Category Theory and Model Theory: Symbiotic Scaffolds
AMS Special Session on Competing Foundations for Mathematics: How Do We Choose?

John T. Baldwin
University of Illinois at Chicago

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- Foundations,Scaffold and Philosophical background
- Distinguishing Category theory and model theory
    1. size
    2. vocabulary, syntax, semantics
- material and structural set theory
- the model theory scaffold
- Conflict and Resolution

Based on *Exploring the generous arena* [Bal2x].
[Bal18] Model theory and the Philosophy of Mathematical Practice

Helpful comments by Bonnie Gold, A.R.D. Mathias, Ming Ng, and Rehana Patel,

What do we want a foundation to do?

Maddy thinks that whether set theory or category theory is ‘more foundational’ does not make for a productive debate. Rather she urges ‘a concerted study of the methodological questions raised by category theory’.

Pen Maddy *What do we want a foundation to do?* [Mad19]

Criteria for a Successful Foundation

For Maddy, Set theory satisfies

- **shared standard of proof** ‘is a belief that mathematical research, vaguely thought of as carried out in naive set theory, can be reduced to a formal set theoretic foundation.’
- **generous arena** that encompasses all of mathematics.
- **meta-mathematical corral** Allows for the incompleteness of the foundation.
- **risk assessment**
- **elucidation**

Set theory fails the following plausible but not crucial criteria.

- **essential guidance**
- **proof checking** (Homotopy type theory)

What do we want a foundation to do?

> *Mac Lane asserts, 'But I see no need for a single foundation — on any one day it is a good assurance to know what the foundation of the day may be — with intuitionism, linear logic or whatever left for the morrow.'*
> *Saunders Mac Lane [Mat92, 119]*

In a section entitled 'Foundation or Organization' [Mac86, 406] Mac Lane considers that each of ZFC and category theory is lacking 'as **foundations** for mathematics'.

He also regards each as a possible organization and says, 'Neither organization is wholly successful.'

We adopt this distinction and use it to refine Maddy's criteria.

What should a scaffold do?

> *Atiyah described mathematics as the science of analogy. In this vein, the purview of category theory is mathematical analogy.*
> *Category theory provides a cross-disciplinary language for mathematics designed to delineate general phenomena, which enables the transfer of ideas from one area of study to another. The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline.*
>
> *Emily Riehl. Category Theory in Context. [Rie16]*

I would cheerfully replace each instance of ‘category theory’ by ‘model theory’.

What is a scaffold?

A scaffold is a particular kind of organization for mathematics.

1. local foundations for mathematical topic X
   1. MT: Formal Theories of X
   2. CT: Category of X
2. promotes *unity* across mathematics by providing a method for transporting concepts and results from one area to another
   1. MT: the classification of theories serves as a unifying principle by isolating combinatorial properties, recognizing that certain disparate areas of mathematics share certain of these properties, and exploiting that commonality.
   2. CT: 'the action of packaging each variety of objects into a category shifts one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them.' Emily Riehl [Rie16, 11]

There is no requirement that the scaffold encompass all of mathematics but only that it makes connections across many areas.

Distinctions between Model Theory and Category Theory

1. the role of formalization
2. quite different interpretations of Dedekind's idea of a 'structure'
3. size

Formalized or Formalism-free

Kennedy discusses a *formalism-free* ([Ken21]) approach to mathematics.

In our context, the mathematician defines a class of objects in *naive* set theory or natural language. Often, the exact vocabulary of the class is unclear.

Juliette Kennedy. Gödel, Tarski, and the Lure of Natural Language.

A central distinction

- Typically, category theory is formalism-free.
- Typically, model theory uses a *full* formalization in an essential way.
  1. **syntax:** A formal vocabulary and set of axioms for various areas of mathematics;
  2. **semantics:** A collection of structures (each definable in set theory) for that vocabulary that satisfy those axioms.

Some categorical logic makes a similar distinction. (e.g. [Joh02, p 807-812].)

Structure and isomorphism

1. CT: Structure (or object) and morphism are undefined terms. Structures are ‘isomorphic’ if they are related by an invertible morphism.
2. MT: A vocabulary is a set τ of relation symbols, function symbols, and constant symbols chosen to represent basic concepts.

