

# Hanf numbers for Extendibility and related phenomena

John T. Baldwin  
University of Illinois at Chicago\*  
Saharon Shelah†  
Hebrew University of Jerusalem

January 12, 2021

## Abstract

This paper contains portions of Baldwin’s talk at the Set Theory and Model Theory Conference (Institute for Research in Fundamental Sciences, Tehran, October 2015) and a detailed proof that in a suitable extension of ZFC, there is a complete sentence of  $L_{\omega_1, \omega}$  that has maximal models in cardinals cofinal in the first measurable cardinal and, of course, never again.

In this paper we discuss two theorems whose proofs depend on extensions of the Fraïssé method. We prove here the Hanf number for the property that every model of a (complete) sentence of  $L_{\omega_1, \omega}$  with cardinality  $\kappa$  is extendible<sup>1</sup> is (modulo some mild set theoretic hypotheses that we remove in [BS2x]) the first measurable cardinal. And we outline the description of an explicit  $L_{\omega_1, \omega}$ -sentence  $\phi_n$  characterizing  $\aleph_n$  for each  $n$ . We provide some context for these developments as outlined in the lectures at IPM<sup>2</sup>.

The phrase ‘Fraïssé construction’ has taken many meanings in the over 60 years since the notion was born [Fra54] (and earlier in an unpublished thesis). There are two major streams. We focus here on variants in the original construction, which usually use the standard notion of substructure. We don’t deal here directly with ‘Hrushovski constructions’ where a specialized notion of strong submodel varying with the case plays a central role. An annotated bibliography of developments of the Hrushovski variant until 2009 appears at [Bal].

The first variant we want to consider is the vocabulary. Fraïssé worked with a *finite, relational* vocabulary. While model theory routinely translates between functions and their graphs and there is usually little distinction between finite and countable vocabularies; in the infinite vocabulary case such extensions for the Fraïssé construction yield weaker but still very useful consequences. The second is a distinction in goal: the construction of *complete* sentences of  $L_{\omega_1, \omega}$  (equivalently studying the *atomic* models of a complete first order theory) rather than constructing  $\aleph_0$ -categorical theories. This second shift raises new questions about the cardinality of the resulting models. The result in Section 4 pins down more precisely the existence spectra for *complete* sentences of  $L_{\omega_1, \omega}$ . Section 3 expresses the role of large cardinal axioms in more algebraic terms. Rephrased, it says that, consistently with the existence of a measurable cardinal, there is a

---

\*Research partially supported by Simons travel grant G5402, G3535.

†Item 1092 on Shelah’s publication list. Partially supported by European Research Council grant 338821, and by National Science Foundation grant 136974.

<sup>1</sup>We say  $\mathbf{K}$  is *universally extendible in  $\kappa$*  if  $M \in \mathbf{K}$  with  $|M| = \kappa$  has a proper  $\prec_{\mathbf{K}}$ -extension in the class. Here, this means has an  $\infty, \omega$ -elementary extension.

<sup>2</sup>Due to our tardiness in preparing this paper it could be included in the special volume dedicated to the 2015 conference.

nicely defined (by a complete sentence of  $L_{\omega_1, \omega}$ ) class of models that has non-extendible (maximal) models cofinally below the first measurable. The previous upper bound for such behavior was  $\aleph_{\omega_1}$ .

We acknowledge helpful comments by the referee and by Joel Berman, Sakai Fuchino, Menachim Magidor, Ioannis Soukatas and especially Will Boney.

## 1 Hanf numbers and Spectrum functions in infinitary logic

Recent years have brought a number of investigations of the spectrum (cardinals in which a property occurs) for various phenomena and various sorts of infinitary definable classes. Some of the relevant phenomena are existence, amalgamation, joint embedding, maximal models etc. The class might be defined as an abstract elementary class, the models of a (complete) sentence of  $L_{\omega_1, \omega}$ , etc.

Hanf observed [Han60] that for any property  $P(\mathbf{K}, \lambda)$ , where  $\mathbf{K}$  ranges over a *set* of classes of models, there is a cardinal  $\kappa = H(P)$  such that  $\kappa$  is the least cardinal satisfying: if  $P(\mathbf{K}, \lambda)$  holds for some  $\lambda \geq \kappa$  then  $P(\mathbf{K}, \lambda)$  holds for arbitrarily large  $\lambda$ .  $H(P)$  is called the Hanf number of  $P$ . e.g.  $P(\mathbf{K}, \lambda)$  might be the property that  $\mathbf{K}$  has a model of power  $\lambda$ .

Morley [Mor65] showed for an arbitrary sentence of  $L_{\omega_1, \omega}(\tau)$  the Hanf number for existence is  $\beth_{\omega_1}$  when  $\tau$  is countable (More generally, it is  $\beth_{(2^{|\tau|})^+}$ . [She78]); the situation for *complete* sentences is much more complicated. Knight [Kni77] found the first complete sentence characterizing  $\omega_1$  (i.e. has a model in  $\omega_1$  but no larger) by building on the construction of many non-isomorphic  $\aleph_1$ -like linear orderings. Hjorth found, by a procedure generalizing the Fraïssé -construction, for each  $\alpha < \omega_1$ , a set  $S_\alpha$  (finite for finite  $\alpha$ ) of complete  $L_{\omega_1, \omega}$ -sentences<sup>3</sup> such that some  $\phi_\alpha \in S_\alpha$  characterizes  $\aleph_\alpha$ . It is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. Baldwin, Koerwien, and Laskowski [BKL16] show a modification of the Laskowski-Shelah example (see [LS93, BFKL16]) gives a family of  $L_{\omega_1, \omega}$ -sentences  $\phi_r$ , which characterize  $\aleph_r$  for  $r < \omega$ . In Section 4 we sketch the new notion of n-disjoint amalgamation that plays a central role in [BKL16].

Further results by [BKS09, KLH16, BKS16], where the hypothesis are weakened to allow incomplete sentences of  $L_{\omega_1, \omega}$  or even AEC (Abstract Elementary Classes  $(\mathbf{K}, \leq)$  where the properties of *strong substructure*,  $\leq$  are defined axiomatically) are placed in context in [BB17]. Analogous results were proved earlier for *incomplete* sentences by [BKS16] who code certain bipartite graphs in way that determine specific inequalities between the cardinalities of the two parts of the graph; in this case all models have cardinality less than  $\beth_{\omega_1}$ .

All the exotica mentioned here and described in more detail in [BB17] occurs below  $\beth_{\omega_1}$ . Baldwin and Boney [BB17] have shown that the Hanf number for amalgamation is no more than the first strongly compact cardinal. This immense gap motivated the current paper. We show that for the case of universally extendable (every model has a proper extension), there is a smaller gap. There is a *complete* sentence of  $L_{\omega_1, \omega}$  which has a maximal model in cardinals cofinal in the first measurable (if such exists), but no larger maximal model. Is the same true of amalgamation? That is, can amalgamation eventually behave very differently than it does in small cardinalities? At the end of this paper we point to the only known example where amalgamation (for a complete  $L_{\omega_1, \omega}$ -sentence) holds on an initial segment then fails, then holds again; then there are no larger models.

---

<sup>3</sup>Inductively, Hjorth shows at each  $\alpha$  and each member  $\phi$  of  $S_\alpha$  one of two sentences,  $\chi_\phi, \chi'_\phi$ , works as  $\phi_{\alpha+1}$  for  $\aleph_{\alpha+1}$ .

## 2 Disjoint Amalgamation

### 2.1 Classes determined by finitely generated structures

The original Fraïssé construction took place in a *finite relational* vocabulary and the resulting infinite structure was  $\aleph_0$ -categorical for a first order theory. We explore here several ways to construct a countable atomic model for a first order theory and thus a complete sentence in  $L_{\omega_1, \omega}$ .

Recall (e.g. chapter 7 of [Bal09]) that the models of a complete sentence of  $L_{\omega_1, \omega}(\tau)$  are the reducts to  $\tau$  of the atomic (every finite sequence realizes a principal type) models of a complete first order theory in a vocabulary  $\tau'$  extending  $\tau$ . We discuss classes determined by a countable set of finitely generated models. In Sections 3 and 4, we describe the examples of such classes used to prove our main results.

**Definition 2.1.1.** Fix a countable vocabulary  $\tau$  (possibly with function symbols). Let  $(\mathbf{K}_0, \subseteq)$  denote a countable collection of finite  $\tau$ -structures and let  $(\widehat{\mathbf{K}}, \subseteq)$  denote the abstract elementary class containing all structures  $M$  such that every finite generated substructure of  $M$  is in  $\mathbf{K}_0$ .

These classes have syntactic characterizations.

- Lemma 2.1.2.**
1.  $\widehat{\mathbf{K}}$  is defined by an  $L_{\omega_1, \omega}$ -sentence  $\phi$ .
  2. If  $\mathbf{K}_0$  is closed under substructure then  $\phi$  may be taken universal [Mal69].
  3.  $(\mathbf{K}_0, \subseteq)$  satisfies the axioms for AEC (except for unions under chains.)

While traditional Fraïssé classes are closed under substructure and produce  $\aleph_0$ -categorical first order structures, which are *uniformly* locally finite, the search for atomic models [Hjo07, BFKL16, BKL16, BS19] does not always require closure under substructure and produces a generic structure which is locally finite but not uniformly so. In Section 3, we expand the subject further by using countable collections of *finitely generated* rather than finite structures as the ‘Fraïssé class’.

**Definition 2.1.3.** Fix a countable vocabulary  $\tau$  (possibly with function symbols). Let  $(\mathbf{K}_0, \leq)$  denote a countable collection of finite  $\tau$ -structures with  $(\widehat{\mathbf{K}}, \leq)$  as in Definition 2.1.1.

1. A model  $M \in \widehat{\mathbf{K}}$  is *rich* or  $\mathbf{K}_0$ -homogeneous if for all  $A$  and  $B$  in  $\mathbf{K}_0$  with  $A \leq B$ , every embedding  $f : A \rightarrow M$  extends to an embedding  $g : B \rightarrow M$ . We denote the class of rich models in  $\widehat{\mathbf{K}}$  as  $\mathbf{R}$ .
2. The model  $M \in \widehat{\mathbf{K}}$  is *generic* if  $M$  is rich and  $M$  is an increasing union of a countable chain of finitely generated substructures, each of which is in  $\mathbf{K}_0$ .
3. We let  $\mathbf{R}$  denote the subclass of  $\widehat{\mathbf{K}}$  consisting of rich models.

In the examples considered here the generic models will always be countable.

**Definition 2.1.4.** An AEC  $(\mathbf{K}, \leq)$  has  $(< \lambda, 2)$ -disjoint amalgamation if for any  $A, B, C \in \mathbf{K}$  with cardinality  $< \lambda$  and  $A$  strongly embedded in  $B, C$ , there is a  $D$  and strong embedding of  $B, C$  into  $D$  that agree on  $A$  and such that the intersection of their ranges is their image of  $A$ .

$\mathbf{K}$  has 2-amalgamation if the ranges of the embedding are allowed to intersect outside of  $f(A)$ .

$\mathbf{K}$  has the joint embedding property (JEP) if any two models can be embedded in some larger  $D$ .

Fraïssé's theorem asserted that if a class of finite models in a finite relational language is closed under substructure and satisfies AP and JEP then there is a generic model whose theory is  $\aleph_0$ -categorical and quantifier eliminable. The following extension of Fraïssé's theorem is well-known [Hod93] and the proof is essentially the same.

**Lemma 2.1.5.** *Suppose  $\tau$  is countable and  $\mathbf{K}_0$  is a countable class of finite or countable  $\tau$ -structures that satisfies 2-amalgamation, in particular  $(\leq \aleph_0, 2)$ -disjoint amalgamation, and JEP, then*

1. *A  $\mathbf{K}_0$ -generic (and so rich)  $\tau$ -structure  $M$  exists.*
2. *if  $\mathbf{K}_0$  is closed under substructure, the generic is ultra-homogeneous (every isomorphism between arbitrary finitely generated substructures extends to an automorphism).*

A key distinction from the Fraïssé situation is that in the first order case  $\widehat{\mathbf{K}}$  doesn't really play a role while in the infinitary case it is an important intermediary between the finitely generated structures and  $\mathbf{R}$ . Fraïssé passes to the first order theory of the generic since it is  $\aleph_0$ -categorical in *first order logic*. In our more general situation the generic may be  $\aleph_0$ -categorical only in  $L_{\omega_1, \omega}$ . The Scott sentence of the rich model gives the  $L_{\omega_1, \omega}$  sentence we study. As noted at the beginning of this section we may regard the models as reducts of atomic models of a first order theory. Thus  $\widehat{\mathbf{K}}$  may have arbitrarily large models while  $\mathbf{R}$  does not; this holds of some examples in [Hjo07, BFKL16, BKL16].

**Corollary 2.1.6.** *Suppose  $(\mathbf{K}_0, \leq)$  satisfies the hypotheses of Lemma 2.1.5. Fix  $\lambda \geq \aleph_0$ . If  $\widehat{\mathbf{K}}$  has  $(\leq \lambda, 2)$ -amalgamation and has at most countably many isomorphism types of countable structures, then every  $M \in \widehat{\mathbf{K}}$  of power  $\lambda$  can be extended to a rich model  $N \in \widehat{\mathbf{K}}$ , which is also of power  $\lambda$ .*

*Proof.* Given  $M \in \widehat{\mathbf{K}}$  of power  $\lambda$ , construct a continuous chain  $\langle M_i : i < \lambda \rangle$  of elements of  $\widehat{\mathbf{K}}$ , each of size  $\lambda$ . At a given stage  $i < \lambda$ , focus on a specific finite substructure  $A \subseteq M_i$  and a particular finite extension  $B \in \widehat{\mathbf{K}}$  of  $A$ . If there is an embedding of  $B$  into  $M_i$  over  $A$ ,  $M_{i+1} = M_i$ . If not, we may assume  $B \cap M_i = A$ . Let  $M_{i+1}$  be the disjoint amalgamation of  $M_i$  and  $B$  over  $A$ . As there are only  $\lambda$ -possible extensions, we can, by iterating, organize this construction so that  $N = \bigcup \{M_i : i < \lambda\}$  is rich.  $\square_{2.1.6}$

Crucially, in Section 3.2.22 the class  $\widehat{\mathbf{K}}$  under consideration will not satisfy two-amalgamation even with finite models; but there will be amalgamation of free structures with finite.

## 2.2 Atomic Models of First order theories

We discuss here classes generated by finite (not finitely generated) structures. Suppose a generic  $\tau$ -model  $M$  exists. When is  $M$  an atomic model of its first-order  $\tau$ -theory? As remarked in Section 2 of [BKL16] this second condition has nothing to do with the choice of embeddings on the class  $\mathbf{K}_0$ , but rather with the choice of vocabulary. The following condition is needed when, for some values of  $n$ ,  $\mathbf{K}_0$  has infinitely many isomorphism types of structures of size  $n$

We denote the class of atomic models of a complete first order theory by  $\mathbf{At}$ .

**Definition 2.2.1.** *A class  $\mathbf{K}_0$  of finite structures in a countable vocabulary is separable if, for each  $A \in \mathbf{K}_0$  and enumeration  $\mathbf{a}$  of  $A$ , there is a quantifier-free first order formula  $\phi_{\mathbf{a}}(\mathbf{x})$  such that:*

- $A \models \phi_{\mathbf{a}}(\mathbf{a})$  and
- for all  $B \in \mathbf{K}_0$  and all tuples  $\mathbf{b}$  from  $B$ ,  $B \models \phi_{\mathbf{a}}(\mathbf{b})$  if and only if  $\mathbf{b}$  enumerates a substructure  $B'$  of  $B$  and the map  $\mathbf{a} \mapsto \mathbf{b}$  is an isomorphism.

In practice, we will apply the observation that if for each  $A \in \mathbf{K}_0$  and enumeration  $\mathbf{a}$  of  $A$ , there is a quantifier-free formula  $\phi'_{\mathbf{a}}(\mathbf{x})$  such that there are only finitely many  $B \in \mathbf{K}_0$  with cardinality  $|A|$  that under some enumeration  $\mathbf{b}$  satisfy  $\phi'_{\mathbf{a}}(\mathbf{b})$ , then  $\mathbf{K}_0$  is separable.

