Category Theory and Model Theory: Symbiotic Scaffolds
Carnegie Mellon University logic seminar

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Based on *Exploring the generous arena* [Bal2x].

What do we want a foundation to do?

> *So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides* Risk Assessment *for mathematical theories, a* Generous Arena *where the branches of mathematics can be pursued in a unified setting with a* Shared Standard of Proof*, and a* Meta-mathematical Corral *so that formal techniques can be applied to all of mathematics at once.*
>
> *Pen Maddy* What do we want a foundation to do? *[Mad19]*

Maddy thinks that whether set theory or category theory is 'more foundational' does not make for a productive debate. Rather she urges 'a concerted study of the methodological questions raised by category theory'.

Criteria for a Successful Foundation

Set theory satisfies

- **shared standard of proof** ‘is a belief that mathematical research, vaguely thought of as carried out in naive set theory, can be reduced to a formal set theoretic foundation.’
- **Generous Arena** that encompasses all of mathematics.
- **risk assessment**
- **meta-mathematical corral**
- **elucidation**

Set theory fails the following plausible but not crucial criteria.

- **essential guidance**
- **proof checking** (Homotopy type theory)

What do we want a foundation to do?

> *Mac Lane asserts, 'But I see no need for a single foundation — on any one day it is a good assurance to know what the foundation of the day may be — with intuitionism, linear logic or whatever left for the morrow.'*
> *Saunders Mac Lane [Mat92, 119]*

In a section entitled 'Foundation or Organization' [Mac86, 406] Maclane regards either ZFC or category theory as either **foundations** or **not wholly successful organizations** for mathematics. We adopt this distinction and use it to refine Maddy's criteria.

What should a scaffold do?

> *Atiyah described mathematics as the science of analogy. In this vein, the purview of category theory is mathematical analogy. Category theory provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another. The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline.*
>
> *Emily Riehl. Category Theory in Context. [Rie16]*

I would cheerfully replace each instance of 'category' by 'model'.
That is what I mean by a scaffold.

Scaffolds

What is a scaffold?

1. local foundations for mathematics
   1. MT: Formal Theories of X
   2. CT: Category of X
2. promotes *unity* across mathematics by providing a method for transporting concepts and results from one area to another
   1. MT: the classification of theories serves as a unifying principle to treat different areas of mathematics by isolating combinatorial principles that transfer across fields.
   2. CT: 'the action of packaging each variety of objects into a category shifts one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them.' Emily Riehl [Rie16, 11]

There is no requirement that the scaffold encompass all of mathematics but only that it makes connections across many areas.

Structuralism in Mathematics

Two slogans

1. “mathematics is the general study of structures”
2. in pursuing such study, we can “abstract away from the nature of objects instantiating those structures

Erich Rech `https://plato.stanford.edu/entries/structuralism-mathematics/`

Formalized or Formalism-free

A *formalism-free* ([Ken21]) approach to mathematics defines in *naive* set theory or natural language a class of objects. Often, the exact vocabulary of the class is unclear.
Juliette Kennedy. Gödel, Tarski, and the lure of Natural Language.

In contrast the model theoretic notion of an axiomatic system requires a distinction between i) semantics: a class of structures (defined set theoretically) and ii) syntax: a formal language in which axioms are stated and iii) a formal definition of the connection between them.

A central distinction

- Category theory is formalism-free.
- Most of model theory makes essential use of formalization.
  1. A formal language and set of axioms for various areas of mathematics;
  2. A collection of structures that satisfy those axioms.

Structure and isomorphism

1. CT: Structure (or object) and morphism are undefined terms. Structures are ‘isomorphic’ if they are related by an invertible morphism.
2. MT: A vocabulary is a set τ of relation symbols, function symbols, and constant symbols chosen to represent basic concepts.

   A τ-*structure* with universe A assigns (e.g., to each n-ary relation symbol R an $R^A \subseteq A^n$),etc.