   A τ-*structure*, whose universe is a set A, assigns (e.g., to each n-ary relation symbol R an $R^A \subseteq A^n$), etc.

   Two structures in a vocabulary τ are isomorphic if there is a bijective function between their domains preserving instances of relations and functions in τ.

Definability

Macintyre [Mac03]

‘It seems to me now uncontroversial to see the fine structure of definitions as becoming the central concern of model theory, to the extent that one can easily imagine the subject being called Definability Theory in the near future.’

We see definability as one of the pillars of the model theoretic approach; in particular, definability is key to establishing the classification.

Structuralism in Mathematics

One theme in the philosophy of mathematics explores Dedekind's notion of structure.
In pursuing such a study, we can "abstract away from the nature of objects instantiating those structures" [Rec19]

Typically, category theory abstracts away by considering structures as 'atoms'.

Model theory, via definability, studies the anatomy of individual structures defined in set theory as well as morphisms between them.

A bad and a good argument for category theory

A silly argument against set theoretic foundations

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A good argument for category theory

The diagram definitions of notions like product and co-product emphasize the transferable notion.

In contrast the extensional set theoretic definitions are harder to generalize.

A shared standard of proof for the two scaffolds

Undergraduate mathematics is developed in ‘naive set theory’.

Model theory explicitly works in ZFC (sometimes + large cardinals) set theory.

Category theory continues to work in naive set theory to describe basic structures especially in analysis.
But alternative ‘category-theory based set-theoretic foundations’ exist.

In the next two sections we explore some motivations for the alternative set theoretic foundations and their connections with ZFC.

Size

The Homotopy Category

Let $\mathcal{H}$ be the homotopy category, whose objects are topological spaces and morphisms are homotopy classes of continuous functions. It is easy to see that a morphism in $\mathcal{H}$ may be a proper class.

For any cardinal κ, let X_κ be the κ-pointed star consisting of κ copies of the unit interval that intersect in a single point. Each such space is contractible (it can be continuously shrunk to a point).

So each continuous maps from one of these spaces into another (including the same space)is homotopic to a constant map.

Thus, this morphism has a proper class of members.

Small and large

Category Theory

1. A category is small if both the collection of objects and the collection of arrows (morphisms) are sets. Otherwise, large.
2. A category $\mathcal{C}$ is concrete if there is a faithful functor $F : \mathcal{C} \to \textbf{Sets}$.

Freyd 1970

'$\mathcal{H}$ is not concrete. There is no interpretation of the objects of $\mathcal{H}$ so that the maps may be interpreted as functions (in a functorial way, at least). $\mathcal{H}$ has always been the best example of an abstract category, historically and philosophically. Now we know that it was of necessity abstract, mathematically' [Fre04, 1].

Two ways to compare size

Eilenberg and Mac Lane formalized the foundations of category theory in the 1945 first paper in Von Neumann-Gödel-Bernays set theory.

Two solutions to small and large

1. Absolute: a clear distinction between sets, 'small' and proper classes 'large' (NBG) or [Mac71] a single Grothendieck universe.
2. Relative: Zermelo-Fraenkel set theory (with choice) (ZFC): There are only sets but they can get very big.

In ZFC, 'a proper class' is a definable collection of sets that is NOT contained in a set; proper classes are large.

In Mac Lane set theory, $\aleph_\omega$ is a definable collection but not provably a set.

Cantor's Attic

https://neugierde.github.io/cantors-attic/#cantors-attic

Cantor's Attic

- the gods: NBG - proper classes and sets are two types of entity.
- **The upper attic** is the realm of large cardinals and the higher infinite, extending from inaccessibility to inconsistency.
- The **middle attic**: surveys the infinite cardinals whose existence can be proved in, or is at least equiconsistent with, the ZFC axioms of set theory.
- The mezzanine contains those cardinals in the middle attic that are sets in weak set theories. Those less than $\aleph_\omega$.
- The **lower attic** classifies various large countable ordinals.

Grothendieck universes: Upper Attic

> *Grothendieck's number theory links large structures to small. Notably, each single scheme has a large category of sheaves. The point is not to study vastly many sheaves but to give a unifying framework for general theorems. Grothendieck gave a set theoretic foundation using universes, which he described informally as sets large enough that the habitual operations of set theory do not go outside them (SGA 1 VI.1 p. 146).*
> *[McL20, 1]*

Grothendieck was aware that the existence of a universe was equivalent to:

1. the existence of a (strongly) inaccessible cardinal;
2. and the existence of a strongly inaccessible cardinal implies the consistency of ZFC.