**Lemma 2.2.2.** [BKL16] *Suppose  $\tau$  is countable and  $\mathbf{K}_0$  is a class of finite  $\tau$ -structures that is closed under substructure, satisfies amalgamation, and JEP, then a  $\mathbf{K}_0$ -generic (and so rich) model  $M$  exists. Moreover, if  $\mathbf{K}_0$  is separable,  $M$  is an atomic model of  $\text{Th}(M)$ . Further,  $\mathbf{R} = \mathbf{At}$ , i.e., every rich model  $N$  is an atomic model of  $\text{Th}(M)$ .*

Proof: Since the class  $\mathbf{K}_0$  of finite structures is separable it has countably many isomorphism types, and thus a  $\mathbf{K}_0$ -generic  $M$  exists by the usual Fraïssé construction. To show that  $M$  is an atomic model of  $\text{Th}(M)$ , it suffices to show that any finite tuple  $\mathbf{a}$  from  $M$  can be extended to a larger finite tuple  $\mathbf{b}$  whose type is isolated by a complete formula. Coupled with the fact that  $M$  is  $\mathbf{K}_0$ -locally finite, we need only show that for any finite substructure  $A \leq M$ , any enumeration  $\mathbf{a}$  of  $A$  realizes an isolated type. Since every isomorphism of finite substructures of  $M$  extends to an automorphism of  $M$ , the formula  $\phi_{\mathbf{a}}(\mathbf{x})$  isolates  $\text{tp}(\mathbf{a})$  in  $M$ .

The final sentence follows since any two rich models are  $L_{\infty, \omega}$ -equivalent.  $\square_{2.2.2}$

### 3 Hanf number for All Models Extendible

We say an abstract elementary class (the models of a complete sentence in  $L_{\omega_1, \omega}$ ) is *universally extendible in  $\kappa$*  if every model of cardinality  $\kappa$  has a proper strong extension ( $L_{\infty, \omega}$ -elementary extension). In this section we prove the following theorem.

**Theorem 3.0.1.** *There is a complete sentence  $\phi$  of  $L_{\omega_1, \omega}$  that has arbitrarily large models. But under reasonable set theoretic conditions (specified below), we show that for arbitrarily large  $\lambda < \mu$ , where  $\mu$  is the first measurable cardinal, and unboundedly many  $\lambda$  if there is no measurable cardinal,  $\phi$  has a maximal model (with respect to substructure, which in this case means  $\prec_{\infty, \omega}$ ) with cardinality between  $\lambda$  and  $2^\lambda$ .*

We remove in [BS2x] the set theoretic hypotheses by adapting techniques from [She, GS05] but at the cost of weakening the freeness of the  $P_0$ -maximal model; see Remark 3.3.13.

If  $|M|$  is at least the first measurable  $\mu$ , then for any  $\aleph_1$ -complete non-principal ultrafilter  $\mathcal{D}$  on  $\mu$ ,  $M^\mu / \mathcal{D}$  is a proper extension of  $M$ . This holds because we can find an  $f \in M^\mu$  which hits each element  $a \in M$  at most once. Thus the equivalence class of  $f$  cannot be that of any constant map on  $M$  (since  $\mathcal{D}$  is non-principal). On the other hand, by the Łos theorem for  $L_{\omega_1, \omega}$ , since  $\mathcal{D}$  is  $\aleph_1$ -complete, the ultrapower is a proper  $L_{\omega_1, \omega}$ -elementary extension of  $M$ . Thus, we have shown the Hanf number for extendability is at most  $\mu$ :

**Lemma 3.0.2.** *If  $\mu$  is measurable, for any  $\phi \in L_{\mu, \mu}$ , in particular in  $L_{\omega_1, \omega}$ , no model of cardinality  $\geq \mu$  is maximal.*

The proof of the converse (Theorem 3.0.1) fills the remainder of this section. If we only demand the result for an arbitrary sentence of  $L_{\omega_1, \omega}$  there are easy examples. We learned an example in terms of  $\omega$ -models (which is easily reinterpreted into  $L_{\omega_1, \omega}$ ) from Magidor [Mag16]. The following sketch of such an example will suggest some of the key points of the main argument. Note that we write  $\mathcal{P}(X)$  for the powerset of  $X$ .

**Example 3.0.3.** Consider a class  $\mathbf{K}$  of 3-sorted structures where:  $P_0$  is a set,  $P_1$  is a boolean algebra of subsets of  $P_0$  (given by an extensional binary  $E$ ) and  $P_2$  is just a set;  $\{F_n : n < \omega\}$  is a family of unary functions which assigns to each  $c \in P_2$ , a sequence  $F_n(c) \in P_1$ . Demand:  $\bigwedge_n F_n(c) = F_n(d)$  implies  $d = c$ . Let  $\psi \in L_{\omega_1, \omega}$  axiomatize  $\mathbf{K}$ . We claim  $M$  is a maximal model of  $\text{mod}(\psi)$  with cardinality  $\lambda$  if  $\lambda < \text{first measurable}$ ,  $|P_0^M| = \lambda$ ,  $P_1^M = \mathcal{P}(P_0^M)$ , and  $P_2^M$  codes each sequence in  ${}^\omega(P_1^M)$  via the  $F_n$ .

Suppose for contradiction that  $N$  with  $M \overset{\omega_1, \omega}{\prec} N$  witnesses non-maximality, then the choice of  $M$  and the demand imply that there must be an element  $a^* \in P_0^N - P_0^M$ . Then  $D = \{b \in P_1^M : E(a^*, b)\}$  is a non-principal ultrafilter on  $\lambda = P_0^M$ . To see that  $D$  is non-principal, note that if some  $b' \in P_1^M$  generated  $D$ , then  $b' \prec a^*$ , contrary (by elementary extension) to their both being atoms.

Since  $D$  is  $\aleph_1$ -incomplete (as  $\lambda$  is not measurable) there exists a sequence  $\langle b_n : n < \omega \rangle$  of elements of  $P_1^M$  with empty intersection. Since each countable sequence of subsets of  $P_0^M$  is coded as  $\langle F_n^M(c) : n < \omega \rangle$  for some  $c \in P_2^M$ , there is a  $d \in P_2^M$  with  $F_n(d) = b_n$  for each  $n$ . Thus,  $M \models \neg(\exists x) \bigwedge E(x, F_n(d))$ , while  $N \models \bigwedge E(a^*, F_n(d))$ . This contradicts  $M \overset{\omega_1, \omega}{\prec} N$ .

There are  $2^{\aleph_0}$  2-types over the empty set, given, for each  $X \subset \omega$ , via  $(c, d)$  realizes  $p_X$  iff  $X = \{n : F_n(c) \cap F_n(d) \neq \emptyset\}$ . This implies no sentence satisfied by  $M$  can be complete, since a minor variant of Scott's characterization of countable models shows that a sentence  $\psi$  is complete if and only if only countably many  $L_{\omega_1, \omega}$ -types over  $\emptyset$  are realized in models of  $\psi$ . In Section 3.2 we modify this example to obtain a complete sentence.

### 3.1 Some preliminaries on Boolean Algebras

There are a number of slightly different jargons among set theorists, model theorists, category theorists, and Boolean algebraists. In this section we will spell some of them out, indicate some translations, specify our notation, and prove some properties of Boolean algebras that will be used in the proof.

An ultrafilter of a Boolean algebra  $B$  is a maximal filter (i.e. a subset of  $B$  that is closed up, under intersection and contains either  $a$  or  $a^-$  – the complement of  $a$ ). An ultrafilter on a set  $X$  is a subset of its power set and so is an ultrafilter of the Boolean algebra  $\mathcal{P}(X)$ .

We begin with some basic properties of independence in Boolean algebras. A key fact is an equivalence of two notions of independence on countably infinite Boolean algebras that disappears in the uncountable. That is, a countable Boolean algebra is  $\aleph_0$ -categorical if and only if it is free on countably many generators in the sense of 3.1.1 if and only if it is generated by an independent set in the sense of 3.1.3. But this equivalence fails in the uncountable.

**Definition 3.1.1.** 1. For  $X \subseteq B$  and  $B$  a Boolean algebra,  $\overline{X} = X_B = \langle X \rangle_B$  be the subalgebra of  $B$  generated by  $X$ .

2. A set  $Y$  is independent (or free) from  $X$  modulo an ideal  $\mathfrak{S}$  (with domain  $I$ ) in a Boolean algebra  $B$  if and only if for any Boolean-polynomial  $p(v_0, \dots, v_k)$  (that is not identically 0), and any  $a \in \langle X \rangle_B - \mathfrak{S}$ , and distinct  $y_i \in Y$ ,  $p(y_0, \dots, y_k) \wedge a \notin \mathfrak{S}$ .
3. Such an independent  $Y$  is called a basis for  $\langle X \cup Y \cup I \rangle$  over  $\langle X \cup I \rangle$ .

There is no requirement that  $\mathfrak{S}$  be contained in  $X$ . Observe the following:

**Observation 3.1.2.** 1. If  $\mathfrak{S}$  is the 0 ideal, (i.e.,  $Y$  is independent from  $X$ ), the condition becomes: for any  $a \in \langle X \rangle_B - \{0\}$ ,  $B \models p(y_0, \dots, y_k) \wedge a > 0$ . That is, every finite Boolean combination of elements of  $Y$  meets each non-zero  $a \in \langle X \rangle_B$ .

2. Let  $\pi$  map  $B$  to  $B/\mathfrak{S}$ . If ‘ $Y$  is independent from  $X$  over  $\mathfrak{S}$ ’ then the image of  $Y$  is free from the image of  $X$  (over  $\emptyset$ ) in  $B/\mathfrak{S}$ . Conversely, if  $\pi(Y)$  is independent over  $\pi(X)$  in  $B/\mathfrak{S}$ , for any  $Y'$  mapping by  $\pi$  to  $\pi(Y)$ ,  $Y'$  is independent from  $X$  over  $\mathfrak{S}$ .

So, if  $X$  is empty, the condition ‘ $Y$  is independent over  $\mathfrak{S}$ ’ implies the image of  $Y$  is an independent subset of  $B/\mathfrak{S}$ .

3. If a set  $Y$  is independent (or free) from  $X$  over an ideal  $\mathfrak{S}$  in a Boolean algebra  $B$  and  $Y_0$  is a subset of  $Y$ , then  $Y - Y_0$  is independent (or free) from  $X \cup Y_0$  ( $\langle X \cup Y_0 \rangle_B$ ) over the ideal  $\mathfrak{S}$  in the Boolean algebra  $B$ .

From left to right in item 2), note that if for any nontrivial term<sup>4</sup>  $\sigma(\mathbf{v})$ , and any  $\mathbf{y} \in Y$  there is an  $a$  with  $\sigma(\mathbf{y}) \wedge a \notin I$  then  $\pi(\sigma(\mathbf{y}) \wedge a)$  is not 0 in  $B/I$ . Conversely, if some  $\sigma(\pi(\mathbf{y})) \neq 0$  then if  $\mathbf{y}'$  is in  $\pi^{-1}(\pi(\mathbf{y}))$ , then  $\sigma(\mathbf{y}') \notin I$ .

The notion of independence above corresponds to the closure system generated by subalgebra [Grä79, chapter 12]; it does *not* satisfy the axioms for a matroid (combinatorial geometry); exchange fails. It is an independence system (the empty set can be considered independent and subsets of independent sets are independent.). But given  $X$  and  $Y$  independent with  $|Y| > |X|$ , in general there is no guarantee that some element of  $Y - X$  can be added to  $X$  and maintain independence. But, see Lemma 3.1.9.

The contrast between the notion of independence above and the following is crucial for the construction here.

**Definition 3.1.3.** Let  $X, Y$  be sets of elements from a Boolean algebra of sets.  $X$  is independent over  $Y$  if for any infinite  $A$  that is a non-trivial finite Boolean combination of elements of  $X$  and any  $B$  which is a non-empty finite Boolean combination of elements of  $Y$ ,  $A \cap B$  and  $A^c \cap B$  are each infinite.

Both kinds of independence will occur in the models in Section 3.3. There are models in  $\mathbf{K}_1$ , Definition 3.2.2, that are constructed in Construction 3.3.9 with a homomorphism from  $P_1^M$  into  $\mathcal{P}(P_0^M)$  that does *not* transfer from ‘independence in the boolean algebra sense’ (Definition 3.1.1.2 to ‘set independence’ (Definition 3.1.3). In  $\mathbf{K}_2$ , there is an isomorphism from  $P_1^M$  into  $\mathcal{P}(P_0^M)$  that correctly transfers ‘independence’. (See Lemma 3.2.21.)

**Definition 3.1.4.** A pushout consists of an object  $P$  along with two morphisms  $i_1 : X \rightarrow P$  and  $i_2 : Y \rightarrow P$  which complete a commutative square with two given morphisms  $f$  and  $g$  mapping an object  $Z$  to  $X$  and  $Y$  respectively such that any morphisms  $j_1, j_2$  from  $X$  and  $Y$  to a  $Q$  must factor through  $P$ .

In [FG90], it is shown by a category theoretic argument that for distributive lattices the abstract embeddings into the pushout (Notation 3.1.6) are 1-1 and if  $A$  is a Boolean algebra, the images of the embedding intersect in image of  $A$ . Thus the variety of Boolean algebras has *disjoint<sup>5</sup> amalgamation*.

We now connect this notion with our version of independence in Definition 3.1.1.

**Lemma 3.1.5.** Let  $D = A \otimes_C B$  be the Boolean algebra obtained as the pushout (Definition 3.1.4) of  $A$  and  $B$  over  $C$ . Suppose  $\mathfrak{S}$  is an ideal of  $D$  and  $I_2 \subset A - C$  such that  $\langle I_2 \rangle_D \cap \mathfrak{S} = \emptyset$  and  $B - \mathfrak{S} \neq \emptyset$ . Then  $I_2$  is independent from  $B$  over  $\mathfrak{S}$ .

<sup>4</sup>A trivial term (or polynomial) is one which is identically 0.

<sup>5</sup>Called strong in [FG90].

Proof. Fix a Boolean-polynomial  $p(v_0, \dots, v_k)$  (that is not identically 0), and suppose for contradiction there is a  $d \in B - \mathfrak{S}$  and distinct  $y_i \in I_2$  with  $p(y_0, \dots, y_k) \wedge d \in \mathfrak{S}$ . Any morphisms  $f_1, f_2$  from  $A, B$  to any  $D'$  must factor through  $D$ . In particular, we can extend  $\mathfrak{S} \cap A$  and  $\mathfrak{S} \cap B$  to maximal ideals omitting  $p(y_0, \dots, y_k) \in A$  and  $d \in B$ ; the resulting map from  $D$  that commutes with the induced  $f_i$  to the 2-element algebra sends all of  $\mathfrak{S}$  and so  $p(y_0, \dots, y_k) \wedge d$ , but not  $d$  or  $p(y_0, \dots, y_k)$  to 0. But there is no such homomorphism.  $\square_{3.1.5}$

There are several sets of confusing terminology arising from various perspectives in the study of Boolean algebra and misleading analogies with, for example, the study of groups. For example, consider the notion of the product of two Boolean algebras,  $A, B$ . That is, the structure on the *Cartesian (direct) product* of  $A$  and  $B$ , obtained by defining the operations coordinate-wise. Note that, while there are isomorphic copies of  $A$  and  $B$  in the product, the natural injections into  $A \times \{0\}, \{0\} \times \{B\}$ , map to ideals not sub-Boolean algebras.

A generalization of the dual of the direct product operation is often called the ‘*free product with amalgamation*’; we will call the free amalgamation of Boolean algebras  $B$  and  $A$  over  $C$  the one that is obtained by the pushout/free product construction of Notation 3.1.6; it is the coproduct in the category-theoretic language.

**Notation 3.1.6.** *Let  $C \subseteq A, B$  be Boolean algebras. The disjoint amalgamation  $D = A \otimes_C B$  is the Boolean algebra obtained as the pushout [AB11] of  $A$  and  $B$  over  $C$ . It is characterized internally by the following condition. For  $a \in A - C, b \in B - C$ :  $a \leq b$  in  $D$  if and only if there is a  $c \in C$  with  $a < c < b$  (and symmetrically).  $D$  is generated as a Boolean algebra by  $A \cup B$  where  $A$  and  $B$  are sub-Boolean algebras of  $D$ .*

We will distinguish certain subsets of our models in terms of atoms.

