Category Theory and Model Theory: Symbiotic Scaffolds
SIU logic seminar

John T. Baldwin
University of Illinois at Chicago

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Based on *Exploring the generous arena* [Bal2x].

What do we want a foundation to do?

So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides Risk Assessment *for mathematical theories, a* Generous Arena *where the branches of mathematics can be pursued in a unified setting with a* Shared Standard of Proof*, and a* Meta-mathematical Corral *so that formal techniques can be applied to all of mathematics at once.*

Pen Maddy What do we want a foundation to do? *[Mad19]*

Maddy thinks that whether set theory or category theory is 'more foundational' does not make for a productive debate. Rather she urges 'a concerted study of the methodological questions raised by category theory'.

Criteria for a Successful Foundation

For Maddy, Set theory satisfies

- **shared standard of proof** 'is a belief that mathematical research, vaguely thought of as carried out in naive set theory, can be reduced to a formal set theoretic foundation.'
- **Generous Arena** that encompasses all of mathematics.
- **risk assessment**
- **meta-mathematical corral**
- **elucidation**

Set theory fails the following plausible but not crucial criteria.

- **essential guidance**
- **proof checking** (Homotopy type theory)

What do we want a foundation to do?

> *Mac Lane asserts, 'But I see no need for a single foundation*
> *— on any one day it is a good assurance to know what the*
> *foundation of the day may be — with intuitionism, linear logic*
> *or whatever left for the morrow.'*
> *Saunders Mac Lane [Mat92, 119]*

In a section entitled 'Foundation or Organization' [Mac86, 406]
Maclane regards either ZFC or category theory as either **foundations**
or **organizations** for mathematics.

We adopt this distinction and use it to refine Maddy's criteria.

He says, 'Neither organization is wholly successful.'

What should a scaffold do?

Atiyah described mathematics as the science of analogy. In this vein, the purview of category theory is mathematical analogy.

Category theory provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another.

The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline.

Emily Riehl. Category Theory in Context. [Rie16]

I would cheerfully replace each instance of 'category theory' by 'model theory'.

A scaffold is a particular kind of organization.

Scaffolds

What is a scaffold?

1. local foundations for mathematical topic X
   1. MT: Formal Theories of *X*
   2. CT: Category of *X*
2. promotes *unity* across mathematics by providing a method for transporting concepts and results from one area to another
   1. MT: the classification of theories serves as a unifying principle to treat different areas of mathematics by isolating combinatorial principles that transfer across fields.
   2. CT: 'the action of packaging each variety of objects into a category shifts one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them.'   Emily Riehl [Rie16, 11]

There is no requirement that the scaffold encompass all of mathematics but only that it makes connections across many areas.

Structuralism in Mathematics

Two slogans

1. "mathematics is the general study of structures"
2. in pursuing such study, we can "abstract away from the nature of objects instantiating those structures

Erich Rech https://plato.stanford.edu/entries/structuralism-mathematics/

Distinctions between Model Theory and Category Theory

1. the role of formalization
2. internal vs external structure
3. size

Formalized or Formalism-free

A *formalism-free* ([Ken21]) approach to mathematics defines in *naive* set theory or natural language a class of objects. Often, the exact vocabulary of the class is unclear.
Juliette Kennedy. Gödel, Tarski, and the lure of Natural Language.

A central distinction

- Category theory is formalism-free.
- Most of model theory makes essential use of formalization.
    1. **syntax:** A formal language and set of axioms for various areas of mathematics;
    2. **semantics:** A collection of structures that satisfy those axioms.

Structure and isomorphism

1. CT: Structure (or object) and morphism are undefined terms. Structures are 'isomorphic' if they are related by an invertible morphism.

2. MT: A vocabulary is a set τ of relation symbols, function symbols, and constant symbols chosen to represent basic concepts.

   A τ-*structure* with universe A assigns (e.g., to each n-ary relation symbol R an $R^A \subseteq A^n$),etc.

   Two structures in a vocabulary τ are isomorphic if there is a bijective function between their domains preserving relations and functions in τ.

A bad and a good argument for category theory

A silly argument against set theoretic foundations

A mathematician might ask 'Is $2 \in 3$?'

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The notion of fixing a vocabulary describes exactly when this question makes sense – only if $\in$ is in the vocabulary.
Mathematicians don't actually make this mistake.

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A good argument for category theory

The diagram definitions of notions like product and co-product emphasize the transferable notion.

