = v$. Now $b \triangle b_1 \in P_1^M$ and $R(M, b') \cap P_0^{M_{\alpha^*}} = \emptyset$. So by the choice of α_*, $N \models \neg R(a^*, b')$. So, $N \models \neg R(a^*, b)$ if and only $N \models \neg R(a_*, b_{\beta_1})$. But, we have $v \in \mathcal{D}$ and $R(M, b_{\beta_1}) \cap P_0^{M_{\alpha_*}} = v$, so $N \models R(a^*, b_{\beta_1})$ and thus $N \models R(a_*, b)$ as required. $\square_{3.3.13}$

Now we can find $\delta^* > \alpha^*$ such that $\delta^* \in S_A \cap C$, the universe of M_{δ^*} is δ^*, $A \cap \delta^* = A_{\delta_v}$, and choose an increasing sequence $\langle \gamma_n^{\delta^*} : n < \omega\rangle$ with limit δ^* and $\gamma_n^{\delta^*} \notin S$. Further we can choose $b_{\gamma_n^{\delta^*}}$ so that $R(M, b_{\gamma_n^{\delta^*}}) \cap M_{\alpha^*}) = v_n$ and the sequence $\{b_{\gamma_n^{\delta^*}}\}$ is independent over $P_1^{\delta^*} \cup P_4^M$. But the existence of such a sequence violates Claim 3.3.12 so we finish case ii) and thus Lemma 3.3.4. $\square_{3.3.4}$

Corollary 3.3.14. *Under the hypotheses of Theorem 3.3.4, there is a maximal model of $\mathbf{R}$ of cardinality at most 2^λ.*

Proof. Fix a P_0-maximal model N_0 of cardinality λ from Theorem 3.3.4. Build for as long as possible a continuous $\subseteq$-increasing chain of $N_\alpha \in \mathbf{R}$ such that each $N_\alpha \neq N_{\alpha+1}$. Recall that by Lemma 3.2.14 the relation R is extensional. So, each $|P_1^{N_\alpha}| \leq 2^{|P_0^{N_0}|} = 2^\lambda$. So this construction must stop and the final, maximal in $\mathbf{R}$, model has cardinality at most 2^λ. $\square_{3.3.14}$

Remark 3.3.15. *We can not directly show M_λ is free as there is no assumption that λ does not reflect. In fact, by the contrapositive of Corollary 3.2.21 the final model in the chain built in Theorem 3.3.14, which might be M, is* not *free.*

4 Hanf Number for Existence

As mentioned in the introduction, we improved in [BKL16] Hjorth's result [Hjo02] by exhibiting for each $n < \omega$ a complete sentence ψ_n such that ψ_n characterizes $\aleph_n$. This improvement is achieved by combining the combinatorial idea of Laskowski-Shelah in [LS93] with a new notion of n-dimensional amalgamation. We explain the main definition and theorem here (as in the Tehran lectures) and refer to [BKL16] for the proofs. The combinatorial fact is:

Fact 4.0.16. *For every $k \in \omega$, if cl is a locally finite closure relation on a set X of size $\aleph_k$, then there is an independent subset of size $k+1$.*

Fix a vocabulary τ_r with infinitely many r-ary relations R_n and infinitely many $r+1$-ary functions f_n. We consider the class $\boldsymbol{K}_0^r$ of finite τ_r-structures (including the empty structure) that satisfy the following three conditions; closure just means subalgebra closure with respect to the functions.

- The relations $\{R_n : n \in \omega\}$ partition the $(r+1)$-tuples;
- For every $(r+1)$-tuple $\boldsymbol{a} = (a_0, \ldots, a_r)$, if $R_n(\boldsymbol{a})$ holds, then $f_m(\boldsymbol{a}) = a_0$ for every $m \geq n$;
- There is no independent subset of size $r+2$.