**Notation 3.1.7.** *An atom is an element  $a$  of a Boolean algebra such that for every  $c$  either  $c \wedge a = a$  or  $c \wedge a = 0$ . The element  $a$  is a non-trivial atom if it is neither 0 nor 1. For any Boolean algebra  $B$ ,  $\text{At}(B)$  denotes the set of atoms of  $B$ .*

We work in a Boolean algebra  $P_1^M$  and use  $P_{4,1}^M$  for  $\text{At}(P_1^M)$ . We will denote by  $P_4^M$  the set of finite joins of atoms and  $P_{4,n}^M$  for those elements that are the join of exactly  $n$  atoms.  $P_4^M$  is always an ideal of  $P_1^M$  but it is only a Boolean algebra if it is finite, and even then it will not be a sub-Boolean algebra. A Boolean algebra is atomic, or in anachronistic terminology, atomistic if every element is an arbitrary join of atoms<sup>6</sup>.

For  $M$  in the class of finitely generated structures  $\mathbf{K}_0$ , below, the ideal  $P_4^M$  will be atomistic when viewed as a Boolean algebra (with  $b^* = 1$  and complement as relative complement below  $b^*$ .) and the maximal such. For  $M$  in the class  $\mathbf{K}_2$  the entire Boolean algebra  $P_1^M$  will be atomistic but this will be false for all  $M$  in  $\mathbf{K}_{<\aleph_0}^{-1}$  (since it has only finitely many atoms) and for some  $M$  in  $\mathbf{K}^1$  which are not in  $\mathbf{K}_{<\aleph_0}^{-1}$ . We will use the next remark in proving Lemma 3.2.12.

**Lemma 3.1.8.** *Let  $B_0 \subseteq B_1 \subseteq B_2$  be Boolean algebras. Suppose  $I_i$  for  $i < 3$  form a sequence of ideals in the respective  $B_i$  with  $I_1 \cap B_0 = I_0$  and  $I_2 \cap B_1 = I_1$ . If, for  $i = 0, 1$ ,  $J_i \subset B_{i+1}$  is independent from  $B_i$  modulo  $I_i$  in  $B_{i+1}$ , then  $J = J_0 \cup J_1$  is independent from  $B_0$  modulo the ideal  $I_2$ .*

Proof. Let  $\mathbf{b}$  be a finite sequence of distinct elements from  $J$ . Suppose  $\sigma(\mathbf{y})$  is a non-zero term in the same number of variables as the length of  $\mathbf{b}$ . For any  $d \in B_0 - I_2$ , we must show  $\sigma(\mathbf{b}) \wedge d \notin I_2$ . Writing  $\sigma$

<sup>6</sup>Equivalently for Boolean algebras, if every non-zero element is above at least one atom. The conditions are not equivalent on an arbitrary distributive lattice.



in disjunctive normal form it suffices to show some disjunct  $\tau$  (which is just a conjunction of literals  $y_i$  and  $y_i^-$ ) satisfies  $\tau(\mathbf{b}) \wedge d \notin I_2$ . Decompose  $\tau(\mathbf{b})$  as  $\tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1)$  where  $\mathbf{b}_i \in J_i$ . Since  $J_0$  is independent from  $B_0$  modulo  $I_1$ ,  $\tau_0(\mathbf{b}_0) \wedge d \notin I_1$  and clearly it is some  $d_1 \in B_1$ . Similarly, since  $J_1$  is independent from  $B_1$  modulo  $I_2$ ,  $\tau_1(\mathbf{b}_1) \wedge d_1 \notin I_2$ . So  $\tau(\mathbf{b}) \wedge d_1 = \tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1) \wedge d_1 \notin I_2$  as required.  $\square_{3.1.8}$

Although our notion of independence does not satisfy exchange, we are able to show that under certain conditions each suitable element is a member of a basis.

**Lemma 3.1.9.** *If  $B$  is a countable atomless Boolean algebra, then for any  $b \neq 0, 1 \in B$ , there is a basis  $J$  of  $B$  that contains  $b$ .*

*Proof.* Observe that by quantifier elimination all non-constant elements of  $B$  realize the same 1-type. But then if  $A = \langle a_i : i < \omega \rangle$  is a basis for  $B$ , the automorphism  $\alpha$  of  $B$  (guaranteed by  $\aleph_0$ -categoricity) which takes  $a_1$  to  $b$  takes  $A$  to  $\alpha(A)$  which is a basis containing  $b$ .  $\square_{3.1.9}$ .

The next result is used in step 2 of the proof of Claim 3.3.5.

**Lemma 3.1.10.** *Let  $B_1 \subseteq B_2$  be countable Boolean algebras and suppose  $I_2$  is an ideal of  $B_2$  and  $J_1$  is a countable subset of  $B_2$  such that  $J_1$  is independent from  $B_1$  modulo  $I_2$ . If  $b$  is also independent from  $B_1$  modulo  $I_2$  and  $b \in \langle J_1 \cup I_2 \rangle_{B_2}$ , then there is a  $J'_1$  such that  $b \in J'_1$ ,  $J'_1$  is independent from  $B_1$  modulo  $I_2$  and each of  $J_1$  and  $J'_1$  generates (with  $I_2$ ) the same subalgebra of  $B_2$ .*

*Proof.* Let  $b^*$  be the image of  $b$  when  $\pi$  projects  $B_2$  onto  $B_2/I_2$  and  $B_3$  denote the image of  $\pi(\langle J_1 \cup I_2 \rangle_{B_2})$ . By Lemma 3.1.9, there is a  $J''_1 \subset B_2/I_2$  with  $b^* \in J''_1$  that freely generates  $B_3$ . Now choose  $J'_1$  by choosing a preimage for each element of  $J''_1$  and the result follows by Observation 3.1.2.2.  $\square_{3.1.10}$

## 3.2 Defining the Complete Sentence

In this subsection we construct a complete  $L_{\omega_1, \omega}$ -sentence  $\phi$ , essentially the ‘existential-completion’ of Example 3.0.3. We show in Section 3.3 in an extension of *ZFC*, that  $\phi$  has maximal models in  $\lambda$  for arbitrarily large  $\lambda$  less than the first measurable cardinal.

Each model is a member of the class  $\mathbf{K}$  of Example 3.0.3; but Definition 3.2.2 describes the finitely generated models. This section is devoted to the construction of a countable generic structure for that class; the details of the construction will be essential for the main argument in the next section. Our goal is to build this generic structure as a Fraïssé-style limit of finitely generated structures; in each of these structures  $P_0^M$  and  $P_4^M$  will be finite.

**Definition 3.2.1.**  $\tau$  is a vocabulary with unary predicates  $P_0, P_1, P_2, P_4$ , binary  $R, E, \wedge, \vee$ , unary functions  $\bar{\phantom{x}}, G_1$ , constants  $0, 1$  and unary (partial) functions  $F_n$ , for  $n < \omega$ .

We originally introduced the properties of  $\mathbf{K}_1$ , in two stages ( $\mathbf{K}^{-1}$  and  $\mathbf{K}_1$ ) simply to allow the reader to absorb the definition more slowly. It turned out in [BS2x], that the class  $\mathbf{K}_{-1}$  plays an independent role.

We will study several classes  $\mathbf{K}$  with various subscripts and subscripts. In general for a class  $\mathbf{K}_{< \aleph_0}^i$  denotes a class of finitely generated structures and either  $\hat{\mathbf{K}}^i$  or  $\mathbf{K}_i$  denotes the class of all direct limits of models from  $\mathbf{K}_{< \aleph_0}^i$ .

We use the word finitely generated in the usual sense. We have a vocabulary with function symbols; each element of a model is given by a term in the finite set of generators. Thus, if  $M$  is finitely generated  $P_0^M$  and  $P_2^M$  must be finite and  $P_1^M$  is countable

**Definition 3.2.2.**  $\mathbf{K}_{< \aleph_0}^{-1}$  is the class of finitely generated structures  $M$  satisfying.

1.  $P_0^M, P_1^M, P_2^M$  partition  $M$ .
2.  $(P_1^M, 0, 1, \wedge, \vee, <, \bar{\phantom{x}})$  is a Boolean algebra ( $\bar{\phantom{x}}$  is complement).
3.  $R \subset P_0^M \times P_1^M$  with  $R(M, b) = \{a : R^M(a, b)\}$  and the set of  $\{R(M, b) : b \in P_1^M\}$  is a Boolean algebra.  $f^M : P_1^M \mapsto \mathcal{P}(P_0^M)$  by  $f^M(b) = R(M, b)$  is a Boolean algebra homomorphism into  $\mathcal{P}(P_0^M)$ .  
Note that  $f$  is not<sup>7</sup> in  $\tau$ ; it is simply a convenient abbreviation for the relation between the Boolean algebra  $P_1^M$  and the set algebra on  $P_0$  by the map  $b \mapsto R(M, b)$ .
4.  $P_{4,n}^M \subseteq P_1^M$  is the set containing each join of  $n$  distinct atoms from  $M$ ;  $P_4^M$  is the union of the  $P_{4,n}^M$ ;  $P_4^M$  has a maximum element often denoted<sup>8</sup> by  $b^*$ . That is,  $P_4^M$  is the set of all finite joins of atoms (in  $P_1^M$ ). If  $b_1 \neq b_2$  are in  $P_4^M$  then  $R(M, b_1) \neq R(M, b_2)$ .
5.  $G_1^M$  is a bijection from  $P_0^M$  onto  $P_{4,1}^M$ , which by 4) is the non-trivial atoms of  $P_1^M$ , such that  $R(M, G_1^M(a)) = \{a\}$ .
6.  $P_2^M$  is finite (and may be empty). Further, for each  $c \in P_2^M$  the  $F_n^M(c)$  are functions from  $P_2^M$  into  $P_1^M$ .
7. If  $a \in P_{4,1}^M$  and  $c \in P_2^M$  then for all but finitely many  $n$ ,  $a \notin_{\mathcal{M}} F_n^M(c)$ . This implies for each  $x \in P_0^M$ ,  $\bigcap_n \{x : (G_1(x) \in F_n^M(c))\} = \emptyset$ .
8.  $P_1^M$  is generated as a Boolean algebra by  $P_4^M \cup \{F_n^M(c) : c \in P_2^M, n \in \omega\} \cup X$  where  $X$  is a finite subset of  $P_1^M$ .

We denote by  $\mathbf{K}^{-1}$  the class of direct limits of models in  $\mathbf{K}_{<\aleph_0}^{-1}$ .

We now add requirements to Definition 3.2.2, to ensure that no elements of  $P_1^M$  are needed as generators and to lay the ground for the study of free extensions. (See Definition 3.2.11.) We refine the class  $\mathbf{K}_{<\aleph_0}^{-1}$  from Definition 3.2.2 to a class  $\mathbf{K}_{<\aleph_0}^1$ ; here the structure is witnessed by a family of witnesses  $\langle n_*, \mathbf{B}, b^* \rangle$ . The class of direct limits of these finitely generated structures generate will be denoted  $\mathbf{K}^1$ . From  $\mathbf{K}^1$  we will derive the rich class  $\mathbf{K}_2 = \mathbf{R}$  in Definition 3.2.19.

**Definition 3.2.3.**  $M$  is in the class of structures  $\mathbf{K}_{<\aleph_0}^1$  if  $M \in \mathbf{K}_{<\aleph_0}^{-1}$  and there is a witness  $\langle n_*, \mathbf{B}, b^* \rangle$  such that:

1.  $b^* \in P_1^M$  is the supremum of the finite joins of atoms in  $P_1^M$ . Further, for some  $k$ ,  $\bigcup_{j \leq k} P_{4,j}^M = \{c : c \leq b^*\}$  and for all  $n > k$ ,  $P_{4,n}^M = \emptyset$ .
2.  $\mathbf{B} = \langle B_n : n \geq n_* \rangle$  is an increasing sequence of finite Boolean subalgebras of  $P_1^M$ .
3.  $B_{n_*} \supseteq \{b \in P_1^M : b \leq b^*\} = P_4^M$ ; it is generated by the subset  $P_4^M \cup \{F_n^M(c) : n < n_*, c \in P_2^M\}$ .  
Moreover, the Boolean algebra  $B_{n_*}$  is free over the ideal  $P_4^M$  (equivalently,  $B_{n_*}/P_4^M$  is a free Boolean algebra<sup>9</sup>.)
4.  $\bigcup_{n \geq n_*} B_n = P_1^M$ .

<sup>7</sup>The subsets of  $P_0^M$  are not elements of  $M$ .

<sup>8</sup>But  $b^*$  is not constant in the vocabulary; as the models are extended,  $b^*$  changes.

<sup>9</sup>A further equivalence:  $|Atom(B_{n_*})| - |P_{4,1}^M|$  is a power of two.

5. For each  $c \in P_2^M$  the  $F_n^M(c)$  for  $n < \omega$  are distinct and independent over  $\{0\}$ .
6. The set  $\{F_m(c) : m \geq n_*, c \in P_2^M\}$  (the enumeration is without repetition) is free from  $B_{n_*}$  over<sup>10</sup>  $\{0\}$ .  $B_{n_*} \supseteq P_4^M$  and  $F_m(c) \wedge b^* = 0$  for  $m \geq n_*$ . (In this definition,  $0 = 0^{P_1^M}$ .)
- In detail, let  $\sigma(\dots x_{c_i} \dots)$  be a Boolean algebra term in the variables  $x_{c_i}$  (where the  $c_i$  are in  $P_2^M$ ) which is not identically 0. Then, for finitely many  $n_i \geq n_*$  and a finite sequence of  $c_i \in P_2^M$ :

$$\sigma(\dots F_{n_i}(c_i) \dots) > 0$$

and some  $n < \omega$ . Further, for any non-zero  $d \in B_{n_*}$  with  $d \wedge b^* = 0$ , (i.e.  $d \in B_n - P_M^4$ ),

$$\sigma(\dots F_n(c) \dots) \wedge d > 0.$$

7. For every  $n \geq n_*$ ,  $B_n$  is generated by  $B_{n_*} \cup \{F_m(c) : n > m \geq n_*, c \in P_2^M\}$ . Thus  $P_1^M$  and so  $M$  is generated by  $B_{n_*} \cup P_2^M$ .

**Remark 3.2.4.** The first part of Condition 6 of Definition 3.2.3 implies condition 8 of Definition 3.2.2. The second part of condition 6 implies, in particular, that if  $b \in P_1^M - P_4^M$ , there are infinitely many elements below  $b$  in  $P_1^M$ . Note that the free generation condition of 6) is not preserved by arbitrary direct limits; in particular it will fail in the  $P_0$ -maximal model of cardinality  $\lambda$ . However, our construction in an extension of ZFC of  $\mathbf{K}_1$ -free extensions (Definition 3.2.11) will guarantee the  $\mathbf{K}_1$ -freeness of submodels that are less than  $\lambda$ -generated.

Note that if  $\langle n_*, \mathbf{B}, b^* \rangle$  witnesses  $M \in \mathbf{K}_{< \aleph_0}^1$  then for any  $m \geq n_*$ , so does  $\langle m, \mathbf{B}, b^* \rangle$ .

The following lemma shows the prototypical models in  $\mathbf{K}_{< \aleph_0}^1$  in fact exhaust the class. Note that each  $P_1^M$  is an atomic Boolean algebra.

**Lemma 3.2.5.** For any  $M \in \mathbf{K}_{< \aleph_0}^1$ ,  $P_1^M$  has a natural decomposition as a product of an atomic and an atomless Boolean algebra.