   Two structures in a vocabulary τ are isomorphic if there is a bijective function between their domains preserving relations and functions in τ.

A bad and a good argument for category theory

A silly argument against set theoretic foundations

A mathematician might ask 'Is $2 \in 3$?'

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The notion of fixing a vocabulary describes exactly when this question makes sense – only if $\in$ is in the vocabulary.
Mathematicians don't actually make this mistake.

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A good argument for category theory

The diagram definitions of notions like product and co-product emphasize the transferable notion.

In contrast the extensional set theoretic definitions are harder to generalize.

How do the scaffolds get ‘shared standard of proof’?

Model theory explicitly works in set theory.

‘In practice’, so does category theory but we will discuss categorical foundations below.

‘In practice’

E.g., a text such as Hartshorne’s algebraic geometry begins by studying
‘old-fashioned varieties in affine or projective space. They provide the geometric intuition which motivates all further developments. . . . Only after that do I develop systematically the language of schemes, coherent sheaves, and cohomology.’

Lawvere, F.W. and Schanuel, S.H. Conceptual Mathematics Cambridge (1997) high school text [**?**]

Set Theories

The dual role of set theory

Set theories: two roles

1. As a foundation
2. Each scaffold can give local foundations for set theory and then study set theory.

local foundations of set theory

Two conceptions of set theory

1. material set theory: element and set are fundamental.
   MST: elements are sets.
2. structural set theory: function and set are fundamental.
   SST: The elements of a set X are not sets; they are functions from a terminal object 1 to the object (set) X.
   terminal object 1: For every object C, there is a unique morphism from C to 1.

Axiomatizing set theory

Both set theories can be axiomatized as first order theories.

1. material set theory: The vocabulary is $\{\epsilon, =\}$.
2. structural set theory: The vocabulary has symbols for: objects, arrows, domain, codomain, equality, and composition.

Material and Structural Set Theories as Foundations

Weak set theories

A well-known family of bi-interpretable weak set theories include these four.

1. structural set theories:
   1. a well-pointed topos with a natural numbers object and with the axiom of choice;
   2. ETCS, Elementary Theory of the Category of Sets; ([Law64]).
2. material set theories:
   1. BZC, bounded Zermelo with choice
   2. Mac Lane set theory (finite order arithmetic [McL20]).

Each of these weak theories omit the axiom of replacement. Thus they do not have the cardinals $\aleph_\omega$ or $\beth_\omega$.

More algebraically, [Mat01, 9.32] notes that Mac Lane set theory *cannot* prove that for every n, one can iterate the process of taking the dual vector space n-times (starting with $\Re[x]$).

But they extend

Shulman [Shu19] is a precise reference for the known ability to extend the axioms of ETCS by
considering axioms which vastly strengthen ETCS: including
full separation, collection, and replacement axioms.

These extensions require explicit set theoretic syntax to formulate axiom schemes. He concludes that ETCS plus 'structural replacement', is equi-consistent and indeed mutually interpretable with ZFC ([Shu19, Cor. 8.53]).

Moreover, it is possible to define large cardinals in similar ways and so replicate the hierarchy of set theories within the category theory framework

Size

Grothendieck universes

Grothendieck's number theory links large structures to small. Notably, each single scheme has a large category of sheaves. The point is not to study vastly many sheaves but to give a unifying framework for general theorems. Grothendieck gave a set theoretic foundation using universes, which he described informally as sets large enough that the habitual operations of set theory do not go outside them (SGA 1 VI.1 p. 146).
[McL20, 1]

Logicians are wary because from the existence of universes one can prove the consistency of ZFC.
Grothendieck was aware that the existence of a universe was equivalent to the existence of a (strongly) inaccessible cardinal.
κ is (strongly) inaccessible if

1. No $\mu < \kappa$ is cofinal in κ.
2. $\mu < \kappa$ implies $2^{\mu} < \kappa$.

Small and large

Category Theory

1. A category is small if both the collection of objects and the collection of arrows (morphisms) is a set. Otherwise, large.
2. A category is locally small if for any objects B, C $HOM(B, C)$ is a set.
3. A category $\mathcal{C}$ is concrete if there is a faithful functor $F : \mathcal{C} \to$ **Sets**.