In contrast the extensional set theoretic definitions are harder to generalize.

How do the scaffolds get 'shared standard of proof'?

Model theory explicitly works in set theory.

'In practice', so does category theory but we will discuss categorical foundations below.

'In practice'

E.g., a text such as Hartshorne's algebraic geometry begins by studying
'old-fashioned varieties in affine or projective space. They provide the geometric intuition which motivates all further developments. . . . Only after that do I develop systematically the language of schemes, coherent sheaves, and cohomology.'

Lawvere, F.W. and Schanuel, S.H. Conceptual Mathematics Cambridge (1997) high school text [LS97]

Size

The Homotopy Category

Let $\mathcal{H}$ be the homotopy category, whose objects are topological spaces and morphisms are homotopy classes of continuous functions. It is easy to see that a morphism in $\mathcal{H}$ may be a proper class.
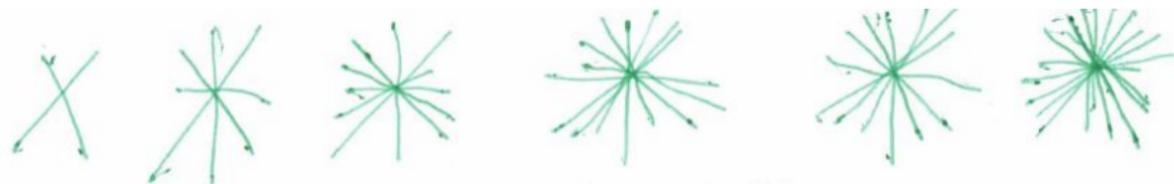
For any cardinal κ, let X_κ be the κ-pointed star consisting of κ copies of the unit interval. Each such space is contractible (it can be continuously shrunk to a point).

So any pair of continuous maps from one of these spaces into another (including the same space) are homotopic to a constant map.

Thus, this morphism has a proper class of members.

Small and large

Category Theory

1. A category is small if both the collection of objects and the collection of arrows (morphisms) is a set. Otherwise, large.

2. A category $\mathcal{C}$ is concrete if there is a faithful functor $F : \mathcal{C} \to$ **Sets**.

Freyd 1970

'$\mathcal{H}$ is not concrete. There is no interpretation of the objects of $\mathcal{H}$ so that the maps may be interpreted as functions (in a functorial way, at least). $\mathcal{H}$ has always been the best example of an abstract category, historically and philosophically. Now we know that it was of necessity abstract, mathematically' [Fre04, 1].

Two ways to compare size

Eilenberg and Mac Lane formalized the foundations of the first paper on category theory in 1945 in Von Neumann-Gödel-Bernays set theory.

Two solutions to small and large

1. Absolute: a clear distinction between sets, 'small' and proper classes 'large' (NBG) or below an 'inaccessible cardinal'.

2. Relative: Zermelo-Fraenkel set theory (with choice) (ZFC): There are only sets but they can get very big.

In ZFC, 'a proper class' is a definable collection of sets that is NOT contained in a set; they are large.

In Mac Lane set theory, $\aleph_\omega$ is a proper class.

Cantor's Attic

cantors-attic

Climb into Cantor's Attic, where you will find infinities large and small. We aim to provide a comprehensive resource of information about all notions of mathematical infinity.

View the Project on GitHub
neugierde/cantors-attic

The upper attic
The upper attic is the realm of large cardinals and the higher infinite, extending from inaccessibility to inconsistency.

The middle attic
The middle attic surveys the infinite cardinals whose existence can be proved in, or is at least equiconsistent with, the ZFC axioms of set theory.

The lower attic
The lower attic classifies various large countable ordinals.

The parlour
Aspiring to the attic, the parlour contains the truly vast finite numbers.

Grothendieck universes: Upper Attic I

Grothendieck's number theory links large structures to small. Notably, each single scheme has a large category of sheaves. The point is not to study vastly many sheaves but to give a unifying framework for general theorems. Grothendieck gave a set theoretic foundation using universes, which he described informally as sets large enough that the habitual operations of set theory do not go outside them (SGA 1 VI.1 p. 146).
[McL20, 1]

Grothendieck was aware that the existence of a universe was

1. equivalent to the existence of a (strongly) inaccessible cardinal;

2. the existence of a strongly inaccessible cardinal implies the consistency of ZFC.

Upper Attic II: Friedman's program

Finite functions and the necessary use of large cardinals

'We present a coherent collection of finite mathematical theorems some of which can only be proved by going well beyond the usual axioms for mathematics.'