Proof. Let  $M \in \mathbf{K}_{< \aleph_0}^1$ , witnessed by  $\langle n_*, \mathbf{B}, b^* \rangle$ . Then the atomic part,  $P_4^M$ , is the collection of elements of  $P_1^M$  that are  $\leq b^*$ . And the independent generation by the  $F_n^M(c_i)$  for  $n \geq n_*$  and  $c_i \in P_2^M$  shows the quotient  $P_1^M/P_4^M$  is atomless.  $\square_{3.2.5}$

Condition Definition 3.2.3.3 guarantees:

**Lemma 3.2.6.** Each structure in  $\mathbf{K}_{< \aleph_0}^1$  is finitely generated by  $P_0^M \cup P_2^M$ .

**Lemma 3.2.7.**  $\mathbf{K}_{< \aleph_0}^1$  is countable.

Proof. Let  $M$  be in  $\mathbf{K}_{< \aleph_0}^1$ , witnessed by  $\langle n_*, \mathbf{B}, b^* \rangle$ . The isomorphism type of  $M$  is determined by the structure on  $P_4^M$  induced by the  $F_n(c_i)$  and  $c_i \in P_2^M$ . If  $m \geq n_*$ ,  $F_m(c_i) \wedge b^* = 0$  so they leave no trace on  $P_4^M$ . Since this tail,  $\{F_m(c_i) : m \geq n_*\}$  generates an atomless boolean algebra in the sense of  $P_1^M$ , that boolean algebra is  $\aleph_0$  categorical. But there can be only countably many structures induced on the finite  $P_4^M$  by the countable set  $F_n(c_i)$  through the formulas  $x < F_n(c_i)$  which determine the values of  $R$  on  $P_4^M$  since only the  $F_m(c_i)$  for  $m < n_*$  have non-empty intersection with  $P_4^M$  (i.e. are above atoms) and  $P_2^M$  is finite.  $\square_{3.2.7}$

<sup>10</sup>As in Definition 3.1.1.2 with  $X = \emptyset$ .

**Definition 3.2.8.** The class  $\mathbf{K}_1 = \hat{\mathbf{K}}_{<\aleph_0}^1$  is the collection of all direct limits of models in  $\mathbf{K}_{<\aleph_0}^1$ .

**Lemma 3.2.9.** There is a minimal model  $M_{min}$  of  $\mathbf{K}_{<\aleph_0}^1$ , that can be embedded in any model of  $\mathbf{K}_1$ .

Proof. Let  $P_0^{M_{min}}$  be empty, so  $P_4^{M_{min}} = \{0\}$ . Also, let  $P_2^{M_{min}}$  be empty.

**Lemma 3.2.10.** If  $M_0 \subseteq M_1$  are both in  $\mathbf{K}_{<\aleph_0}^1$ , witnessed by  $\langle n_*^i, \mathbf{B}^i, b_*^i \rangle$ , for  $i = 0, 1$ , then for sufficiently large  $n$ ,  $B_n^0 = B_n^1 \cap P_1^{M_0}$ .

Proof. Recall  $\mathbf{B}^i = \langle B_n^i : n < \omega \rangle$ . Since the  $B_n^1$  exhaust  $P_1^{M_1}$ ,  $B_{n_*^0}^0$  is finite, and for  $c \in P_2^{M_0}$  and all  $r$ ,  $F_r^{M_1}(c) = F_r^{M_0}(c)$ , for all sufficiently large  $n$ ,  $B_n^1$  contains the  $F_r^{M_0}(c)$  for  $r < n$  and thus  $B_n^0$ . But if some  $b \in B_n^1 \cap P_1^{M_0}$ , but is not in  $B_n^0$  then for some  $k$ ,  $b \in B_{k+1}^0 - B_k^0$ . But then  $B_{k+1}^0$  is not generated by  $B_n^{0*}$  along with the  $F_r^{M_0}(c)$  for  $r < k$ .  $\square_{3.2.10}$

Note that if the conclusion of Lemma 3.2.10 holds for  $n$ , it holds for all  $m \geq n$ .

We now introduce some special notation for this paper by defining  $\mathbf{K}_1$ -free over ( $\mathbf{K}_1$ -free extension of) for models in  $\mathbf{K}_1$ .  $M_2$  is a  $\mathbf{K}_1$ -free extension of  $M_1$  if not only is the image of  $P_1^{M_2}$  in the Boolean algebra  $P_1^{M_2}/P_4^{M_2}$  a free extension of the image of  $P_1^{M_1}$  but the  $F_n(c)$  satisfy technical conditions which allow the preservation of this condition under unions of chains.

**Definition 3.2.11.** When  $M_1 \subseteq M_2$  are both in  $\mathbf{K}_1$ , we say  $M_2$  is  $\mathbf{K}_1$ -free over or is a  $\mathbf{K}_1$ -free extension of  $M_1$  and write  $M_1 \subseteq_{fr} M_2$ , witnessed by  $(I, H)$  when

1.  $I \subset P_1^{M_2} - (P_1^{M_1} \cup P_4^{M_2})$  satisfies i)  $I \cup P_1^{M_1} \cup P_4^{M_2}$  generates  $P_1^{M_2}$  and ii)  $I$  is independent from  $P_1^{M_1}$  modulo  $P_4^{M_2}$  in  $P_1^{M_2}$ . (Definition 3.1.1.2.)
2. There is a function  $H$  from  $P_2^{M_2} \setminus P_2^{M_1}$  to  $\mathbb{N}$  such that the  $F_n(c)$  for  $n \geq H(c)$  are distinct and

$$\{F_n^M(c) : c \in P_2^{M_2} \setminus P_2^{M_1} \text{ and } n \geq H(c)\} \subset I$$

and for every  $c \neq d \in P_2^{M_2}$ ,  $\{n : (\exists m) F_m^{M_2}(c) = F_m^{M_2}(d)\}$  is finite.

We say  $M$  is  $\mathbf{K}_1$ -free over the empty set or simply  $\mathbf{K}_1$ -free if  $M$  is a  $\mathbf{K}_1$ -free extension of  $M_{min}$ .

**Lemma 3.2.12.** 1. If  $M_1 \subseteq_{fr} M_2$  by  $(I_1, H_1)$  and  $M_2 \subseteq_{fr} M_3$  by  $(I_2, H_2)$  then  $M_1 \subseteq_{fr} M_3$  by  $(I_1 \cup I_2, H_1 \cup H_2)$ . Thus,  $\subseteq_{fr}$  is a partial order.

2. More generally, if  $M_\alpha$  with  $\alpha < \delta$  is a continuous  $\subseteq_{fr}$ -increasing sequence then  $M = \bigcup M_\alpha$  satisfies  $M_\alpha \subseteq_{fr} M$  witnessed by  $(\bigcup_{\alpha < \beta < \delta} I_\beta, \bigcup_{\alpha < \beta < \delta} H_\beta)$ .

Proof. By Lemma 3.1.8 (taking the ideals as  $P_4^{M_2}$  and  $P_4^{M_3}$ ),  $I_1 \cup I_2$  is free from  $P_1^{M_1}$  over  $P_4^{M_3}$ .  $H_1 \cup H_2$  is well-defined since the  $H_i$  are defined on disjoint sets. Part 2 follows by induction. Successors are similar, while limits are automatic.  $\square_{3.2.12}$

**Remark 3.2.13.** In an increasing chain such as that of Lemma 3.2.12.2, if some  $b \in P_1^{M_{\alpha+1}}$  is free from  $P_1^{M_\alpha}$  modulo  $P_4^{M_{\alpha+1}}$  then  $b$  is also free from  $P_1^{M_\alpha}$  over  $P_4^{M_\beta}$  for any  $\beta > \alpha$  since  $P_4^{M_\beta} \cap P_1^{M_{\alpha+1}} = P_4^{M_{\alpha+1}}$ .

The next lemma uses the requirement that the  $B_n$  in the witnessing sequence are free Boolean algebras.

**Lemma 3.2.14.** If  $M_0 \subset M_1$  are both in  $\mathbf{K}_{<\aleph_0}^1$  then  $M_0 \subset_{fr} M_1$ .

Proof. We can assume by Lemma 3.2.10 that the  $n_*^i$  for  $i < 2$  are equal and that  $B_{n_*^1}^1 \cap P_1^{M_0} = B_{n_*^0}^0$ . Since the  $B_{i,n_*}$  are free from  $P_4^{M_i}$  over  $\emptyset$ , we can choose bases  $I_0, I_1$  for  $B_{0,n_*^0}$  and  $B_{1,n_*^1}$  respectively. Now  $I_0 \cup I_1 \cup \{F_n^{M_i}(c) : i < 2, n \geq n_{n_*^i}^i, c \in P_2^{M_i}\}$  is a free basis of  $P_1^{M_i}$  over  $P_4^{M_i}$ . Hence  $(I_2 \setminus I_1) \cup \{F_n^{M_i}(c) : i < 2, n \geq n_{n_*^i}^i, c \in P_2^{M_i}\}$  is the required  $I$  from Definition 3.2.11 with  $H(c) = n_*$  for all  $c$ .  $\square_{3.2.14}$

Crucially, Lemma 3.2.14 fails in general if  $\mathbf{K}_{<N_0}^1$  is replaced by  $\mathbf{K}_1$ . Lemma 3.2.14 immediately yields.

**Corollary 3.2.15.** *Each model  $N$  in  $\mathbf{K}_{<N_0}^1$  is  $\mathbf{K}_1$ -free over the empty set.*

To find large  $\mathbf{K}_1$ -free models we apply Lemma 3.2.12.2 to construct a sequence of  $\mathbf{K}_1$ -free extensions. We now show that if  $M_1$  is  $\mathbf{K}_1$ -free,  $N_1 \subseteq M_1$  and  $N_1 \subseteq N_2$  with  $N_2$  a finitely generated extension of the finitely generated substructure  $N_1$ , then  $M_1$  and  $N_2$  can be amalgamated over  $N_1$ . Note that by Lemma 3.2.14, on  $\mathbf{K}_{<N_0}^1$ ,  $\subseteq$  is the same as  $\subseteq_{fr}$ . There are three key ingredients in the amalgamation proof:  $N_1$  and  $N_2$  must be finitely generated; this is reflected positively in the ability to employ the witnessing sequences  $\mathbf{B}^i$  in the proof but also by the key role in the proof of the finite set  $P_{4,1}^{N_2} - P_{4,1}^{N_1}$ . Secondly,  $M_1$  must be  $\mathbf{K}_1$ -free. Thirdly, we must ensure that ‘atomicity’ is preserved in constructing extensions of Boolean algebra so the definitions of  $P_4$  and  $P_{4,1}$  are ‘absolute’ between models. It is this third condition which drives the complexity of steps 1 to 3 in the following proof. The free amalgam  $D = A \otimes_C B$ , where either of  $A, B$  has only finitely many atoms must destroy the atomicity of some elements. (If  $a$  is an atom of  $A$  and  $b_1, \dots, b_n$  are the atoms of  $B$ , for at least one  $i$ ,  $A \otimes_C B \models 0 < a \wedge b_i < a$ .) Thus we will have to construct a quotient algebra of the free amalgam in step 3 below in order to find an amalgam which does not destroy atoms.

**Theorem 3.2.16.** *Suppose  $M_1 \in \widehat{\mathbf{K}} = \mathbf{K}_1$  is  $\mathbf{K}_1$ -free and  $N_1 \subset M_1$ . Let  $N_1 \subset N_2$  with both in  $\mathbf{K}_{<N_0}^1$ . Then there are an  $M_2$  and an  $f$  such that:*

1.  $M_2 \in \mathbf{K}_1$ ,  $M_1 \subseteq_{fr} M_2$  and so  $M_2$  is  $\mathbf{K}_1$ -free.
2.  $f$  maps  $N_2$  into  $M_2$  over  $N_1$ . Moreover, the image in  $M_2$  of  $N_2$  is  $\mathbf{K}_1$ -free over  $N_1$ .

Proof. We lay out the situation in more detail.  $M_1$  is  $\mathbf{K}_1$ -free means that  $M_1$  is  $\mathbf{K}_1$ -free over  $M_{min}$  by  $(I_1, H_1)$ . For  $i = 1, 2$ , let  $\langle n_*^i, \mathbf{B}^i, b_*^i \rangle$  witness that  $N_i \in \mathbf{K}_{<N_0}^1$ . Suppose  $N_1 \subseteq_{fr} N_2$  is witnessed by  $(I_2, H_2)$ . Invoking Lemmas 3.2.10 and 3.2.4, we can rename  $n_*^i$  and rechoose  $n_*$  for  $N_2$  so that  $n_*^1 = n_*^2 = n_*$  and  $B_n^1 = B_n^2 \cap N_1$  for  $n \geq n_*$ , and (since  $P_2^{N_2}$  is finite) for each  $c \in P_2^{N_1}$ ,  $H_1(c) \leq n_*$ . Let  $J_1 \subset B_{n_*}^1$  be the pre-image of the basis of  $B_{n_*}^1 / P_4^{N_1}$ . Then, since  $J_1 / P_4^{N_1}$  is a generating set of  $B_{n_*}^1 / P_4^{N_1}$ , for each  $b \in B_{n_*}^1$ , there is a Boolean combination  $b'$  of elements of  $J_1$  such that  $b' \triangle b \in P_4^{N_1}$ . Note also, that by our choice of  $n_*$  (Definition 3.2.3.6), if  $b \in P_1^{N_2}$  is above an atom of  $P_1^{N_2}$ ,  $b \in B_{n_*}^1$ . Let  $k = |P_{4,1}^{N_2} - P_{4,1}^{N_1}|$ , fix  $a_0 \dots a_{k-1}$  listing a new set  $A$ , and let  $f$  be 1-1 function from  $P_{4,1}^{N_2} - P_{4,1}^{N_1}$  onto  $A$ ;  $A$  contains an image of each new atom in  $N_2$ .

**Step 1:** Construct a Boolean algebra  $\mathbb{B}_1$  that is generated by  $P_1^{M_1} \cup A$  and so that the atoms of  $\mathbb{B}_1$  are  $P_{4,1}^{M_1} \cup A$ . For this demand, let  $\mathcal{D}_\ell$ , for each  $\ell < k$ , be an ultrafilter of the Boolean algebra  $P_1^{M_1}$ , disjoint from  $I_1 - J_1$  such that for  $b \in P_1^{N_1}$ ,  $b \in \mathcal{D}_\ell$  if and only if  $N_2 \models f^{-1}(a_\ell) \leq b$ . (Such an ultrafilter exists as the set  $\{b \in P_1^{N_1} : f^{-1}(a_\ell) \leq b\}$ , as noted in last paragraph, contains no element of  $I_1 - J_1$  and is a filter on  $P_1^{N_1}$  that can be extended to an ultrafilter on the Boolean algebra  $P_1^{M_1}$ .)

Now let  $X$  be the union of the Stone space of  $P_1^{M_1}$ , denoted  $S(P_1^{M_1})$  with  $A$ . For  $b \in P_1^{M_1}$ , let

$$X_b = \{d \in S(P_1^{M_1}) : b \in d\} \cup \{a_\ell : b \in \mathcal{D}_\ell\}.$$

Now let  $\mathbb{B}_1$  be the subalgebra of  $\mathcal{P}(X)$  generated by the  $\{X_b : b \in P_1^{M_1}\} \cup A$ . Now, generalizing the Stone representation theorem, we embed  $P_1^{M_1} \cup A$  into  $\mathbb{B}_1$  by a map  $g$ ; let  $g(b) = X_b$  for  $b \in P_1^{M_1}$  and  $g(a) = \{a\}$  for  $a \in A$ .

Since  $P_{4,k}^{N_2} \cap P_1^{N_1} = P_{4,k}^{N_1}$ , there can be no non-zero  $b \in P_4^{N_1}$  and so no non-zero  $b \in P_4^{N_2}$  with  $N_2 \models b < f^{-1}(a_i)$ . Note i) that for  $b \in P_1^{M_1}$ ,  $b \in \mathcal{D}_\ell$  iff  $\mathbb{B}_1 \models f^{-1}(a_i) \leq b$  and ii) that  $e$  is an atom of  $P_1^{M_1}$  if and only if  $X_e$  is a principal ultrafilter in  $\mathbb{B}_1$ . Thus, the atoms of  $\mathbb{B}_1$  are exactly  $P_{4,1}^{M_1} \cup A$ .