Set/Model Theory

κ is (strongly) inaccessible if

1. No $\mu < \kappa$ is cofinal in κ.
2. $\mu < \kappa$ implies $2^\mu < \kappa$.

In ZFC, 'a proper class' is a definable collection of sets.

Two solutions to small and large

Eilenberg and Mac Lane formalized the foundations of the first paper on category theory in 1945 in Neumann-Gödel-Bernays set theory.

Two solutions to small and large

1. Absolute: a clear distinction between sets, ‘small’ and proper classes ‘large’ (NBG) or below an ‘inaccessible cardinal’.
2. Relative: Zermelo-Fraenkel set theory (with choice) (ZFC): There are only sets but they can get very big.

Why must category theory distinguish large and small?

Naive Answer

There is not a set of all groups.

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Naive Answer

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Actual Answer

Here is a category where every object is a set but a single morphism is a proper class.

Freyd wrote in 1970,
'$\mathcal{H}$ is not concrete. There is no interpretation of the objects of $\mathcal{H}$ so that the maps may be interpreted as functions (in a functorial way, at least). $\mathcal{H}$ has always been the best example of an abstract category, historically and philosophically. Now we know that it was of necessity abstract, mathematically' [Fre04, 1].

The Homotopy Category

Let $\mathcal{H}$ be the homotopy category, whose objects are topological spaces and morphisms are homotopy classes of continuous functions. It is easy to see that a morphism in $\mathcal{H}$ may be a proper class.

For any cardinal κ, let X_κ be the κ-pointed star consisting of κ copies of the unit interval which are disjoint except for one point common to all.

Each such space is contractible (it can be continuously shrunk to a point).

So any pair of continuous maps from one of these spaces into another (including the same space) are homotopic to a constant map.

Thus, this morphism has a proper class of members.

Freyd shows there is no other representation that avoids this problem.

Why doesn't model theory distinguish large and small?

Cardinality Matters

Consistently, there are regular cardinals (e.g. $\kappa = \aleph_1$) and groups A ($|A| = \kappa$) such that every strictly smaller group is free but A is not free.
Shelah's singular cardinal theorem: This fails for any singular cardinal κ.
[Vas2x] transfers the result via internal size to accessible categories.

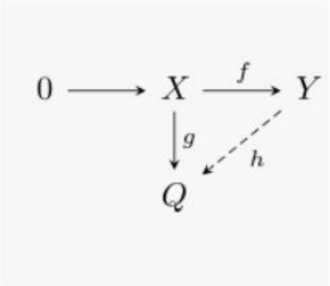
The Model Theory Scaffold

Universal objects

In earlier mathematics

Hausdorff [Hau05, H 1908] introduced a form of universality that is seminal for model theory: M is κ-universal for a class $\boldsymbol{K}$ if $|M| = \kappa$, if $N \in \boldsymbol{K}$ and $|N| \leq \kappa$ then there is an embedding of N into M. Hausdorff introduced the generalized continuum hypothesis $\kappa^+ = 2^\kappa$ and showed it implied there is a κ^+ universal model in every κ.

$(\exists Q)(\forall X, Y)(\exists h) g'' X \subseteq h \circ f$



Universal objects in model theory

Homogenous universal object in Model Theory

Let T be a first order theory.
A model M is (model) homogeneous universal if N models T, $|N| < |M|$ $N \prec N' \models T$, $|N'| < |M|$, there is an elementary embedding of N' into M over N.

Theorem

If $\mu < \kappa$ implies $2^\mu \leq \kappa$ then there is a (unique) homogenous-universal saturated model in cardinality κ.
E.g For all κ under GCH or if κ is a strong limit cardinal.