It is easy to see from Fact 4.0.16 that every model in $\aleph_r$ is maximal. The main effort is to show there is a complete sentence ϕ_r satisfying those conditions which has model in $\aleph_r$. For this we introduce a notion patterned on excellence[6] but weaker. We pass from a class $\boldsymbol{K}_0^r$ of, now, locally finite structures to the associated class $\widehat{\boldsymbol{K}}$ as in Definition 2.1.1.

Definition 4.0.17. *For $k \geq 1$, a* k-configuration *is a sequence $\overline{M} = \langle M_i : i < k \rangle$ of models (not isomorphism types) from $\boldsymbol{K}$. We say $\overline{M}$* has power λ *if $\| \bigcup_{i<k} M_i \| = \lambda$. An* extension *of $\overline{M}$ is any $N \in \boldsymbol{K}$ such that every M_i is a substructure of N.*

Informally a (λ, k)-disjoint amalgamation holds when for any sequence of k models, at least one with λ elements, there is common extension, which properly extends each model in the sequence. Crucially, there is no notion of a universal model yet. Here is the precise formulation.

Definition 4.0.18. *Fix a cardinal $\lambda = \aleph_\alpha$ for $\alpha \geq -1$. We define the notion of a class $(\boldsymbol{K}, \leq)$ having* (λ, k)-disjoint amalgamation *in two steps:*

1. *$(\boldsymbol{K}, \leq)$ has $(\lambda, 0)$-disjoint amalgamation if there is $N \in \boldsymbol{K}$ of power λ;*
2. *For $k \geq 1$, $(\boldsymbol{K}, \leq)$ has $(\leq \lambda, k)$-disjoint amalgamation if it has $(\lambda, 0)$-disjoint amalgamation and every k-configuration $\overline{M}$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ such that every M_i is a proper substructure of N.*

For $\lambda \geq \aleph_0$, we define $(< \lambda, k)$-disjoint amalgamation by: has $(\leq \mu, k)$-disjoint amalgamation for each $\mu < \lambda$.

Whether or not a given k-configuration $\overline{M}$ has an extension depends on more than the sequence of isomorphism types of the constituent M_i's, as the pattern of intersections is relevant as well. For example, when (as here) strong substructure is just substructure), a 2-configuration $\langle M_0, M_1 \rangle$ with neither contained in the other has an extension if and only if the triple of structures $\langle M_0 \cap M_1, M_0, M_1 \rangle$ has an extension amalgamating them disjointly. Thus we abuse notation a bit and write $(< \lambda, 2)$ amalgamation for both the notion defined here and the one in Definition 2.1.4. But there is no existing analog of our disjoint $(< \lambda, k)$-amalgamation for $k > 2$.

[6] Shelah's theory of excellence concerns unique free disjoint amalgamations of infinite structures in ω-stable classes of models of complete sentences in $L_{\omega_1,\omega}$.

Now we modify a theme familiar from the theory of excellence. If the cardinality increases by one the the number of models that can be amalgamated drops by one. In Shelah's context [She09] (chapter 21 of [Bal09]) there is a reliance on Fodor's lemma to obtain compatible filtrations of the models in κ^+ to prove the version of Proposition 4.0.19. A very different approach was needed to go from the finite to the countable. Instead of the kth level concerning finding an embedding into an upper corner for a given 2^{k-1} vertices of a k-cube, we consider actual containment for k-models and do not worry about their intersections.

Lemma 4.0.19 (Proposition 2.20 of [BKL16])**.** *Fix locally finite $(\boldsymbol{K}, \leq)$ with JEP. For all cardinals $\lambda \geq \aleph_0$ and for all $k \in \omega$, if $\boldsymbol{K}$ has $(< \lambda, k+1)$-disjoint amalgamation, then it also has $(\leq \lambda, k)$-disjoint amalgamation.*

Together, these propositions yield 1)-3) of the next result. Recall from Definition 2.1.4, that by 2-amalgamation, we mean the usual notion that allows identifications. We say 2-amalgamation is *trivially* true in a cardinal κ in all models in κ are maximal.