**Step 2:** Find a sub-Boolean algebra  $\mathbb{B}^*$  of  $\mathbb{B}_1$  that is a suitable base for amalgamating  $\mathbb{B}_1$  with  $P_1^{N_2}$ . For this, denote by  $\mathbb{B}^*$  the sub-Boolean algebra of  $\mathbb{B}_1$  generated by  $g(P_1^{N_1} \cup A)$ . Denote by  $\mathbb{B}^*$  the sub-Boolean algebra of  $P_1^{N_2}$  generated by  $P_1^{N_1} \cup f^{-1}(A)$ .

Compose  $g$  with the identity on  $P_1^{N_1}$  with the map  $f$  given in the first paragraph of the proof using the operations of  $N_2$  to give a map from  $P_1^{N_1} \cup (P_{4,1}^{N_2} - P_{4,1}^{N_1})$  into  $\mathbb{B}_1$  that takes  $\mathbb{B}^*$  to  $\mathbb{B}^*$ . We also denote this map by  $f$ .

To ease notation, we will suppress  $g$  and pretend that  $P_1^{N_1} \cup A$  is actually<sup>11</sup> contained in  $\mathbb{B}_1$ .

**Step 3:** Construct a Boolean algebra  $\mathbb{B}_2$  that is an amalgam of  $P_1^{M_1}$  and  $P_1^{N_2}$  over  $f(\mathbb{B}^*) = \mathbb{B}^*$  such that the atoms of  $\mathbb{B}_2$  are  $P_{4,1}^{\mathbb{B}_1} \cup A$ .  $\mathbb{B}_2$  is a quotient of the pushout  $\mathbb{B}'_2$  of  $\mathbb{B}_1$  and  $P_1^{N_2}$  over the sub-Boolean algebra  $\mathbb{B}^*$  of  $\mathbb{B}_1$  generated by  $P_1^{N_1}$  and  $A$ . The crux of the proof is the specification of the atoms of  $\mathbb{B}_2$ ; it allows us to extend the amalgam of Boolean algebras to an amalgam in  $\mathbf{K}_1$ .

By standard properties of the coproduct (Lemma 3.1.6),  $\mathbb{B}_1$  and  $P_1^{N_2}$  are disjointly embedded over  $\mathbb{B}^*$  into their coproduct  $\mathbb{B}'_2$ . We will regard the embedding of  $\mathbb{B}_1$  as the identity and denote by  $f$  the embedding of  $P_1^{N_2}$  extending our earlier  $f$  mapping the sub-Boolean algebra  $\mathbb{B}^*$  of  $P_1^{N_2}$  into  $\mathbb{B}_1$ . Crucially, while  $\mathbb{B}_1$  and  $f(P_1^{N_2})$  are sub-Boolean algebras of  $\mathbb{B}'_2$ ; they are *not* ideals.

The atoms of the amalgamation base  $\mathbb{B}^*$  remain atoms in  $\mathbb{B}'_2$  as: if  $a$  is an atom of  $\mathbb{B}^*$  then every  $b_1 \in \mathbb{B}_1$  satisfies  $b_1 \wedge a = 0$  or  $b_1 \wedge a = a$  and similarly for  $b_2 \in P_1^{N_2}$  and therefore also for  $b_1 \wedge b_2$ ; using disjunctive normal form, no element of  $\mathbb{B}'_2$  contradicts the atomicity of an atom of  $\mathbb{B}^*$ . Recall  $N_1 \subseteq_{fr} N_2$  is witnessed by  $(I_2, H_2)$ . To guarantee the atoms of  $\mathbb{B}_1 \setminus \mathbb{B}^*$  (i.e.  $P_{4,1}^{M_1} - P_{4,1}^{N_1}$ ) are atoms of  $\mathbb{B}_2$ , we divide  $\mathbb{B}'_2$  by the ideal<sup>12</sup>,  $\mathfrak{S}$ , generated by

$$\mathfrak{S}_0 = \{a \wedge f(b) : a \in P_{4,1}^{\mathbb{B}_1} \setminus P_{4,1}^{N_1}, b \in I_2, a \wedge f(b) < a\}.$$

(\*) Since each element of  $\mathfrak{S}$  is *strictly* below a finite join of atoms in  $\mathbb{B}'_2$  (actually in  $\mathbb{B}_1$ ),  $\mathfrak{S}$  is a proper ideal of  $\mathbb{B}_1$  bounded by elements of  $P_4^{\mathbb{B}_1}$ ; but  $\mathfrak{S} \cap P_4^{\mathbb{B}_1} = \emptyset$ . Indeed, by freeness of the coproduct,  $\mathfrak{S} \cap \mathbb{B}_1 = \emptyset$ . Note that the subalgebra of  $\mathbb{B}'_2$  generated by  $f(I_2)$  is a subset of  $\mathbb{B}_1$  so it is disjoint from  $\mathfrak{S}$ .

Let  $\pi$  map  $\mathbb{B}'_2$  onto  $\mathbb{B}_2 =_{def} \mathbb{B}'_2 / \mathfrak{S}$ . By (\*), no element of  $\mathbb{B}_1 \cup f(I_2)$  is collapsed by the map  $\pi : \mathbb{B}'_2 \rightarrow \mathbb{B}_2$ . Thus,  $\pi$  is 1-1 on  $\mathbb{B}_1 \cup f(P_1^{N_2})$  and  $\mathbb{B}_2$  is a disjoint amalgamation of the Boolean algebras  $\pi(\mathbb{B}_1)$  and  $\pi(f(P_1^{N_2}))$ . Since  $\mathbb{B}'_2$  is generated by  $\mathbb{B}_1 \cup f(P_1^{N_2})$ , without loss of generality, we can assume the preimage of a potential atom of  $\mathbb{B}_2$  has the form  $a \wedge f(b)$  where  $a \in \mathbb{B}_1 - \mathbb{B}^*$  is an atom of  $\mathbb{B}_1$  and  $b \in P_1^{N_2} - \mathbb{B}^*$ . By the freeness property of coproducts<sup>13</sup>,  $\mathbb{B}'_2 \models a \wedge f(b) < a$ , so  $\pi(a \wedge f(b)) = 0$  and  $\pi(a) = a$  is an atom.

**Step 4:** The actual  $\tau$ -amalgam. Now to define the extension  $M_2$ , let  $P_1^{M_2} = \mathbb{B}_2$ ,  $P_{4,1}^{M_2} = P_{4,1}^{M_1} \cup A$ ;  $P_4^{M_2}$  is the set of finite joins of these atoms. Then, let  $P_2^{M_2} = P_2^{M_1} \cup P_2^{N_2}$  and the  $F_n^{M_2}(c)$  be as in whichever of  $M_1, N_2$  in which  $c$  lies. Define  $P_0^{M_2}$  to be a set in 1-1 correspondence with  $P_4^{M_2}$  and call the correspondence  $G_1^{M_2}$ . Finally, we must define  $R^{M_2}$ : for each  $b \in P_1^{M_2}$ , let  $R(M_2, b) = \{a \in P_0^{M_2} : G_1^{M_2}(a) \leq^{M_2} b\}$ .

<sup>11</sup>Clearly, this could be achieved by choosing a new copy of  $\mathbb{B}_1$ .

<sup>12</sup>Abusing notation, since  $\mathbb{B}_1$  is not a  $\tau$ -structure, we write  $P_{4,1}^{\mathbb{B}_1}$  for the set of atoms of  $\mathbb{B}_1$  and  $P_4^{\mathbb{B}_1}$  for their finite joins.

<sup>13</sup> $\mathbb{B}'_2$  is freely generated as a Boolean algebra by (isomorphic copies of)  $\mathbb{B}_1$  and  $P_1^{N_2}$  over  $\mathbb{B}^*$ .

By Lemma 3.1.5,  $I_2$  is independent from  $\mathbb{B}_1$  over  $\mathfrak{S}$  in  $\mathbb{B}'_2$  and so, by (\*),  $\pi(f(I_2))$  is independent from  $P_1^{M_2} = \mathbb{B}_2$  over  $P_4^{M_2}$  in  $M_2$ . So  $M_1 \subset_{\text{fr}} M_2$  with  $H_{M_2}(c) = n^*$  for  $c \in P_2^{M_2}$ .  $\square_{3.2.16}$

Note the  $M_0, M_1, M_2, M_3$  in the next argument are  $N_0, M_1, N_2, M_2$  in Lemma 3.2.16.

**Corollary 3.2.17.**  $(\mathbf{K}_{<\aleph_0}^1, \subseteq)$  has the disjoint amalgamation property.

Proof. We know every member of  $\mathbf{K}_{<\aleph_0}^1$  is  $\mathbf{K}_1$ -free over the empty set. So the amalgamation becomes a special case of Lemma 3.2.15 when we add a proof that the amalgam is in  $\mathbf{K}_{<\aleph_0}^1$ . We have the following situation.  $M_0$  is  $\mathbf{K}_1$ -free over the minimal model  $M_{\text{min}}$ . That is, there are  $J_0, I_0, H_0$  such that  $J_0$  generates  $B_{0, n_0^q}$  and  $(J_0 \cup I_0, H_0)$  witness that  $M_0$  is a  $\mathbf{K}_1$ -free extension of the minimal model  $M_{\text{min}}$ . Similarly there are for  $i = 1, 2$ ,  $J_i, I_i, H_i$  such that  $J_i$  generates  $B_{i, n_i^q}$  and  $(J_i \cup I_i, H_i)$  that witness that  $M_i$  is a  $\mathbf{K}_1$ -free extension of the minimal model  $M_0$ .

Choose  $n_*$  as the maximum of  $n_i^q$  for  $i < 3$ ; we can assume the  $n_i^q$  for  $i < 3$  are equal and that  $B_{2, n_*^q} \cap B_{0, n_*^q} = B_{0, n_*^q}$  for  $i = 1, 2$ . Rechoosing  $n_*$  by Lemma 3.2.10 we can assume for all  $n \geq n_*$ ,  $B_n^1 \cap P_1^{M_0} = B_n^0 = B_n^2 \cap P_1^{M_0}$ .

Choose  $M_3$  by Lemma 3.2.16. Let  $b_{n_*}^3 = b_{n_*}^1 \wedge b_{n_*}^2$ . Now let  $B_{3, n_*}$  be the subboolean algebra of  $M_3$  generated by  $J_0 \cup J_1 \cup J_3$  and for  $n \geq n_*$ ,  $B_{3, n}$  be generated by  $B_{1, n} \cup B_{2, n}$ . This is the required witnessing sequence.  $\square_{3.2.17}$

Since  $\mathbf{K}_{<\aleph_0}^1$  has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model. This construction can be rearranged in order type  $\omega$  so by Theorem 3.2.16 and Lemma 3.2.12 the generic is  $\mathbf{K}_1$ -free.

**Corollary 3.2.18.** *There is a countable generic model  $M$  for  $\mathbf{K}_0$ . We denote its Scott sentence by  $\phi_M$ . Moreover  $M$  is  $\mathbf{K}_1$ -free.*

Aligning our notation with earlier sections of the paper we note the models of  $\phi_M$  are rich in the sense defined there.

**Definition 3.2.19.** *We say a model  $N$  in  $\mathbf{K}_1$  is rich if for any  $N_1, N_2 \in \mathbf{K}_{<\aleph_0}^1$  with  $N_1 \subseteq N_2$  and  $N_1 \subseteq M$ , there is an embedding of  $N_2$  into  $N$  over  $N_1$ . We denote the class of rich models in  $\mathbf{K}_1$  as  $\mathbf{K}_2$  or  $\mathbf{R}$ .*

Lemma 3.2.16 finds a  $\mathbf{K}_1$ -free extension of each  $\mathbf{K}_1$ -free model in  $\mathbf{K}_1$ ; more strongly:

**Corollary 3.2.20.** *Let  $M_1$  be  $\mathbf{K}_1$ -free. There exists an  $M_2 \in \mathbf{K}_2$  which is a proper  $\mathbf{K}_1$ -free extension of  $M_1$ .*

Proof. Iterate Corollary 3.2.16 as in Corollary 2.1.6 to obtain a rich model; note that  $\mathbf{K}_1$ -freeness is preserved at each stage.  $\square_{3.2.20}$

The crucial distinction from Corollary 2.1.6 is that here we extend only ‘ $\mathbf{K}_1$ -free models’ in  $\mathbf{K}_1$  to  $\mathbf{K}_2$ . While *this* construction applied to models in  $\mathbf{K}_2$  will necessarily increase  $P_0$  (case 2 of Construction 3.3.9), we can find extensions in  $\mathbf{K}_1$  which do not extend  $P_0$  or  $P_1$  but only  $P_2$  (case 4 of Construction 3.3.9).

For the construction in Section 3.3 we require two crucial properties of the generic model.

**Lemma 3.2.21.** *If  $M$  is the generic model then*

*$i$  if  $b_1 \neq b_2$  are in  $P_1^M - P_4^M$  then  $R(M, b_1) \neq R(M, b_2)$ , i.e. the map  $f$  from Definition 3.2.2.3 is injective.*

ii For any  $a \in P_0^M$ ,  $b \in P_1^M$ ,  $M \models R(a, b) \vee R(a, b^-)$ . Indeed,  $P_1^M$  is an atomic Boolean algebra.

iii For each  $b \in P_1^M - P_4^M$ ,  $R^M(M, b)$  is infinite and coinfinite.

Proof. For i) fix a finitely generated model  $M_0$  containing  $b_1, b_2$ ; there is a finitely generated extension  $M_1$  in  $\mathbf{K}_{<\aleph_0}^1$  by adding  $a \in P_0^{M_1}$  with  $R^{M_1}(a, b_1) \wedge \neg R^{M_1}(a, b_2)$ . This shows the injectivity; the other conditions are similar.  $\square_{3.2.21}$

**Lemma 3.2.22.** *If  $M, N \in \mathbf{K}_2$ ,  $M \equiv_{\infty, \omega} N$  so they satisfy the Scott sentence  $\Phi_M$ . Moreover, if  $M \subset N$  and are both in  $\mathbf{K}_2$ ,  $M \prec_{\infty, \omega} N$ .*

Proof: Suppose  $M$  and  $N$  are in  $\mathbf{K}_2$ . We define a back-and-forth between  $M$  and  $N$  for  $\mathbf{a} \in M^n$ ,  $\mathbf{b} \in N^n$  by  $\mathbf{a} \equiv \mathbf{b}$  if they realize the same first order type over the  $\emptyset$  with respect to  $T$ . Fix such  $\mathbf{a} \equiv \mathbf{b}$  and choose  $c \in M$ . The interest is when  $c$  is not in  $A = \text{acl}(\mathbf{a})$ , If  $c \in P_1^M - A$ , let  $A_1 = \langle A \cup \{c\} \rangle_M$ . Since  $M \in \mathbf{K}_1$ ,  $A_1 \in \mathbf{K}_1$ . Now let  $B = \langle \mathbf{b} \rangle_N$  that is equivalent to  $A$ . By richness there exists  $B_1$  isomorphic to  $A_1$  with  $B \subset B_1 \subset N$ .

If  $M \subset N$  and both are in  $\mathbf{K}_2$ , then  $\text{acl}_M(\mathbf{a}) = \text{acl}_N(\mathbf{a})$  for  $\mathbf{a} \in M$ ; this yields the moreover.  $\square_{3.2.22}$

This completes our description of the class  $\mathbf{K}_2$  of rich models and its Scott sentence. At this point we show any  $\mathbf{K}_1$ -free-member of  $\mathbf{K}_2$  has a proper  $\mathbf{K}_1$ -free-extension in  $\mathbf{K}_1$ . In case 2 of Construction 3.3.9, we apply Corollary 3.2.20 to regain a member of  $\mathbf{K}_2$ .