Types as descriptions: stability

Fix $M \models T$ and $A \subset M$.
A complete n-type in $S_n(\mathrm{Th}(M, A)$ is a description of an n-tuple $\mathbf{b}$ over A.
I.e., the collection of all formulas $\phi(\boldsymbol{a}, \mathbf{x})$ with $\boldsymbol{a} \in A$ such that $M \models \phi(\boldsymbol{a}, \mathbf{b})$.

Definition

Write $S_n(M, A)$ for $S_n(\mathrm{Th}(M, A))$.

The complete theory T is λ-stable if for every $M \models T$ and every $A \subset M$,

$$|A| \leq \lambda \Rightarrow S_n(M, A) \leq \lambda.$$

If M is saturated $S_n(\mathrm{Th}(M, A))$ corresponds to the orbits of G_A on M^n.

Classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T is stable in no λ.

   T has the order property; some formula $\phi(\mathbf{x}, \mathbf{y})$ defines a linear order on an infinite subset of M^n.
2. (strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$ and has saturated models exactly in those cardinals.

   (stable) For every formula ϕ, there is an integer n and a formula ϕ_n asserting 'there is no sequence of n-elements with the ϕ-order property'.
3. (superstable) T is stable in those $\lambda \geq 2^{\aleph_0}$.
4. (ω-stable) T is stable in all infinite λ.

The syntactical characterization

A theory T is unstable if there is a formula with the order property. This formula may change from theory to theory.

1. In a dense linear order one ϕ is $x < y$;
2. In a real closed field one is $(\exists z)(x + z^2 = y)$,
3. In the theory of $(\mathbb{Z}, +, 0, \times)$ one is $(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. In the theory of complex exponentiation $(\mathcal{C}, +, \times, \exp)$, one first notices that $\exp(u) = 0$ defines a substructure which is isomorphic to $(\mathbb{Z}, +, 0, \times)$ and uses the formula from arithmetic.
5. In infinite boolean algebras an unstable formula is $x \neq y$ & $(x \wedge y) = x$; here the domain of the linear order is *not* definable.

It is this flexibility, grounded in the formal language, which underlies the wide applicability of stability theory.

Wild vs Tame

An informal notion of *tame/wild mathematics* developed during the 20th century.

Roughly, wild mathematics includes the 'wilderness' of point set topology; Pillay in ([BKPS01]) includes any area exhibiting the Gödel phenomena, undecidability and coding of pairs (thus, no notion of dimension).

We now explain sufficient model theoretic conditions for tameness.

Geometry and model theoretic tameness

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

This dimension generalizes such concepts as Krull/Weil dimension in algebraically closed field, transcendence degree, dimensions in differential fields.

It imposes a dimension on the realizations of 'regular' types.

For models of appropriate stable theories it assigns a dimension to the model.

o-minimality imposes a similar geometry on certain ordered structures. This is the key to being able to describe structures.

Examples

1. ω-stable (very very tame) (Z, S), algebraically closed fields, differentially closed fields, divisible abelian groups
2. superstable (very tame)
   $(Z, +)$ $(Z, +, \Gamma)$, all predicates unary, certain families of equivalence relations
   as AEC: left modules over left Noetherian, or left pure-semisimple rings
3. stable (tame): all modules, separably closed fields, free group on $n > 1$ generators.
4. (Not the independence property) (domesticated); any *o*-minimal, *ACFV*, *Q* with finitely many linear orders
5. Both strict order property and independence property (wild) $(Z, +, \cdot)$, set theory, boolean algebras