Harvey Friedman 2001 Annals of Math [Fri98].

Middle Attic

There is interesting mathematics in the higher infinite.

Group Theory

Shelah's singular cardinal theorem: If κ is singular, and *A* is an abelian group such that every strictly smaller group is free then *A* is free.

Consistently, there are regular cardinals (e.g. $\kappa = \aleph_1$) and groups *A* ($|A| = \kappa$) such that every strictly smaller group is free but *A* is not free.

[Vas2x] transfers the result via internal size to accessible categories.

Model Theory

Cardinals far smaller than inaccessible but vastly larger than those permitted in Mac Lane set theory play significant roles in the classification of theories.

Material and Structural Set Theories

The dual role of set theory

Set theories: two roles

1. As a foundation
2. Each scaffold can give local foundations for set theory and then study set theory.

local foundations of set theory

Two conceptions of set theory

1. material set theory: element and set are fundamental.
   MST: elements are sets.

2. structural set theory: function and set are fundamental.
   SST: The elements of a set X are not sets; they are functions from a terminal object 1 to the object (set) X.
   terminal object 1: For every object C, there is a unique morphism from C to 1.

Axiomatizing set theory

Both set theories can be axiomatized as first order theories.

1. material set theory: The vocabulary is $\{\epsilon, =\}$.

2. structural set theory: The vocabulary has symbols for: objects, arrows, domain, codomain, equality, and composition.

Material and Structural Set Theories as Foundations

Weak set theories

A well-known family of bi-interpretable weak set theories include these four.

1. structural set theories:
   1. a well-pointed topos with a natural numbers object and with the axiom of choice;
   2. ETCS, Elementary Theory of the Category of Sets; ([Law64]).
2. material set theories:
   1. BZC, bounded Zermelo with choice
   2. Mac Lane set theory

Each of these weak theories omit the axiom of replacement. Thus they do not have the cardinals $\aleph_\omega$ or $\beth_\omega$.

More algebraically, [Mat01, 9.32] notes that Mac Lane set theory *cannot* prove that for every *n*, one can iterate the process of taking the dual vector space *n*-times (starting with $\Re[x]$).

Narrowing the material/structural divide

There are stronger structural set theories which are bi-interpetable with

1. ZFC
2. the large cardinal hierarchy

Shulman [Shu19] is a precise reference.
These extensions require explicit set theoretic syntax to formulate
axiom schemes.

The Model Theory Scaffold

Wild vs Tame

An informal notion of *tame/wild mathematics* developed during the 20th century.

1. Roughly, wild mathematics includes the 'wilderness' of point set topology;
2. Model theoretic version: (e.g. [BKPS01]) includes any area exhibiting the Gödel phenomena, undecidability, or coding of pairs (thus, no notion of dimension).

Classification of first order theories

https://forkinganddividing.com/#_01_1
Gabe Conant's diagram

Examples

1. ω-stable (very very tame) (Z, S), algebraically closed fields, differentially closed fields, divisible abelian groups

2. superstable (very tame)
   $(Z, +)$ $(Z, +, \Gamma)$, all predicates unary, certain families of equivalence relations
   as AEC: left modules over left Noetherian, or left pure-semisimple rings

3. stable (tame): all modules, separably closed fields, free group on $n > 1$ generators.

4. (Not the independence property) (domesticated); any *o*-minimal, *ACFV*, *Q* with finitely many linear orders

5. Both strict order property and independence property (wild) $(Z, +, \cdot)$, set theory, boolean algebras

The syntactical characterization

A theory *T* is unstable if there is a formula with the order property. This formula may change from theory to theory.

1. In a dense linear order one ϕ is $x < y$;
2. In a real closed field one is $(\exists z)(x + z^2 = y)$,
3. In the theory of $(\mathbb{Z}, +, 0, \times)$ one is $(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. In the theory of complex exponentiation $(\mathcal{C}, +, \times, \exp)$, one first notices that $\exp(u) = 0$ defines a substructure which is isomorphic to $(\mathbb{Z}, +, 0, \times)$ and uses the formula from arithmetic.
5. In infinite boolean algebras an unstable formula is $x \neq y \,\&\, (x \wedge y) = x$; here the domain of the linear order is *not* definable.
6. In the random graph the formula is $R(x, y)$ and it is NOT a linear order.

It is this flexibility, grounded in the formal language, which underlies the wide applicability of stability theory.