**Lemma 3.2.23.** *If  $M \in \mathbf{K}_2$ , there is an  $N$  such that  $M \subset_{fr} N$ , both are in  $\mathbf{K}_1$ ,  $P_2^N = P_2^M$ ,  $P_0^N = P_0^M$ , and  $P_1^N$  is generated by  $P_1^M \cup \{b\}$  and  $b \in N'$  with  $N \prec N'$ . Moreover given  $u \subseteq P_0^M$ , we can require  $R(N, b) = u$  and  $b$  is free from  $P_1^M$  over  $P_4^N$ . Finally, if  $M$  is  $\mathbf{K}_1$ -free then so is  $N$ .*

Proof: Let  $p(x)$  be the type of an element satisfying  $P_1(x) \wedge \neg P_4(x)$ :

$$\{x \geq G_1(a) : a \in u\} \cup \{G_1(a) \wedge x = 0 : a \in P_4^N \setminus u\} \cup \{b \wedge \sigma(x) \neq a : b \in P_1^M \setminus P_4^M, a \in P_4^M\},$$

where  $\sigma(x)$  ranges over nontrivial Boolean polynomials. Each finite subset  $q$  of  $p$  is satisfied in  $M$  because  $M \in \mathbf{K}_2$ . Thus there is an elementary extension  $N'$  of  $M$  where  $p$  is realized by some  $b$ . Let  $\mathbb{B}$  be the boolean subalgebra of  $P_1^{N'}$  generated by  $P_1^M \cup \{b\}$ . Since  $N'$  satisfies the first order properties of  $\mathbf{K}_2$ , the atoms of  $M$  are atoms of  $\mathbb{B}$ .

Define a  $\tau$ -structure  $N$  with  $P_1^N = \mathbb{B}$ . Interpret  $P_2$  and the  $F_n$  in  $N$  as in  $M$ . Extend  $G_1^M$  and  $P_0^M$  so that  $P_0^N = (G_1^N)^{-1}(Y)$ . The structure  $N$  is well-defined; we must prove it is in  $\mathbf{K}_1$ .

Let  $\langle (M_i, Z_i) : i < |M|, Z_i \subset_\omega Y \rangle$  list the pairs of finitely generated  $M_i \subset M$  in  $\mathbf{K}_{<\aleph_0}^1$  and finite subsets  $Z_i$  of  $Y$ . (The  $M_i$  will be repeated.) Let  $N_i \subset N$  with  $P_0^{N_i} = P_0^{M_i} \cup Z_i$ ,  $P_2^{N_i} = P_2^{M_i}$ , and  $P_1^{N_i}$  be the universe of the Boolean subalgebra of  $N$   $\tau$ -generated by  $P_2^{N_i} \cup \{b\} \cup Z_i$ . It is easy to check each  $N_i \in \mathbf{K}_{<\aleph_0}^1$ . Now  $N$  is the direct limit of the finitely generated  $\{N_i : i < |M|\}$  so it is in  $\mathbf{K}_1$ .

Finally  $b$  is free from  $P_1^M$  over  $P_4^N$  since no nontrivial unary polynomial  $\sigma$  satisfies maps  $\sigma(b) \wedge a \in P_4^N$  with  $a \in P_1^M - P_4^M$ . The moreover follows by Definition 3.2.11 from the independence of  $b$ .  $\square_{3.2.23}$

### 3.3 Constructing maximal models in an extension of ZFC

We show that for arbitrarily large cardinals below a measurable cardinal, assuming a mild set theoretic hypothesis described below,  $\mathbf{K}_2$  has maximal models. We begin by defining a pair of set theoretic notions and some specific notions of maximal model.



**Definition 3.3.1** ( $\diamond_S$ ). Given a cardinal  $\kappa$  and a stationary set  $S \subseteq \kappa$ ,  $\diamond_S$  is the statement that there is a sequence  $\langle A_\alpha : \alpha \in S \rangle$  such that

1. each  $A_\alpha \subseteq \alpha$ ;
2. for every  $A \subseteq \kappa$ ,  $\{\alpha \in S : A \cap \alpha = A_\alpha\}$  is stationary in  $\kappa$ .

**Definition 3.3.2** ( $S$  reflects). Let  $\kappa$  be a regular uncountable cardinal and let  $S$  be a stationary subset of  $\kappa$ . For  $\alpha < \kappa$  with uncountable cofinality,  $S$  reflects at  $\alpha$  if  $S \cap \alpha$  is stationary in  $\alpha$ .  $S$  reflects if it reflects at some  $\alpha < \kappa$ .

**Definition 3.3.3.** 1. A model  $M \in \mathbf{K}_2 = \mathbf{R}$  is  $P_0$ -maximal (for  $\mathbf{K}_1$ ) if  $M \subseteq N$  and  $N \in \mathbf{K}_2$  ( $\in \mathbf{K}_1$ ) implies  $P_0^M = P_0^N$ .

2. A model  $M \in \mathbf{K}_2$  is maximal for  $\mathbf{K}_2$  if  $M \subseteq N$  and  $N \in \mathbf{K}_2$  implies  $M = N$ .

Let  $S_{\aleph_0}^\lambda$  denote the stationary set  $\{\delta < \lambda : \text{cf}(\delta) = \aleph_0, \delta \text{ is divisible by } |\delta|\}$ .

We now define a crucial notion.

**Definition 3.3.4** ( $A$ -good defined). Suppose that  $N_n \subset_{\text{fr}} N_{n+1}$  for  $n < \omega$ , is sequence of models,  $\overline{N}$ , in  $\mathbf{K}_1$ . We say a sequence  $\mathbf{b} = \langle b_n : n < \omega \rangle$  is

1. good for  $\overline{N}$  if
  - (a)  $P_2^{N_{n+1}} - P_2^{N_n}$  is infinite;
  - (b) for each  $n$ ,  $b_n \in P_1^{N_{n+1}}$  and  $\{b_n\}$  is free from  $P_1^{N_n}$  over  $P_4^{N_{n+1}}$ ;
  - (c) if  $a \in P_0^{N_i}$ , then for all but finitely many  $n \geq i$ ,  $a \notin R(N_{n+1}, b_n)$ .
2. for  $A \subset \bigcup \overline{N}$ ,  $\mathbf{b}$  is  $A$ -good if each  $b_n \in A$ .
3. and labeled if there is a pair  $(N^{\mathbf{b}}, c^{\mathbf{b}})$  with  $N^{\mathbf{b}} \in \mathbf{K}_1$  and  $N^{\mathbf{b}} \supseteq N_\omega = \bigcup N_n$  such that for each  $n$ ,  $F_n^{N^{\mathbf{b}}}(c^{\mathbf{b}}) = b_n$ . By the definition of  $\mathbf{K}_1$ ,  $\bigcap_n R(N^{\mathbf{b}}, F_n^{N^{\mathbf{b}}}(c^{\mathbf{b}})) = \emptyset$ .

Note that for every  $c \in N_m \subsetneq N_\omega$ , at most finitely many of any good sequence  $\langle b_k : k < \omega \rangle$  occur in the sequence  $F_n^{N^{\mathbf{b}}}(c)$  for  $n < \omega$  (as  $F_n^{N^{\mathbf{b}}}(c) \in N_m$  and for  $k > m$ ,  $b_k \notin N_m$ ).

Any proper  $P_0$ -extension of a model  $M$  induces a non-principal ultrafilter  $A$  on  $P_1^M$ . Claim 3.3.5 is instrumental via case 5 in constructing, for the particular  $M$  under consideration, an ostensibly non-principal  $\aleph_1$ -complete ultrafilter on  $\mathcal{P}(P_0^M)$  which contradicts that  $\lambda$  is not measurable. See 3.3.11.

**Claim 3.3.5.** Suppose that for  $n < \omega$ ,  $\overline{N} = \langle N_n \subset_{\text{fr}} N_{n+1} \rangle$  are in  $\mathbf{K}_1$ . For  $A \subseteq N_\omega$ , if Condition A) holds then so does condition B).

A) There is an  $A$ -good sequence for  $\overline{N}$ .

B) There is a labeled  $A$ -good sequence for  $\overline{N}$ .

Proof. The following construction is for the fixed  $A$ -good sequence  $\mathbf{b}$ . Let  $N = N_\omega \bigcup_{n < \omega} N_n$ . Note that each  $P_1^{N^{\mathbf{b}}} = P_1^N$ ; the extension  $N^{\mathbf{b}}$  only adds an element  $c$  to  $P_2^{N_\omega}$  and interprets the  $F_m^{N^{\mathbf{b}}}(c)$ . The difficulty is that while we know each  $N_{n+1}$  is  $\mathbf{K}_1$ -free over  $N_n$ , witnessed by some  $(I_n, H_n)$ , we don't know  $b_n \in I_n$ . We need to find  $I'_n$  which witnesses both  $N_n \subset_{\text{fr}} N_{n+1}$  and  $b_n \in I'_n$ . After this construction we will choose an  $N^{\mathbf{b}}$  extending  $N$  witnessing goodness.

To find  $I'_n$ , we first find  $(X_n, J_n)$  such that:

1.  $X_n \subseteq P_1^{N_n}$  is finite.
2.  $J_n \subset I_n$  is countable.
3. If  $c \in P_2^{N_{n+1}} - P_2^{N_n}$  then for sufficiently large  $m$ ,  $F_m^{N_{n+1}}(c) \notin J_n$ .
4.  $b_n \in BA(X_n \cup J_n)$ , the Boolean algebra generated by  $X_n \cup J_n$  in  $P_1^{M_{n+1}}$ .

*First step:* First, we construct such an  $(X_n, J_n)$ . Note that  $b_n$  is in a subalgebra generated by a finite subset  $X_n$  of  $P_1^{N_n}$  and a finite subset  $J'_n$  of  $I_n$ .

Now, by 1a) of Definition 3.3.4, fix a sequence  $\langle c_i : i < \omega \rangle$  of distinct elements of  $P_2^{N_{n+1}} - P_2^{N_n}$ . Note that for  $i, j < \omega$  if  $n_i > H_n(c_i)$  and  $n_j > H_n(c_j)$  then  $F_{n_i}^{N_{n+1}}(c_i) \neq F_{n_j}^{N_{n+1}}(c_j)$ . Now we can construct a  $J''_n = \{d_{n,k} : k < \omega\}$  from  $I_n - J'_n$  by  $d_{n,k} = F_m^{N_{n+1}}(c_k)$  for some  $m > H_n(c_k)$ . We now have a countably infinite  $J''_n$  contained in  $I_n - J'_n$  such that for each  $c \in P_2^{N_{n+1}} - P_2^{N_n}$  all but finitely many of the  $F_m^{N_{n+1}}(c)$  are in  $I_n - (J'_n \cup J''_n)$ . Set  $J_n = J'_n \cup J''_n$ .

*Second step:* Now apply Lemma 3.1.10<sup>14</sup> to find  $J_n^*$  with  $J_n^*$  independent from  $P_1^{M_n}$  over  $P_4^{M_{n+1}}$  such that  $\langle J_n^* \cup P_4^{M_{n+1}} \rangle_{P_1^{M_{n+1}}} = \langle J_n \cup P_4^{M_{n+1}} \rangle_{P_1^{M_{n+1}}}$  but  $b_n \in J_n^*$ . Now,  $I'_n$  can be taken as  $(I_n - J_n) \cup J_n^*$ . To ensure that  $N_n \subseteq_{fr} N_{n+1}$  with basis  $I'_n$ , replace  $H_N(c_n)$  by  $H_{N_n}(c_n) + r_n$  where (by Definition 3.2.3) some  $r_n$  bounds the number of  $m$  such that  $F_m^{N_n}(c_n) \in \langle J_n \rangle_{P_1^{M_{n+1}}}$ .

Having found an appropriate basis for  $N = \bigcup N_n$ , we extend  $N$  to  $N^b$  by adding an element  $c^b$  to  $P_2^{N_2}$  and defining  $F_n^{N^b}(c^b) = b_n$ . The sentence immediately before Claim 3.3.5 guarantees that  $N^b$  is  $\mathbf{K}_1$ -free; set  $H^{N^b}(c^*) = 0$ ; thus,  $N^b \in \mathbf{K}_1$ . Since the same  $b_n$  were used, it is clear the labeled sequence is  $A$ -good. (Note that there is no requirement that  $m, n < \omega$ ,  $c \in P_2^{M_0}$ ,  $d \in P_2^{M_1}$  imply  $F_n^{M_1}(c) \neq F_m^{M_1}(d)$ ; we only require that there be only finitely many such conflicts.)  $\square_{3.3.5}$

We now state precisely the main theorem.

**Theorem 3.3.6.** Fix  $\mathbf{K}^1_{<\aleph_0}$ ,  $\mathbf{K}_1 = \hat{\mathbf{K}}^1_{<\aleph_0}$ , and  $\mathbf{K}_2 = \mathbf{R}$  as in Definitions 3.2.3, 3.2.8 and 3.2.19. There is a  $P_0$ -maximal for  $\mathbf{K}_2$  model  $M \in \mathbf{K}_2$  of card  $\lambda$  if there is no measurable cardinal  $\rho$  with  $\rho \leq \lambda$ ,  $\lambda = \lambda^{<\lambda}$ , and there is an  $S \subseteq S^{\aleph_0}_{\aleph_0}$ , that is stationary non-reflecting, and  $\diamond_S$  holds.

Under  $V = L$ , the hypotheses are clearly consistent and imply there are arbitrarily large maximal models of  $\mathbf{R}$  in  $L$ . When a measurable cardinal exists, the consistency of the conditions can be established by forcing; see the article by Cummings in the Handbook of Set Theory [Cum08] or by considering the inner model of a measurable  $L[D]$  where  $D$  is a normal ultrafilter on  $\mu$ .

The argument for Theorem 3.3.6 will have three parts. First, we describe the requirements on a construction of a rich model; then we carry out the construction. Finally, we show the model constructed is  $P_0$ -maximal when  $\lambda$  is below the first measurable and satisfies the other conditions of Theorem 3.3.6.

**Construction 3.3.7 (Requirements).** Fix  $\lambda$  satisfying the cardinal requirements in Theorem 3.3.6. List  $[\lambda]^{<\lambda}$ , the subsets of  $\lambda$  with less than  $\lambda$  elements, as  $\langle U_\alpha : \alpha < \lambda \rangle$  so that each subset is enumerated  $\lambda$  times and  $U_\alpha \subseteq \alpha$ . Since the set of ordinals  $\alpha < \kappa$  such that  $|\alpha|$  divides  $\alpha$  is a cub for any  $\kappa$ , without loss of generality, each  $\alpha \in S$  is a limit ordinal and is divided by  $|\alpha|$ . Let  $\overline{A}^* = \langle A_\delta^* : \delta \in S \rangle$  be a  $\diamond_S$ -sequence.

<sup>14</sup>This is the crucial application of Lemma 3.1.10 which strengthened our notion of independence by getting a standard consequence of exchange, even though exchange fails here.

We will choose  $M_\alpha$  for  $\alpha < \lambda$  by induction to satisfy the following conditions. (Since the universe of  $M$  is a subset of  $\lambda$ , its elements are ordinals so we may talk about their order although the order relation is not in  $\tau$ .)

1.  $M_0$  is isomorphic to the minimal model of  $\mathbf{K}_1$ . For  $\alpha > 1$ ,  $M_\alpha \in \mathbf{K}_2$  has universe an ordinal between  $\alpha$  and  $\lambda$ .
2.  $\langle M_\beta : \beta < \alpha \rangle$  is  $\subseteq$ -continuous.
3. If  $\beta \in \alpha - S$  then  $M_\alpha$  is  $\mathbf{K}_1$ -free over  $M_\beta$ , and  $M_\alpha \in \mathbf{K}_2 = \mathbf{R}$ .
4. If  $\alpha = \beta + 2$  and  $U_\beta \subseteq P_0^{M_\beta}$  then there is a  $b_\beta \in P_1^{M_\alpha}$  such that  $R(M_\alpha, b_\beta) \cap M_{\beta+1} = U_\beta$  and in the Boolean algebra  $P_1^{M_\alpha}$ ,  $\{b_\beta\}$  is free from  $P_1^{M_{\beta+1}}$  modulo  $P_4^{M_\alpha}$ . Moreover  $P_2^{M_\alpha} - P_2^{M_\beta}$  is infinite.
5. If  $\delta \in S$  and  $\alpha = \delta + 1$  then A) implies B), where:
  - A) there is an  $A$ -good sequence  $\bar{\gamma} = \langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$ , where the  $\gamma_{\delta,n}$  are increasing with  $n$  and not in  $S$  such that the  $\langle b_{\delta,n} : n < \omega \rangle$  are good for the  $M_{\gamma_{\delta,n}}$ .
  - B) there is a labeled  $A$ -good sequence  $\hat{\gamma} = \langle \hat{\gamma}_{\delta,n}, \hat{b}_{\delta,n} : n < \omega \rangle$ , for  $\langle M_{\gamma_{\delta,n}} : n < \omega \rangle$  with  $c \in M_{\delta+1}$ .