Theorem 4.0.20 (Theorem 3.2.4 of [BKL16])**.** *For every $r \geq 1$, the class $\mathbf{At^r}$ satisfies:*

1. *there is a model of size $\aleph_r$, but no larger models;*
2. *every model of size $\aleph_r$ is maximal, and so 2-amalgamation is trivially true in $\aleph_r$;*
3. *disjoint 2-amalgamation holds up to $\aleph_{r-2}$;*
4. *2-ap fails in $\aleph_{r-1}$.*
5. *Each of the classes $\hat{\boldsymbol{K}}^r$ and $\mathbf{At}^r$ have $2^{\aleph_s}$ models in $\aleph_s$ for $1 \leq s \leq r$. In addition, $\hat{\boldsymbol{K}}^r$ has $2^{\aleph_0}$ models in $\aleph_0$.*

Parts 4) and 5) require a further refinement of the notion of disjoint amalgamation.

Definition 4.0.21. *Let $(\widehat{\boldsymbol{K}}, \leq)$ be a class of structures defined . Given a cardinal λ and $k \in \omega$, we say that $\boldsymbol{K}$ has* frugal $(\leq \lambda, k)$-disjoint amalgamation *if it has $(\leq \lambda, k)$-disjoint amalgamation and, when $k \geq 2$, every k-configuration $\langle M_i : i < k \rangle$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ with universe $\bigcup_{i<k} M_i$.*

Thus the domain of a frugal amalgamation is just the union of the models amalgamated. It is easy to see that this property holds for the example in [BKL16]. It is essential for the intricate constructions to verify the last two parts of Theorem 4.0.20 and for the work in [BKS16, BS15].

The *finite amalgamation spectrum* of an abstract elementary class $\boldsymbol{K}$ with $LS(\boldsymbol{K}) = \aleph_0$ is the set $X_{\boldsymbol{K}}$ of $n < \omega$ and $\boldsymbol{K}$ satisfies amalgamation[7] in $\aleph_n$. There are many examples[8] where the finite amalgamation spectrum of a complete sentence of $L_{\omega_1,\omega}$ is either $\emptyset$ or ω.

Theorem 4.0.20 gave the first example of such a sentence with a non-trivial spectrum: for each $1 \leq r < \omega$ amalagmation holds up to $\aleph_{r-2}$, but fails in $\aleph_{r-1}$. It holds (trivially) in $\aleph_r$ (since all models are maximal); there is no model in $\aleph_{r+1}$.

This result leaves open whether the property, AP in λ, can be true or false in various patterns as λ increases? Is there even an AEC (and more interestingly a complete sentence of $L_{\omega_1,\omega}$) and cardinals $\kappa < \lambda$ such that amalgamation holds non-trivially in both κ and λ but fails at some cardinal between them?

[7] We say amalgamation holds in κ in the trivial special case when all models in κ are maximal. We say amalgamation fails in κ if there are no models to amalgamate.

[8] Kueker, as reported in [Mal68], gave the first example of a complete sentence failing amalgamation in $\aleph_0$.

Relying on the construction in [BKL16], Baldwin and Souldatos [BS15] show there exist *complete* sentences of $L_{\omega_1,\omega}$ that variously have maximal models a) in two successive cardinals, b) in κ and κ^ω and c) in countably many cardinals. In each case all maximal models of the sentence have cardinality less than $\beth_{\omega_1}$. That proof includes an intricate construction of a complete sentence that has a model in each successor cardinal κ^+ with a definable subset of power κ. The [BS15] result is distinguished from the one here in several ways. It constructs maximal models in designated cardinals rather than an initial segment. The crucial amalgamation properties are quite different. In [BKL16] we establish $(< \lambda, 2)$ amalgamation in all cardinals. In [BS15] The most delicate argument in [BS15] shows that one can amalgamate a model of $\widehat{\boldsymbol{K}}$ of any cardinality with an arbitrary finite model and thus achieve richness.

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