Saturated Models/ universal domains

Models

Let T be a first order theory.
A model M is homogeneous universal if $A \subseteq M$, $|A| < |M|$ $A \subset N' \models T$, $|N'| \leq |M|$, there is an elementary embedding of N' into M over N.

Theorem

1. T is stable in λ if there is a monster model M in λ. That is, Equivalently, for each $A \subseteq M$ with $|A| < |M|$, there are $\leq \lambda$-orbits under $Aut_A(M)$.

2. Stability Spectrum. For every countable T, T is stable in:
   1. (ω-stable): all infinite cardinals ;
   2. (superstable: all cardinals $\geq 2^{|\aleph_0|}$;
   3. (stable) those λ with $\lambda^\omega = \lambda$
   4. (unstable) never.

Geometry and model theoretic tameness

If *T* is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

This dimension generalizes such concepts as Krull/Weil dimension in algebraically closed field, transcendence degree, dimensions in differential fields.

It imposes a dimension on the realizations of 'regular' types.

For models of appropriate stable theories it assigns a dimension to the model.

o-minimality imposes a similar geometry on certain ordered structures. This is the key to being able to describe structures.

Domestication: o-minimality

In 'Esquisse d'un programme', Grothendieck asked for a
tame topology.
Wilkie ([Wil07], [Bal18, 160])argues that o-minimality is a direct
response to Grothendieck's call because o-minimality:

1. is flexible enough to carry out many geometrical and topological
   constructions on real functions and on subsets of real Euclidean
   spaces.

2. builds in restrictions so that we are a priori guaranteed that
   pathological phenomena can never arise. In particular, there is a
   meaningful notion of dimension for all sets under consideration
   and any constructed by the means of 1)

3. is able to prove finiteness theorems that are uniform over fibred
   collections.

Diophantine geometry

Diophantus: Find integer solutions to an equation: e.g. $x^n + y^n = z^n$.

Modern approach: Solve the wild by embedding in the tame

Study a variety $V \subseteq \mathcal{C}^n$ and look at its integral solutions.

The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

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The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

The variety is studied in the very, very,tame structure $\mathcal{C}$ or

(Hrushovski via Pillay) the tame structure $(\mathcal{C}, \Gamma)$ where Γ is a finitely generated subgroup of the $\mathbb{Q}$-points of an algebraic variety.

'The confluence'

The Confluence

denotes tangled web of Fuchsian groups, hyperbolic geometry, number theory, automorphic forms, complex analysis, algebraic geometry etc etc. and recently model theory.

The scaffolds and 'the confluence'

1. **Category Theory**
   Categorical methods underpin many of the modern developments, e.g. the Weil conjectures.

2. **model theoretic contributions are not so well known.**
   1. o-minimal theory: Bounds in analytic number theory yielding Andre-Oort:
      important case: Kobi Peterzil, Jonathan Pila, Sergei Starchenko, and Alex Wilkie
      full conjecture: Pila, Shankar, and, Tsimerman 2022
      ```
      https://www.quantamagazine.org/
      mathematicians-prove-30-year-old-andre-oort-conject
      ```
   2. stable theories:
      1. Mordell-Lang for Function Fields (Hrushovski)
      2. Differentially closed field: Strong minimality as 'not integrable in normal terms': transcendence problems around the Painlevé classification and Fuchsian groups
         Distinct results around the Andre-Oort conjecture and the Ax-Lindemann-Weierstrass theorem
         (Pillay, Nagloo, Freitag, Scanlon, Casale)

Continuous logic and Operator Algebra

Connes Embedding Problem

1. 40 year old problem: every tracial von Neumann algebra embeds (in a trace-preserving way) in an ultrapower $\mathcal{R}^{\mathcal{U}}$ of $\mathcal{R}$.

2. Conjectured answer fails: via operator algebras, complexity theory, quantum logic with the result $MIP^* = RE$.

3. Goldbring-Hart have much shorter proof using continuous logic.

See [Gol21, JNV+21]

Domestication II: other subclasses of NIP

1. NIP relational data bases [BB98, BB00]
2. Distal Theories: combinatorics: Erdŏs-Hajnal property, Elekes-Szabó Property,
3. Monadically stable: Growth rate of universal classes of finite structures:
   (Laskowski, Terry, Braunfeld)

Model Theory Strategies

1. Fix an appropriate vocabulary τ to study the subject.
2. Give a (usually first-order) axiomatization *T* of the area involved.
3. Study definable relations on the structure to obtain tameness.
4. Modify your vocabulary to reduce quantifier complexity of formulas.
5. Use syntactic conditions (stability hierarchy, *o*-minimality) and the dividing line strategy to guide your search for analogies.