**Remark 3.3.8.** Condition 5 asserts that for any  $A \subseteq \bigcup_{n < \omega} M_{\gamma_{\delta,n}}$ : if there is an  $A$ -good sequence then there is a labeled  $A$ -good sequence. In the proof of Claim 3.3.5 we, in fact, took the same sequence so the ‘ $A$ ’ is preserved automatically. But for each  $\delta$  we construct only one pair of a  $c$  labeling a sequence  $b_{\delta,n}$ . We fix the relevant  $A$  for application in the first paragraph of 3.3.11; it will be an ultrafilter on  $P_1^M$  induced by a proper extension.

We now carry out the inductive construction.

**Construction 3.3.9. Details**

- Case 1:*  $\alpha = 0$ . Let  $M_0$  be the minimal model from Lemma 3.2.9. The generic can be taken as  $M_1$ .
- Case 2:*  $\alpha = \beta + 1$  and  $\beta \notin S$ . If  $\beta$  is a limit we only have to choose, by Lemma 3.2.20,  $M_\alpha$  to be a  $\mathbf{K}_1$ -free extension of  $M_\beta$  in  $\mathbf{K}_2$ . If  $\beta$  is a successor, there is an additional difficulty. If  $U_\beta \subset P_0^{M_\beta}$ ; we must choose  $b_\beta$  to satisfy condition 4) and with  $M_\alpha \in \mathbf{K}_2$ . For this, apply Lemma 3.2.23 with  $U_\beta$  as  $U$  and  $M_{\beta+1}$  as  $M$  to construct  $N$  and  $b_\beta$ . Now iterate Corollary 3.2.20  $|M_{\beta+1}|$  times to obtain  $M_\alpha \in \mathbf{K}_2$ . This iteration also ensures  $P_2^{M_\alpha} - P_2^{M_\beta}$  is infinite.
- Case 3:*  $\alpha = \delta$ , a limit ordinal that is not in  $S$ . Set  $M_\delta = \bigcup_{\gamma < \delta} M_\gamma$ . We must prove that if  $\beta \in \delta \setminus S$  then  $M_\delta$  is  $\mathbf{K}_1$ -free over  $M_\beta$ . Since  $S$  does not reflect there exists an increasing continuous sequence  $\langle \alpha_i : i < \text{cf}(\delta) \rangle$  of ordinals less than  $\delta$ , which are not in  $S$  and with  $\alpha_0 = \beta$ . By the induction hypothesis, since  $\alpha_j \notin S$ , for each  $i < j < \text{cf}(\delta)$ ,  $M_{\alpha_j}$  is  $\mathbf{K}_1$ -free over  $M_{\alpha_i}$ . And by Lemma 3.2.12,  $M_\delta$  is  $\mathbf{K}_1$ -free over  $M_\beta$  as required.
- Case 4a:*  $\alpha = \delta + 1$ ,  $\delta \in S$ , and clause 5A fails. This is just as in case 2.
- Case 4b:*  $\alpha = \delta + 1$ ,  $\delta \in S$ , but clause 5A holds.

So, suppose  $\langle M_\beta, b_\beta \rangle$  for  $\beta < \delta$  have been defined. If there exists  $\bar{\gamma}$  as in condition 5A) of Construction 3.3.7 we must construct  $\hat{\gamma} = \langle \hat{\gamma}_{\delta,n}, \hat{b}_{\delta,n}^\delta : n < \omega \rangle$  and  $\hat{c}^\delta$  to satisfy condition 5B). Take any  $\langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$  satisfying 5A. Let the  $M_{\gamma_n}$  be the  $N_n$  from Claim 3.3.5 and by that claim, choose  $M_{\delta+1}, \hat{c}_\delta \in P_2^{M_{\delta+1}}$  such that for each  $n$ ,  $F_n^{M_{\delta+1}}(\hat{c}_\delta) = \hat{b}_{\delta,n}$ .

Case 5: Recall that  $\delta$  is divisible by  $|\delta|$  so we can choose the  $\gamma_n$  so that  $\gamma_{n+1} \geq \gamma_n + \omega$  and each  $\gamma_n$  is not in  $S$ . So, by iterating as in Corollary 3.2.20,  $P_2^{M_{\gamma_{n+1}}} - P_2^{M_{\gamma_n}}$  is infinite. Moreover, again since each  $\gamma_n$  is not in  $S$ ,  $M_{\gamma_{n+1}}$  is  $\mathbf{K}_1$ -free over  $M_{\gamma_n}$  so by Lemma 3.2.12,  $M_\delta$  is  $\mathbf{K}_1$ -free.

*This completes the construction. We fix the domain of  $M$  as the  $\lambda$  chosen for Construction 3.3.7.*

**Claim 3.3.10.** *The structure  $M = \bigcup_{i < \lambda} M_i \in \mathbf{K}_2$ .*

Proof. Since we required the extension to be in  $\mathbf{K}_2 = \mathbf{R}$  in requirement 3 of Construction 3.3.7, for cofinally many  $i$ ,  $M_i \in \mathbf{K}_2$ . By Lemma 3.2.22, they are  $\infty, \omega$ -elementary extensions. Hence  $M \in \mathbf{K}_2$ .  $\square_{3.3.10}$

**Construction 3.3.11.** *Verification that the construction suffices*

Now we now show that  $M$  is  $P_0$ -maximal for  $\mathbf{K}_2$ . Suppose for contradiction there exists  $N$  in  $\mathbf{K}_2$  extending  $M$  such that  $P_0^N \not\supseteq P_0^M$ . Choose  $a^* \in P_0^N - P_0^M$ . Let

$$A = \{b \in P_1^M : R^N(a^*, b)\}.$$

Then, by Lemma 3.2.21.ii, for every  $a \in P_0^N$ , in particular  $a^*$  and every  $b \in P_1^N$  (and so every  $b \in P_1^M$ ) either  $R^N(a^*, b)$  or  $R^N(a^*, b^-)$ . Thus, the subset  $A$  of  $P_1^M$  is a non-principal ultrafilter of the Boolean algebra  $P_1^M$ . For, if  $A$  is principal, it is generated by some atom  $b_0 \in P_4^{M_1}$ . Then  $b_0$  must be in  $P_{4,1}$  and so  $\neg R^N(a^*, b_0)$ , contrary to the hypothesis that  $a \in P_0$ . We will show that  $A$  induces an  $\aleph_1$ -complete ultrafilter on  $\mathcal{P}(P_0^{M_{\alpha^*}})$  for some  $\alpha^* < \lambda$ . But this contradicts that  $\lambda$  is below the first measurable.

Recall that the  $A_\delta$  are the diamond sequence fixed in requirement 3.3.7 and that  $S \subseteq S_{\aleph_0}^\lambda$ . Note

$$S_A = \{\delta \in S : M_\delta \text{ has universe } \delta \ \& \ A_\delta = A \cap \delta\}$$

is a stationary subset of  $\lambda$ . In the construction, we chose  $b_\alpha$  for  $\alpha < \lambda$  which satisfied requirement 4 of Construction 3.3.7. Note

$$C = \{\delta < \lambda : \delta \text{ limit} \ \& \ \alpha < \delta \rightarrow b_\alpha < \delta\}$$

is a club on  $\lambda$ .

There are two cases. We will show the first is impossible and the second implies  $\lambda$  is measurable, contrary to hypothesis. So the construction yields a  $P_0$ -maximal model in  $\mathbf{K}_2$ .

**Case i):** For every  $\alpha < \lambda$  there is a  $b \in P_1^M \cap A$  such that  $R(M, b)$  is disjoint from  $\alpha$  and  $\{b\}$  is independent from  $P_1^{M_\alpha}$  over  $P_4^M$ .

Choose  $\delta^* \in S_A \cap C$ . Since  $\delta^*$  has cofinality  $\omega$  we can choose a sequence  $\langle \hat{\gamma}_n^* : n < \omega \rangle$  such that each is a successor (so not in  $S$ ), and, as we are in case i), with  $b_{\hat{\gamma}_n^*} < \hat{\gamma}_{n+1}^*$ . Since condition 5B) holds there are  $\hat{c}_\delta \in P_2^{M_{\delta+1}}$  such that for each  $n$ ,  $F_n^{M_{\delta+1}}(\hat{c}_\delta) = b_{\hat{\gamma}_n^*}$ . Since  $M_{\delta+1} \in \mathbf{K}_1$ , by clause 8 of Definition 3.2.2,  $M_{\delta+1} \models \neg(\exists x) \bigwedge_n R(x, F_n(c_\delta^*))$ . This contradicts that we chose  $b_{\hat{\gamma}_n^*} \in A$ , since by the definition of  $A$ , for each  $n < \omega$ ,  $R^N(a^*, b_{\hat{\gamma}_n^*})$  holds.

**case ii)** For some  $\alpha^*$ , there is no such  $b$ . That is, if  $b \in P_1^M$  is independent from  $P_1^{M_{\alpha^*}}$  over  $P_4^M$  and  $R(M, b)$  is disjoint from  $\alpha^*$  then  $\neg R(a^*, b)$ . From the list of elements of  $[\lambda]^{<\lambda}$  at the beginning of Construction 3.3.7, we consider the subsequence  $\langle v_\gamma : \gamma < \lambda \rangle$  enumerating  $\mathcal{P}(P_0^{M_{\alpha^*}})$ ; recall each element appears  $\lambda$  times in the list.

We now choose inductively by requirement 4 of Construction 3.3.7 and Lemma 3.2.23 a subsequence<sup>15</sup>  $b_\gamma$  of the  $b_\alpha \in P_1^M$  and  $M_\gamma$  such that  $b_\gamma \in P_1^{M_{\gamma+1}}$  and  $R(M_\gamma, b_\gamma) \cap P_0^{M_{\alpha^*}} = v_\gamma = R(M, b_\gamma)$  and  $\langle b_\beta : \beta \leq \gamma \rangle$  is independent from  $P_1^{M_{\alpha^*}}$  over  $P_4^M$ . In particular,  $b_\beta$  is independent from  $P_1^{M_\beta}$  over  $P_4^{M_{\beta+1}}$  and so by Remark 3.2.13 over  $P_4^M$ .

We claim that if  $\gamma_1 < \gamma_2 \wedge v_{\gamma_1} = v_{\gamma_2}$  then  $R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2})$ . For this, let  $b' = b_{\gamma_1} \triangle b_{\gamma_2}$ . Then  $R(M, b') \cap P_0^{M_{\alpha^*}} = \emptyset$  so by the case choice,  $\neg R(a^*, b')$ . But, as required,  $\neg R(a^*, b')$  implies  $R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2})$ .

Continuing the proof of case ii) we define an ultrafilter  $\mathcal{D}$  on  $\mathcal{P}(P_0^{M_{\alpha^*}})$  by  $v \in \mathcal{D}$  if for some (and hence any)  $b_\gamma$  from our chosen subsequence with  $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}} = v$ ,  $R^N(a^*, b_\gamma)$ . (This is an ultrafilter as each  $u \subset P_0^{M_{\alpha^*}}$  is  $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}}$  for some  $\gamma$  by requirement 4 of Construction 3.3.7.)

Now we show the coding of the elements of  $\mathcal{D}$  extends to the entire original sequence.

**Claim 3.3.12.** *For any  $b \in P_1^M$ , which is one of the original sequence of independent  $b_\alpha$ , if  $v = R(M, b) \cap P_0^{M_{\alpha^*}}$  and  $v \in \mathcal{D}$  then  $N \models R(a^*, b)$ .*

*Proof.* We can choose  $\beta, \beta_1$  so that  $\alpha^* < \beta < \lambda$ ,  $b \in P_1^{M_\beta}$  and  $\beta_1 > \beta$  such that  $v_{\beta_1} = v$ . Now  $\check{b} = b \triangle b_{\beta_1} \in P_1^M$  and  $R(M, \check{b}) \cap P_0^{M_{\alpha^*}} = \emptyset$ . Note that since  $\langle b_\beta : \beta < \lambda \rangle$  is independent from  $P_1^{M_{\alpha^*}}$  over  $P_4^M$  in  $P_1^M$ , in particular  $b$  and  $b_{\beta_1}$  are independent so the singleton  $b \triangle b_{\beta_1}$  is independent from  $P_1^{M_{\alpha^*}}$  over  $P_4^M$  in  $P_1^M$ . So by the choice of  $\alpha_*$ ,  $N \models \neg R(a^*, \check{b})$ . So,  $N \models \neg R(a^*, b)$  if and only if  $N \models \neg R(a^*, b_{\beta_1})$ . But, we have  $v \in \mathcal{D}$  and  $R(M, b_{\beta_1}) \cap P_0^{M_{\alpha^*}} = v$ , so  $N \models R(a^*, b_{\beta_1})$  and thus  $N \models R(a^*, b)$  as required.  $\square_{3.3.12}$

There is no  $\aleph_1$ -complete ultrafilter on  $\mathcal{P}(P_0^{M_{\alpha^*}})$  since  $|P_0^{M_{\alpha^*}}| < \lambda$  is not measurable. So there are  $\langle w_n \subseteq P_0^{M_{\alpha^*}} : n < \omega \rangle$ , each in  $\mathcal{D}$ , that are decreasing and intersect in  $\emptyset$ . Now we can find  $\delta^* > \alpha^*$  such that  $\delta^* \in S_A \cap C$ , the universe of  $M_{\delta^*}$  is  $\delta^*$ ,  $A_{\delta^*} \cap \delta^* = A \cap \delta^*$ , and there is an increasing sequence  $\langle \gamma_n^{\delta^*} : n < \omega \rangle$  with limit at most  $\delta^*$  and each  $\gamma_n^{\delta^*} \notin S$ . Further, by requirement 4 on the construction, we can choose  $\gamma_n^{\delta^*}$  so that  $b_{\gamma_n^{\delta^*}}$  (another subsequence of the original sequence) satisfies  $a \in R(M_{\gamma_n}, b_{\gamma_n})$  if and only if  $a \in w_n$ ,  $b_{\gamma_n^{\delta^*}} \in M_{\gamma_n^{\delta^*+1}}$ , and the sequence  $\{b_{\gamma_n^{\delta^*}}\}$  is independent from  $P_1^{M_{\delta^*}}$  over  $P_4^M$ . Since the  $w_n$  are decreasing with empty intersection, no  $a \in M_{\alpha^*}$  is in more than finitely many of the  $w_n$ . Thus, Definition 3.3.4 1c is satisfied.

So by clause 5) of the Requirements 3.3.7, there is a labeled  $A$ -good sequence  $\hat{b}_{\delta^*, n}$  for  $M_{\delta^*+1}$ ,  $\hat{c}_\delta^* \in P_2^{M_{\delta^*+1}}$  such that for each  $n$ ,  $F_n^{M_{\delta^*+1}}(\hat{c}_\delta^*) = \hat{b}_{\delta^*, n}$ . And by clause 8 of Definition 3.2.2, this contradicts Claim 3.3.12; the intersection of  $R(N, F_n^N(c))$  for  $n < \omega$  must be empty but it contains  $a^*$ . So we finish case ii) and thus Lemma 3.3.6.  $\square_{3.3.6}$

**Remark 3.3.13.** In the construction we showed for limit  $\delta$  that  $M_\delta$  is  $\mathbf{K}_1$ -free using  $S$  does not reflect if  $\delta \notin S$  and that  $\text{cf}(\delta) = \omega$  for  $\delta \in S$ . We have no such tools to show the  $P_0$ -maximal model,  $M = M_\lambda$  built in Theorem 3.3.14 is  $\mathbf{K}_1$ -free. In fact, by the contrapositive of Corollary 3.2.20 the final  $P_0$ -maximal model, which might be  $M$ , is *not*  $\mathbf{K}_1$ -free.

Note that every subset of  $M$  with cardinality  $< \lambda$  is contained in a  $\mathbf{K}_1$ -free substructure; this fails in the ZFC proof [BS2x] of maximal models of  $\mathbf{K}_2$  cofinal in a measurable.