Middle Attic

There is interesting mathematics in the middle attic.

Group Theory

Shelah's singular cardinal theorem: If κ is singular, and if *A* is an abelian group with $|A| = \kappa$ such that every subgroup *B* of *A* with $|B| < \kappa$ is free then *A* is free.
Consistently, there are regular cardinals κ (e.g. $\kappa = \aleph_1$) and groups *A* ($|A| = \kappa$) such that every strictly smaller subgroup of *A* is free but *A* is not free."

Model Theory

Cardinals far smaller than strongly inaccessible but vastly larger than those permitted in Mac Lane set theory play significant roles in the classification of theories.

Material and Structural Set Theories

The dual role of set theory

Set theories: two roles

1. As a foundation
2. Each scaffold can give local foundations for set theory and then study set theory.

3 possible foundational frameworks

1. type theory (e.g. HTT)
2. two local foundations for set theory
   1. material set theory (MST): element and set are fundamental. In MST elements are sets.
   2. structural set theory (SST): function and set are fundamental.

      In SST, the elements of a set X are not sets; they are functions from a terminal object 1 to the object (set) X.

Elements in structural set theory

Definition: terminal object 1:

1 is a terminal object of $\mathcal{C}$ if for every object A, there is a unique morphism from A to 1.

There is a unique terminal object.

If 1 is the terminal object of $\mathcal{C}$ and $f : 1 \to X$ then f is *point* of X.
rough idea: for a set theorist, the image of f is a single element of X.

Lawvere, F.W. and Schanuel, S.H. Conceptual Mathematics Cambridge (1997) high school text [LS97]

Material and Structural Set Theories as Foundations

Axiomatizing set theory

Both set theories can be axiomatized as first order theories.

1. material set theory: The vocabulary is $\{\epsilon, =\}$.
2. structural set theory: The vocabulary has symbols for: objects, arrows, domain, codomain, equality, and composition.

Weak set theories: The Mezzanine

There are long-known (at least 50 years) families of weak set theories, both material and structural, that are bi-interpretable.
e.g SST ETCS, Elementary Theory of the Category of Sets; ([Law64] and MST: Mac Lane set theory.

Each of these weak theories omits the axiom of replacement. thus, the 'cardinals' $\aleph_\omega$ or $\beth_\omega$ are not provably sets.

Narrowing the material/structural divide

There are stronger structural set theories which are bi-interpetable with

1. ZFC
2. the large cardinal hierarchy

Shulman [Shu19] is a precise reference.
These extensions require explicit set theoretic syntax to formulate axiom schemes.

Meta-Mathematical Corral

T

he generous arena should be sufficiently broad to acknowledge that the base system of axioms is incomplete and consider extensions of the theory.

The (J.H.C.) Whitehead Problem

Call A a Whitehead group if any short exact sequence

$$0 \to \mathbb{Z} \to B \to A \to 0$$

splits.
Fact: Any free Abelian group A is Whitehead (and conversely if A is countable).
Whitehead Conjecture: Any Whitehead group is free.

But Shelah showed the conjecture is independent from ZFC + CH.

The Model Theory Scaffold

Wild vs Tame

tame/wild mathematics is an informal 20th century distinction.

1. Grothendieck argued that the 'wilderness' of point set topology arose because it was created for analysis rather than geometry.
2. Model theoretic version: (e.g. [BKPS01]) includes any area exhibiting the Gödel phenomena, undecidability, or coding of pairs (thus, no notion of dimension).

Diophantine geometry

- Diophantus: Find integer solutions to an equation: e.g. $x^n + y^n = z^n$.
- Modern approach: Solve the wild by embedding in the tame. Study a variety $V \subseteq \mathcal{C}^n$ and look at its integral solutions.