Symbiosis

Formalism-Free Model theory I

Abstract Elementary Classes

An abstract elementary class is
a collection **K** of structures for a vocabulary τ with a binary relation $\leq$ refining subset, that

1. partially orders **K**,
2. such that **K** is closed under $\leq$-direct limits,
3. satisfies downward Lowenheim Skolem and
4. coherence: If $A \leq C$, $B \leq C$ and $A \subseteq B$ then $A \leq B$.

Formalism-Free Model theory II: Accessible categories

Definition

1. K is λ-presentable if for any morphism $f : K \to M$ with M a λ-directed colimit $\langle \phi_\alpha : M_\alpha \to M \rangle$, f factors essentially uniquely through one of the M_α, i.e. $f = \phi_\alpha M_\alpha$ for some $f_\alpha : K \to M$.

2. For a regular cardinal λ, a category $\boldsymbol{K}$ is λ-accessible if: (1) $\boldsymbol{K}$ has λ directed colimits. (2) (Smallness condition) There is a set S of λ-presentable objects such that every object of $\boldsymbol{K}$ is a λ-directed colimits of elements of S.

Fact

AEC are certain kinds of concrete accessible categories.

A meeting of the scaffolds

Independence in a categorial setting

Shelah often describes his independence relation (non-forking) as a relation $NF(M_0, M_1, M_2, M_3)$ where $M_0 \subseteq M_1, M_2 \subseteq M_3$.

The definition then involves a number of syntactic relations among the four structures.

Lieberman, Michael and Rosický, Ji and Vasey, Sebastien [LRV19] have defined in arbitrary accessible categories a categorical characterization of squares, which replicates the notion of independence.

http://lagrange.math.siu.edu/calvert/OnlineSeminar/Lieberman210325.pdf

Guidance

Essential Guidance

> *Set theory is a handy vehicle, but its constructions are sometimes artificial. Moreover, it is clearly far too general. As Hermann Weyl once remarked, it contains far too much sand. Mac Lane [Mac86, 407]*

Maddy interprets Mac Lane's complaint as a proposal that a Foundation should provide *Essential Guidance*, 'that would guide mathematicians toward the important structures and characterize them strictly in terms of their mathematically essential features' [Mad19, 19].

She takes this as a plausible Foundational goal, though not a goal of ZFC. As she cogently argues, this goal conflicts with Generous Arena.

Productive Guidance

A *scaffold* as an organization that includes both *local* foundations for various areas of mathematics and productive guidance in how to unify them.

In a scaffold the unification does not take place by a common axiomatic basis but consists of systematic ways of connecting results and proofs in various areas of mathematics.

Assessment

Expansive criteria

Generous Arena

Maddy [Mad19, 10] asks
'If each branch is characterized by its own separate list of axioms, how can work in one branch be brought to bear in another?'
She proposes set theory (ZFC with large cardinals) as a common framework.
A Generous Arena
in which all of mathematics can be developed that provides a *shared standard of proof.*

Meta-Mathematical Corral

A Meta-mathematical Corral, traces the vast reaches of mathematics to a set of axioms so simple that they can then be studied formally.

The Meta-Mathematical Corral in traditional mathematics

The (J.H.C.) Whitehead Problem

Call A is a Whitehead group if any short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow B \rightarrow A \rightarrow 0$$

splits.

Any free Abelian group A is Whitehead (and conversely if A is countable).

(J.H.C.) Whitehead (motivated by complex analysis and algebraic topology) conjectured: Any Whitehead group is free.

Shelah illustrated the relevance of the metamathematical corral for traditional mathematics by showing the Whitehead conjecture is independent of ZFC.

Material/Structural Set Theories

Particular structural and material set theories are bi-interpretable at all levels from *BZ* to *ZFC* + *LargeCardinals*.

Thus, if we consider *Shared Standard of Proof* to be the *mathematical* question of which statements are theorems in the system and *Generous Arena* as giving surrogates for all mathematical entities there is nothing to choose between the set theoretic foundational approaches.

Almost all work in set theory has used material set theory. Both model theory and set theory apply replacement, large cardinals, and forcing to obtain results in traditional mathematics.