Conant's map of the universe

https://forkinganddividing.com/#_01_1

The independence property

Definition

1. $\phi(\mathbf{x}, \mathbf{y})$ has the *independence property* in a model M if for every $n < \omega$ there are $\mathbf{b}_i \in M$, for $i < n$ and $\boldsymbol{a}_s$ for $s \in 2^n$, such that

$$M \models \phi(\boldsymbol{a}_s, \mathbf{b}_i) \text{ iff } i \in s.$$

2. T is *o*-minimal in a vocabulary $(<, \ldots)$ if every definable set is a Boolean combination of intervals.

Clearly o-minimal theories have the NIP (fail the IP).
T is monadically stable/NIP if any expansion of T by arbitrarily many unary predicates remains stable/NIP,

Domestication I: o-minimality

In ‘Esquisse d’un programme’, Grothendieck asked for a tame topology.
Wilkie ([Wil07], [Bal18, 160])argues that o-minimality is a direct response to Grothendieck’s call because o-minimality:

1. is flexible enough to carry out many geometrical and topological constructions on real functions and on subsets of real Euclidean spaces.
2. builds in restrictions so that we are a priori guaranteed that pathological phenomena can never arise. In particular, there is a meaningful notion of dimension for all sets under consideration and any constructed by the means of 1)
3. is able to prove finiteness theorems that are uniform over fibred collections.

Diophantine geometry

Diophantus: Find integer solutions to an equation: e.g. $x^n + y^n = z^n$.

Modern approach: Solve the wild by embedding in the tame

Study a variety $V \subseteq \mathcal{C}^n$ and look at its integral solutions.

The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

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The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

The variety is studied in the very, very,tame structure $\mathcal{C}$ or

(Hrushovski via Pillay) the tame structure $(\mathcal{C}, \Gamma)$ where Γ is a finitely generated subgroup of the $\mathbb{Q}$-points of an algebraic variety.

Model Theory and Number theory

1. stable theory: Mordell-Lang for Function Fields (Hrushovski)
2. Distinct proofs around the Andre-Oort conjecture and the Ax-Lindemann-Weierstrass theorem
   1. o-minimal theory: Bounds in analytic number theory yielding an important case of Andre-Oort:
      Kobi Peterzil, Jonathan Pila, Sergei Starchenko, and Alex Wilkie
   2. Differentially closed field: Strong minimality as 'not integrable in normal terms': transcendence problems around the Painlevé classification and Fuchsian groups
      (Pillay, Nagloo, Frietag, Scanlon, Casale)

Domestication II: other subclasses of NIP

1. NIP relational data bases [BB98, BB00]
2. Distal Theories: combinatorics: Erdős-Hajnal property, Elekes-Szabó. Property,
3. Monadically stable: Growth rate of universal classes of finite structures:
   (Laskowski, Terry, Braunfeld)

Model Theory Strategies

1. Fix an appropriate vocabulary τ to study the subject.
2. Give a (usually first-order) axiomatization T of the area involved.
3. Study definable relations on the structure to obtain tameness.
4. Modify your vocabulary to reduce quantifier complexity of formulas.
5. Use syntactic conditions (stability hierarchy, *o*-minimality) and the dividing line strategy to guide your search for analogies.

Most Striking applications of category theory

Off the top of my head: algebraic topology, homological algebra, etale cohomology (Weil conjectures), homotopical algebra, topological field theory, Mackey functors, Kazhdan-Lusztig theory,
Bruce Westbury
`https://mathoverflow.net/questions/19325/most-striking-applications-of-category-theory`

Symbiosis

Formalism-Free Model theory I

Abstract Elementary Classes

An abstract elementary class is
a collection $\boldsymbol{K}$ of structures with a binary relation $\leq$ refining subset, that

1. partially orders $\boldsymbol{K}$,
2. such that $\boldsymbol{K}$ is closed under $\leq$-direct limits,
3. satisfies downward Lowenheim Skolem and
4. coherence: If $A \leq C$, $B \leq C$ and $A \subseteq B$ then $A \leq B$.