The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

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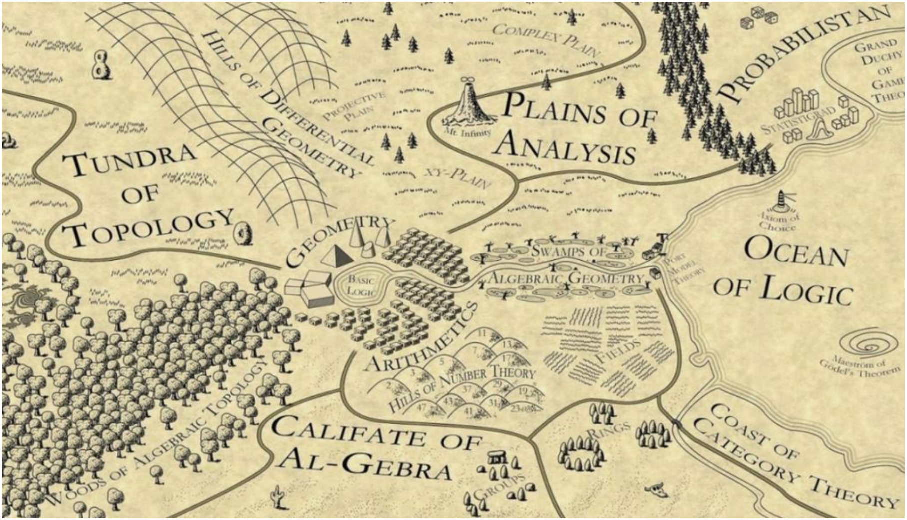
The integer solutions are in a wild structure, $(\mathbb{Z}, +, \times, 0, 1)$.

The variety is studied in the very, very, tame structure $\mathcal{C}$ or

(Hrushovski via Pillay) the tame structure $(\mathcal{C}, \Gamma)$ where Γ is a finitely generated subgroup of the $\mathbb{Q}$-points of an algebraic variety.

Geography of Mathematics



Classification of first order theories: role

The model theoretic classification divides the world of mathematical theories by ‘syntactic’ (set theoretically absolute) characteristics so that methods and results transfer among subjects in quite different areas if those areas are similar under this classification.

The syntactical characterization

A theory T is unstable if there is a formula with the order property.: $\phi(x, y)$ has the order property if there exist a_i, b_i (for $i < \omega$) in a model M of T such that

$$M \models \phi(a_i, b_i) \leftrightarrow i < j.$$

This formula may change from theory to theory.

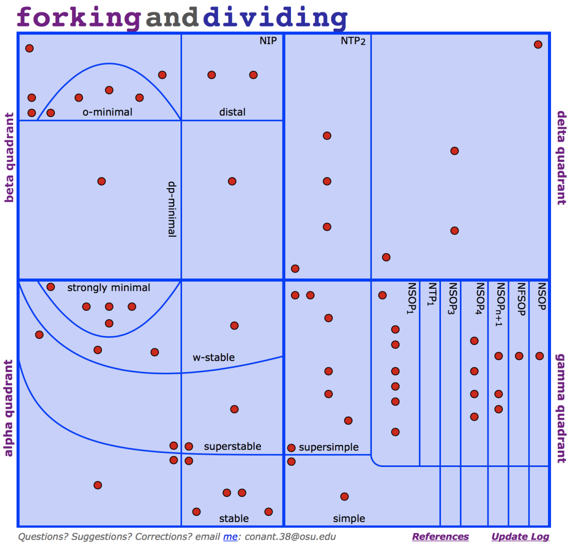
1. In a dense linear order one such ϕ is $x < y$;
2. In the theory of $(\mathbb{Z}, +, 0, \times)$ one such ϕ is $(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
3. random graph: For every finite $X, Y \subseteq X \subset G$ there exists an $a \in G - X$ such that for every $x \in X$:

$$R(a, x) \leftrightarrow x \in Y.$$

   In the random graph one G one such ϕ is the edge relation $R(x, y)$ and it is NOT a linear or even partial order: it witnesses the **independence property**.

It is this flexibility, grounded in the formal language, which underlies the wide applicability of stability theory.

Classification of first order theories



Explaining the diagram

1. tame - stable click stable in upper right
   1. strongly minimal - universe is a geometry
      1. algebraically closed fields
      2. solution of 3rd order pde such as the Weierstass j-function.
   2. ω-stable – often controlled by strongly minimal sets
   3. stable –separably closed fields – why are they understood?
   4. *o*-minimal geometry, definable topology
   5. simple: random graph
2. wild
   1. ZFC, arithmetic
   2. NSOP being explored

Stability Spectrum

Definition

Let T be a first order theory. Fix a *universal domain* $\mathcal{M}$.