Summary

1. These scaffolds complement more than they compete.
2. Each can provide a foundation.

Let many flowers grow!

AMS meeting Wed April 6 8 AM PDT

https://www.jointmathematicsmeetings.org/meetings/
national/jmm2022/2268_program_wednesday.html

References I

📄 John T. Baldwin.
Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism.
Cambridge University Press, 2018.

📄 John T. Baldwin.
Exploring the generous arena.
In J. Kennedy, editor, *Maddy volume*. Cambridge, 202x.
invited: submitted.

📄 John T. Baldwin and M. Benedikt.
Embedded finite models, stability theory, and the impact of order.
In *Proceedings of 13th annual IEEE Symposium on Logic in Computer Science*, pages 490–500, 1998.

References II

📄 John T. Baldwin and M. Benedikt.
Stability theory, permutations of indiscernibles and embedded finite models.
Transactions of the American Mathematical Society, 352:4937–4969, 2000.

📄 S. Buss, A. Kechris, A. Pillay, and R. Shore.
The prospects for mathematical logic in the twenty-first century.
The Bulletin of Symbolic Logic, 7:169–196, 2001.

📄 P. Freyd.
Homotopy is not concrete.
Reprints in Theory and Applications of Categories, 6:1–10, 2004.
First appeared: Streenrods Festschrift: The Steenrod Algebra and its Applications, Lecture Notes in Mathematics, Vol. 168 Springer, Berlin 1970; http://www.tac.mta.ca/tac/reprints/articles/6/tr6.pdf.

References III

📄 Harvey Friedman.
Finite functions and the necessary use of large cardinals.
Ann. of Math., 148:803–893, 1998.

📄 I. Goldbring.
The connes embedding problem: a guided tour.
https://arxiv.org/abs/2109.12682, 2021.

📄 Z. Ji, A. Natarajan, T. Vidick, J. Wright, and J. Yuen.
Mip*= re.
preprint, arxiv 2001.04383, 2021.

📄 Juliette Kennedy.
Gödel, Tarski, and the lure of Natual Language.
Cambridge University Press, Cambridge, 2021.

References IV

📄 F. W. Lawvere.
An elementary theory of the category of sets.
Proceedings of the National Academy of Science of the U.S.A,
52:1506–1511, 1964.

📄 Michael Lieberman, Ji Rosický, and Sebastien Vasey.
Weak factorization systems and stable independence.
2019.
arXiv:1904.05691.

📄 F. W. Lawvere and S. H. Schanuel.
Conceptual Mathematics.
Cambridge University Press, 1997.

📄 Saunders MacLane.
Mathematics: Form and Function.
Springer-Verlag, Heidelberg, 1986.

References V

📄 Penelope Maddy.
What do we want a foundation to do? Comparing set-theoretic, category-theoretic, and univalent approaches.
In S. Centrone, D. Kant, and D. Sarikaya, editors, *Reflections on Foundations: Univalent Foundations, Set Theory and General Thoughts*. Springer-Verlag, 2019.

📄 A.R.D. Mathias.
What is Mac Lane missing?
In Haim Judah, Winfried Just, and Hugh Woodin, editors, *Set Theory of the Continuum*, pages 113–117. Springer-Verlag, 1992.
The link to this paper at
https://www.dpmms.cam.ac.uk/~ardm/scdlistG.html
also contains Mac Lane's reply.

References VI

📄 A.R.D. Mathias.
The strength of MacLane set theory.
Annals of Pure and Applied Logic, 110:107–234, 2001.

📄 C. McLarty.
A finite order arithmetic foundation for cohomology.
Review of Symbolic Logic, 13, 2020.
Published online by Cambridge University Press: 02 August 2019,
pp. 1-30 preprint:http://arxiv.org/pdf/1102.1773v3.pdf.

📄 Emily Riehl.
Category Theory in Context.
Dover, Minneola, New York, 2016.
Kindle edition available; https://web.math.rochester.edu/
people/faculty/doug/otherpapers/Riehl-CTC.pdf.

References VII

Michael Shulman.
Comparing material and structural set theories.
Math Arxiv, Dec. 2018, 2019.

Sebastien Vasey.
Accessible categories, set theory, and model theory: an invitation.
preprint: https://arxiv.org/abs/1904.11307, 202x.

A. Wilkie.
O-minimal structures.
Séminaire Bourbaki, 985, 2007.
http://eprints.ma.man.ac.uk/1745/01/covered/MIMS_
ep2012_3.pdf.