<sup>15</sup>For local intelligibility (and at the risk of global confusion) we use indices  $b_\gamma$  and  $M_\gamma$  rather than  $b_{\alpha_\gamma}$  and  $M_{\alpha_\gamma}$  that would keep more precise track of the subsequence fact.

Recall that a  $P_0$ -maximal model in a class  $\mathbf{K}$  is one that cannot be extended in  $\mathbf{K}$  without extending  $P_0$ . While a maximal model has no extension in  $\mathbf{K}$ . We have constructed a  $P_0$ -maximal model in  $\mathbf{K}_2$ ; we show that it has a  $\mathbf{K}_2$ -maximal extension that is only slightly larger.

**Corollary 3.3.14.** *Under the hypotheses of Theorem 3.3.6, there is a maximal model of  $\mathbf{K}_2$  of cardinality at most  $2^\lambda$ .*

Proof. Fix a  $P_0$ -maximal model  $N_0$  of cardinality  $\lambda$  from Theorem 3.3.6. Build for as long as possible a continuous  $\subseteq$ -increasing chain of  $N_\alpha \in \mathbf{K}_2$  such that each  $P_1^{N_\alpha} \subsetneq P_1^{N_{\alpha+1}}$ . But, necessarily,  $P_0^{N_\alpha} = P_0^{N_{\alpha+1}}$ . Recall that by Lemma 3.2.21.1 the relation  $R$  is injective. So, each  $|P_1^{N_\alpha}| \leq 2^{|P_0^{N_0}|} = 2^\lambda$ . So this construction must stop and the final, maximal in  $\mathbf{K}_2$ , model has cardinality at most  $2^\lambda$ .  $\square_{3.3.14}$

## 4 Hanf Number for Existence

As mentioned in the introduction, we improved in [BKL16] Hjorth's result [Hjo02] by exhibiting for each  $n < \omega$  a complete sentence  $\psi_n$  such that  $\psi_n$  characterizes  $\aleph_n$ . This improvement is achieved by combining the combinatorial idea of Laskowski-Shelah in [LS93] with a new notion of  $n$ -dimensional amalgamation. We explain the main definition and theorem here (as in the Tehran lectures) and refer to [BKL16] for the proofs. The combinatorial fact is:

**Fact 4.0.1.** [LS93] *For every  $k \in \omega$ , if  $\text{cl}$  is a locally finite closure relation on a set  $X$  of size  $\aleph_k$ , then there is an independent subset of size  $k + 1$ .*

Fix a vocabulary  $\tau_r$  with infinitely many  $r$ -ary relations  $R_n$  and infinitely many  $r + 1$ -ary functions  $f_n$ . We consider the class  $\mathbf{K}_0^r$  of finite  $\tau_r$ -structures (including the empty structure) that satisfy the following three conditions; closure just means subalgebra closure with respect to the functions.

- The relations  $\{R_n : n \in \omega\}$  partition the  $(r + 1)$ -tuples;
- For every  $(r + 1)$ -tuple  $\mathbf{a} = (a_0, \dots, a_r)$ , if  $R_n(\mathbf{a})$  holds, then  $f_m(\mathbf{a}) = a_0$  for every  $m \geq n$ ;
- There is no independent subset of size  $r + 2$ .

It is easy to see from Fact 4.0.1 that every model in  $\aleph_r$  is maximal. The main effort is to show there is a complete sentence  $\phi_r$  satisfying those conditions which has model in  $\aleph_r$ . For this we introduce a notion patterned on excellence<sup>16</sup> but weaker. We pass from a class  $\mathbf{K}_0^r$  of, now, locally finite structures to the associated class  $\widehat{\mathbf{K}}$  as in Definition 2.1.1.

**Definition 4.0.2.** *For  $k \geq 1$ , a  $k$ -configuration is a sequence  $\overline{M} = \langle M_i : i < k \rangle$  of models (not isomorphism types) from  $\mathbf{K}$ . We say  $\overline{M}$  has power  $\lambda$  if  $\|\bigcup_{i < k} M_i\| = \lambda$ . An extension of  $\overline{M}$  is any  $N \in \mathbf{K}$  such that every  $M_i$  is a substructure of  $N$ .*

Informally,  $(\lambda, k)$ -disjoint amalgamation holds when for any sequence of  $k$  models, at least one with  $\lambda$  elements, there is common extension, which properly extends each model in the sequence. Crucially, there is no prior assumption of a universal model. Here is the precise formulation.

<sup>16</sup>Shelah's theory of excellence concerns unique free disjoint amalgamations of infinite structures in  $\omega$ -stable classes of models of complete sentences in  $L_{\omega_1, \omega}$ .

**Definition 4.0.3.** Fix a cardinal  $\lambda = \aleph_\alpha$  for  $\alpha \geq -1$ . We define the notion of a class  $(\mathbf{K}, \leq)$  having  $(\lambda, k)$ -disjoint amalgamation in two steps:

1.  $(\mathbf{K}, \leq)$  has  $(\lambda, 0)$ -disjoint amalgamation if there is  $N \in \mathbf{K}$  of power  $\lambda$ ;
2. For  $k \geq 1$ ,  $(\mathbf{K}, \leq)$  has  $(\leq \lambda, k)$ -disjoint amalgamation if it has  $(\lambda, 0)$ -disjoint amalgamation and every  $k$ -configuration  $\overline{M}$  of cardinality  $\leq \lambda$  has an extension  $N \in \mathbf{K}$  such that every  $M_i$  is a proper substructure of  $N$ .

For  $\lambda \geq \aleph_0$ , we define  $(< \lambda, k)$ -disjoint amalgamation by: has  $(\leq \mu, k)$ -disjoint amalgamation for each  $\mu < \lambda$ .

Whether or not a given  $k$ -configuration  $\overline{M}$  has an extension depends on more than the sequence of isomorphism types of the constituent  $M_i$ 's, as the pattern of intersections is relevant as well. For example, when (as here) strong substructure is just substructure), a 2-configuration  $\langle M_0, M_1 \rangle$  with neither contained in the other has an extension if and only if the triple of structures  $\langle M_0 \cap M_1, M_0, M_1 \rangle$  has an extension amalgamating them disjointly. Thus we abuse notation a bit and write  $(< \lambda, 2)$  amalgamation for both the notion defined here and the one in Definition 2.1.4. But there is no existing analog of our disjoint  $(< \lambda, k)$ -amalgamation for  $k > 2$ .

Now we modify a theme familiar from the theory of excellence. If the cardinality increases by one, the number of models that can be amalgamated drops by one. In Shelah's context [She09] (chapter 21 of [Bal09]) there is a reliance on Fodor's lemma to obtain compatible filtrations of the models in  $\kappa^+$  to prove the version of Proposition 4.0.4. A very different approach was needed to go from the finite to the countable. Instead of the  $k$ th level concerning finding an embedding into an upper corner for a given  $2^{k-1}$  vertices of a  $k$ -cube, we consider actual containment for  $k$ -models and do not worry about their intersections.

**Lemma 4.0.4** (Proposition 2.20 of [BKL16]). Fix a locally finite  $(\mathbf{K}, \leq)$  with JEP. For all cardinals  $\lambda \geq \aleph_0$  and for all  $k \in \omega$ , if  $\mathbf{K}$  has  $(< \lambda, k+1)$ -disjoint amalgamation, then it also has  $(\leq \lambda, k)$ -disjoint amalgamation.

Together, these propositions yield 1)-3) of the next result. Recall from Definition 2.1.4, that by 2-amalgamation, we mean the usual notion that allows identifications. We say 2-amalgamation is *trivially* true in a cardinal  $\kappa$  if all models in  $\kappa$  are maximal.

**Theorem 4.0.5** (Theorem 3.2.4 of [BKL16]). For every  $r \geq 1$ , the class  $\mathbf{At}^r$  satisfies:

1. there is a model of size  $\aleph_r$ , but no larger models;
2. every model of size  $\aleph_r$  is maximal, and so 2-amalgamation is trivially true in  $\aleph_r$ ;
3. disjoint 2-amalgamation holds up to  $\aleph_{r-2}$ ;
4. 2-amalgamation fails in  $\aleph_{r-1}$ .
5. Each of the classes  $\widehat{\mathbf{K}}^r$  and  $\mathbf{At}^r$  have  $2^{\aleph_s}$  models in  $\aleph_s$  for  $1 \leq s \leq r$ . In addition,  $\widehat{\mathbf{K}}^r$  has  $2^{\aleph_0}$  models in  $\aleph_0$ .

Parts 4) and 5) require a further refinement of the notion of disjoint amalgamation.

**Definition 4.0.6.** Given a cardinal  $\lambda$  and  $k \in \omega$ , we say that  $\mathbf{K}$  has frugal  $(\leq \lambda, k)$ -disjoint amalgamation if it has  $(\leq \lambda, k)$ -disjoint amalgamation and, when  $k \geq 2$ , every  $k$ -configuration  $\langle M_i : i < k \rangle$  of cardinality  $\leq \lambda$  has an extension  $N \in \mathbf{K}$  with universe  $\bigcup_{i < k} M_i$ .

Thus the domain of a frugal amalgamation is just the union of the models amalgamated. It is easy to see that this property holds for the example in [BKL16]. It is essential for the intricate constructions to verify the last two parts of Theorem 4.0.5 and for the work in [BKS16, BS19].

The *finite amalgamation spectrum* of an abstract elementary class  $\mathbf{K}$  with  $LS(\mathbf{K}) = \aleph_0$  is the set  $X_{\mathbf{K}}$  of  $n < \omega$  such that  $\mathbf{K}$  satisfies amalgamation<sup>17</sup> in  $\aleph_n$ . There are many examples<sup>18</sup> where the finite amalgamation spectrum of a complete sentence of  $L_{\omega_1, \omega}$  is either  $\emptyset$  or  $\omega$ .

Theorem 4.0.5 gave the first example of such a sentence with a non-trivial spectrum: for each  $1 \leq r < \omega$  amalgamation holds up to  $\aleph_{r-2}$ , but fails in  $\aleph_{r-1}$ . It holds (trivially) in  $\aleph_r$  (since all models are maximal); there is no model in  $\aleph_{r+1}$ .

This result leaves open whether the property, AP in  $\lambda$ , can be true or false in various patterns as  $\lambda$  increases? Is there even an AEC (and more interestingly a complete sentence of  $L_{\omega_1, \omega}$ ) and cardinals  $\kappa < \lambda$  such that amalgamation holds non-trivially in both  $\kappa$  and  $\lambda$  but fails at some cardinal between them?

Relying on the construction in [BKL16], Baldwin and Soulatos [BS19] show there exist *complete* sentences of  $L_{\omega_1, \omega}$  that variously have maximal models a) in two successive cardinals, b) in  $\kappa$  and  $\kappa^\omega$  and c) in countably many cardinals. In each case all maximal models of the sentence have cardinality less than  $\aleph_{\omega_1}$ . That proof includes an intricate construction of a complete sentence that has a model in each successor cardinal  $\kappa^+$  with a definable subset of power  $\kappa$ . The [BS19] result is distinguished from the one here in several ways. It constructs maximal models in designated cardinals rather than an initial segment. The crucial amalgamation properties are quite different. The example in [BKL16] satisfies  $(< \lambda, 2)$  amalgamation in all cardinals.

## References

- [AB11] A. Avrils and C. Brech. A Boolean algebra and a Banach space obtained by push-out iteration. *Topology and its Applications*, pages 1534–1550, 2011.
- [Bal] John T. Baldwin. A field guide to Hrushovski constructions. Report: <http://www.math.uic.edu/~jbaldwin/pub/hrutrav.pdf>.
- [Bal09] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, Providence, USA, 2009.
- [BB17] John T. Baldwin and William Boney. Hanf numbers and presentation theorems in AEC. In Jose Iovino, editor, *Beyond First Order Model Theory*, pages 81–106. Chapman Hall, 2017.
- [BFKL16] John T. Baldwin, Sy Friedman, M. Koerwien, and C. Laskowski. Three red herrings around Vaught’s conjecture. *Transactions of the American Math Society*, 368:3673–3694, 2016. Published electronically: November 6, 2015.
- [BKL16] John T. Baldwin, M. Koerwien, and C. Laskowski. Amalgamation, characterizing cardinals, and locally finite AEC. *Journal of Symbolic Logic*, 81:1142–1162, 2016.
- [BKS09] John T. Baldwin, A. Kolesnikov, and S. Shelah. The amalgamation spectrum. *Journal of Symbolic Logic*, 74:914–928, 2009.

<sup>17</sup>We say amalgamation holds in  $\kappa$  in the trivial special case when all models in  $\kappa$  are maximal. We say amalgamation fails in  $\kappa$  if there are no models to amalgamate.

<sup>18</sup>Kueker, as reported in [Mal68], gave the first example of a complete sentence failing amalgamation in  $\aleph_0$ .



- [BKS16] John T. Baldwin, M. Koerwien, and I. Soudatos. The joint embedding property and maximal models. *Archive for Mathematical Logic*, 55:545–565, 2016.
- [BS19] John T. Baldwin and Ioannis Soudatos. Complete  $\mathcal{L}_{\omega_1, \omega}$  with maximal models in multiple cardinalities. *Mathematical Logic Quarterly*, 65(4):444–452, 12 2019.
- [BS2x] John T. Baldwin and S. Shelah. Maximal models up to the first measurable in ZFC. submitted, preprint: Shelah number 1147. <http://homepages.math.uic.edu/~jbaldwin/pub/hmzfcdecsub.pdf>, 202x.
- [Cum08] James Cummings. Iterated forcings and elementary embeddings, vol 3. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer-Verlag, 2008.
- [FG90] E. Fried and G. Grätzer. Strong amalgamation of distributive lattices. *Journal of Algebra*, 128:446–455, 1990.
- [Fra54] R. Fraïssé. Sur quelques classifications des systèmes de relations. *Publ. Sci. Univ. Algeria Sèr. A*, 1:35–182, 1954.
- [Grä79] George Grätzer. *Universal Algebra*. Springer-Verlag, 1979.
- [GS05] R. Göbel and S. Shelah. How rigid are reduced products. *Journal of Pure and Applied Algebra*, 202:230–258, 2005.
- [Han60] William Hanf. Models of languages with infinitely long expressions. In *Abstracts of Contributed papers from the First Logic, Methodology and Philosophy of Science Congress, Vol.1*, page 24. Stanford University, 1960.
- [Hjo02] Greg Hjorth. Knight’s model, its automorphism group, and characterizing the uncountable cardinals. *Journal of Mathematical Logic*, pages 113–144, 2002.
- [Hjo07] Greg Hjorth. A note on counterexamples to Vaught’s conjecture. *Notre Dame Journal of Formal Logic*, 2007.
- [Hod93] W. Hodges. *Model Theory*. Cambridge University Press, Cambridge, 1993.
- [KLH16] Alexei Kolesnikov and Christopher Lambie-Hanson. The Hanf number for amalgamation of coloring classes. *Journal of Symbolic Logic*, 81:570–583, 2016.
- [Kni77] J.F. Knight. A complete  $L_{\omega_1, \omega}$ -sentence characterizing  $\aleph_1$ . *Journal of Symbolic Logic*, 42:151–161, 1977.
- [LS93] Michael C. Laskowski and Saharon Shelah. On the existence of atomic models. *Journal of Symbolic Logic*, 58:1189–1194, 1993.
- [Mag16] M. Magidor. Large cardinals and strong logics: CRM tutorial lecture 1. <http://homepages.math.uic.edu/~jbaldwin/pub//MagidorBarc.pdf>, 2016.
- [Mal68] J. Malitz. The Hanf number for complete  $L_{\omega_1, \omega}$  sentences. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, LNM 72, pages 166–181. Springer-Verlag, 1968.
- [Mal69] Jerome Malitz. Universal classes in infinitary languages. *Duke Math. J.*, 36:621–630, 1969. doi:10.1215/S0012-7094-69-03674-6.

- [Mor65] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.
- [She] S. Shelah. Black boxes. paper 309 archive.0812.0656.
- [She78] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.
- [She09] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications, 2009.
- [Sou13] Ioannis Souldatos. Characterizing the powerset by a complete (Scott) sentence. *Fundamenta Mathematica*, 222:131–154, 2013.