T is stable in λ if for each $A \subseteq \mathcal{M}$ with $|A| \leq \lambda$, there are $\leq \lambda$-orbits under $Aut_A(\mathcal{M})$.

Stability Spectrum Theorem

For every countable T, T is stable in exactly one of the following:

1. (ω-stable): all infinite cardinals;
2. (superstable): all cardinals $\geq 2^{|\aleph_0|}$;
3. (stable): those λ with $\lambda^\omega = \lambda$
4. (unstable): never.

The middle attic is a tool for studying the mezzanine.

Geometry and model theoretic tameness

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

Simplest case For $M \models T$, $a \in \mathrm{acl}_M(B)$ if there is a formula $\phi(x, \mathbf{b})$ such that $M \models \phi(x, \mathbf{b})$ and $\phi(M, \mathbf{b})$ is finite.

This dimension generalizes such concepts as Krull/Weil dimension in algebraically closed fields, transcendence degree in differential fields.

For models of appropriate stable theories it assigns a dimension to the model. This is the key to describing general structures.

But in tame structures, the basic notion of algebraic closure greatly impacts transcendence of many kinds in traditional mathematics.

Tame by o-minimality

In ‘Esquisse d’un programme’, Grothendieck asked for a tame topology.
Wilkie ([Wil07], [Bal18, 160])argues that o-minimality is a direct response to Grothendieck’s call because o-minimality:

1. is flexible enough to carry out many geometrical and topological constructions on real functions and on subsets of real Euclidean spaces.
2. builds in restrictions so that we are a priori guaranteed that pathological phenomena can never arise.

   In particular, there is a meaningful notion of dimension for all sets under consideration.

‘The central valley’

The Central Valley

denotes tangled web of Fuchsian groups, hyperbolic geometry, number theory, automorphic forms, complex analysis, algebraic geometry, algebraic topology etc., etc. and recently model theory.

The scaffolds and 'the central valley'

1. Category Theory
   Categorical methods underpin many of the modern developments, e.g. the Weil conjectures.
2. model theoretic contributions are not so well known.
   1. o-minimal theory: Bounds in analytic number theory yielding special cases and eventually the full Andre-Oort conjecture:
   2. stable theories:
      1. Mordell-Lang for Function Fields
      2. Differentially closed field: Strong minimality as 'not integrable from standard functions': transcendence problems around the Painlevé classification and Fuchsian groups

[Bou99, CFN21, FS18, Pil11, PW06]

Symbiosis

Formalism-Free Model theory I

Abstract Elementary Classes

An abstract elementary class is:
a collection $\boldsymbol{K}$ of structures for a vocabulary τ with a binary relation $\leq$ refining subset, that satisfies natural conditions satisfied by first order logic.

Accessible categories [AR94] are a specific kind of category which pays more attention to cardinality; it considers categories where every object is a directed colimit of objects with a bounded number of generators.

Fact

AEC's are certain kinds of concrete accessible categories.

A meeting of the scaffolds

Independence in a categorical setting

Shelah often describes his independence relation (non-forking) as a relation $NF(M_0, M_1, M_2, M_3)$ where $M_0 \subseteq M_1, M_2 \subseteq M_3$.

The definition then involves a number of syntactic relations among the four structures.

Lieberman, Rosický, and Vasey, n [LRV19, Lie21] have defined in arbitrary accessible categories give by diagram axioms on squares a translation of Shelah's independence.

Conflict and Resolution

Foundations and Scaffolds

Conflicting goals for a foundation

1. Generous Arena: ‘provide a Generous Arena where all of modern mathematics takes place side-by-side and a Shared Standard of what counts as a legitimate construction or proof’
2. Essential Guidance: ‘that would guide mathematicians toward the important structures and characterize them strictly in terms of their mathematically essential features’ [Mad19]

Two ways to resolve the conflict

1. MST and SST provide bi-interpretable set theoretic foundations.
2. Separate Foundations from Scaffolds:
    1. Foundations
        1. type theory
        2. material or structural set theory
    2. Scaffolds:
       Category theory and Model theory are complementary scaffolds
       Scaffolds offer **productive** guidance rather than **essential** guidance.
       Different scaffolds can offer different guidance.

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Let many flowers grow!

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