Formalism-Free Model theory II: Accessible categories

Definition

1. K is λ-presentable if for any morphism $f : K \to M$ with M a λ-directed colimit $\langle \phi_\alpha : M_\alpha \to M \rangle$, f factors essentially uniquely through one of the M_α, i.e. $f = \phi_\alpha M_\alpha$ for some $f_\alpha : K \to M$.
2. For a regular cardinal λ, a category ***K*** is λ-accessible if: (1) ***K*** has λ directed colimits. (2) (Smallness condition) There is a set S of λ-presentable objects such that every object of ***K*** is a λ-directed colimits of elements of S.

Fact

AEC are certain kinds of concrete accessible categories.

A meeting of the scaffolds

Independence in a categorial setting

Shelah often describes his independence relation (non-forking) as a relation $NF(M_0, M_1, M_2, M_3)$ where $M_0 \subseteq M_1, M_2 \subseteq M_3$.
The definition then involves a number of syntactic relations among the four structures.
Lieberman, Michael and Rosický, Ji and Vasey, Sebastien [LRV19] have defined in arbitrary accessible categories a categorical characterization of squares, which replicates the notion of independence.
http://lagrange.math.siu.edu/calvert/OnlineSeminar/Lieberman210325.pdf

Essential Guidance

> *Set theory is a handy vehicle, but its constructions are sometimes artificial. Moreover, it is clearly far too general. As Hermann Weyl once remarked, it contains far too much sand. Mac Lane [Mac86, 407]*

Maddy interprets Mac Lane's complaint as a proposal that a Foundation should provide *Essential Guidance*, 'that would guide mathematicians toward the important structures and characterize them strictly in terms of their mathematically essential features' [Mad19, 19].

She takes this as a plausible Foundational goal, though not a goal of ZFC. As she cogently argues, this goal conflicts with Generous Arena.

Productive Guidance

A *scaffold* as an organization that includes both *local* foundations for various areas of mathematics and productive guidance in how to unify them.

In a scaffold the unification does not take place by a common axiomatic basis but consists of systematic ways of connecting results and proofs in various areas of mathematics.

Expansive criteria

Generous Arena

Maddy [Mad19, 10] asks
'If each branch is characterized by its own separate list of axioms, how can work in one branch be brought to bear in another?'
She proposes set theory (ZFC with large cardinals) as a common framework.
A Generous Arena
in which all of mathematics can be developed that provides a *shared standard of proof.*

Meta-Mathematical Corral

A Meta-mathematical Corral, traces the vast reaches of mathematics to a set of axioms so simple that they can then be studied formally.

The Meta-Mathematical Corral in traditional mathematics

The (J.H.C.) Whitehead Problem

Call A a Whitehead group if for any f, B such that if $f\colon B \rightarrow A$ is a surjective (i.e. onto) group homomorphism whose kernel is isomorphic to the group of integers Z then B is isomorphic to the direct sum of Z and A.
Any free Abelian group A is Whitehead (and conversely if A is countable).
(J.H.C.) Whitehead (motivated by complex analysis and algebraic topology) conjectured: Any Whitehead group is free.

Shelah constructed a specific Whitehead group in the vocabulary $(\in, +, 0)$ that under $V = L$ ($\diamond$) is free as abelian group and under Martin's Axiom is not.
Thus the *metamathematical corral* of independence results available in ZFC extends beyond technical problems about ZFC to problems arising in traditional mathematics.

Material/Structural Set Theories

Particular structural and material set theories are bi-interpretable at all levels from BZ to $ZFC + LC$ 16).

Thus, if we consider *Shared Standard of Proof* to be the *mathematical* question of which statements are theorems in the system and *Generous Arena* as giving surrogates for all mathematical entities there is nothing to choose between the approaches.
Both model theory and set theory apply replacement, large cardinals, and forcing to obtain results in traditional mathematics.
The ease of working in the systems differs.

Summary

1. Each scaffold is more suited to certain areas of mathematics.
2. Each can provide a foundation.

Let many flowers grow!

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