Model Theoretic Perspectives on the Philosophy of Mathematics

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And some infinites are larger than other infinites and some are smaller.

Robert Grosseteste 13th century [Fre54]

We approach the 'practice based philosophy of logic' by examining the practice in one specific area of logic, model theory, over the last century. From this we try to draw lessons not for the philosophy of logic but for the philosophy of mathematics. We argue in fact that the philosophical impact of the developments in mathematical logic during the last half of the twentieth century were obscured by their mathematical depth and by the intertwining with mathematics. That is, that concepts which are normally regarded by both mathematicians and philosophers as 'simply mathematics' have philosophical importance. We make two claims. First is that the mere fact that logical methods have had mathematical impact is important for any investigation of mathematical methodology. Twentieth century logic introduced techniques that were important not just for the problems they were originally designed to solve (arising out of Hilbert's program) but across broad areas of mathematics. But, from a philosophical standpoint, there is a further impact. These methods actually provide tools for the analysis of mathematical methodology.

The longtime standard definition of logic is "the analysis of methods of reasoning". This does not describe the perspective of a contemporary model theorist. A model theorist is a self-conscious mathematician. A model theorist uses various formal languages and semantics to prove mathematical theorems. But there is an inherently metamathematical aspect. The very notion of model theory involves seeking common patterns across distinct areas of mathematical investigation. One of our goals below is to make precise this notion of 'distinct area'.

We view the philosophy of mathematics as a broad inquiry into and critical analysis of the conceptual foundations of actual mathematics work[1]. This investigation also include a study of the basic methodologies and proof techniques of the subject [2]. The foundationalist goal of justifying mathematics is a part of this study. But the study we envision cannot be carried out by interpreting the theory into an über theory such as ZFC; too much information is lost. The coding does not reflect the ethos of the particular subject area of mathematics. The intuition behind fundamental ideas such as homomorphism or manifold disappears when looking at a complicated definition of the notion in a language whose only symbol is ϵ. Tools must be developed for the analysis and comparison of distinct areas of mathematics in a way that maintains meaning; a simple truth

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[1] This is a paraphrase of part of Dutilh-Novaes presentation.

[2] For a broad investigation the philosophy of mathematics including a study of leading contemporary mathematicians (e.g. Grothendieck, Langlands, Shelah, Zilber) see [Zal09]

preserving transformation into statements of set theory is inherently inadequate. The traditional foundationalist approach sacrifices explanation on the altar of justification.

The discussion below has both a sociological and a philosophical aspect. Sociologically, in the remainder of this introduction, we describe recent examples to illustrate the practice of the model theoretic species of logician. Philosophically, in the two main sections of the paper we propose some tools for studying the methodology of mathematics. We aim to sketch a program for using model theoretic concepts for a) formalizing a notion of 'area of mathematics' and b) analyzing basic concepts of mathematics. In Section 1) we sketch the history of model theory in the twentieth century and in particular the development of the notion of a complete theory. We argue for the notion of a first order complete theory as a useful unit of analysis for describing 'an area of mathematics'. We conclude the historical discussion with an introduction to the sophisticated model theoretic methods developed in the last 40 years. In Section 2) we discuss how these methods can provide insight into the way fundamental notions are specified in different areas of mathematics. We discuss in detail the analysis of one particular mathematical notion, dimension, using model theoretic notions.

In both cases, our main point is that model theoretic tools can be brought to bear. We are simply giving introductory sketches of illustrations of that thesis.

For brevity, most of the emphasis is on first order logic. But important extensions to infinitary logic and even 'syntax deprived' model theory will appear later in the paper. In the sociological mode, we now list the titles of papers from the Mid-Atlantic Model Theory conference held in the Fall of 2008 at Rutgers. Our summary of this conference focuses on two currents of 'main-stream' model theory represented at this conference. It does not encompass a number of other areas of model theory such as models of arithmetic, finite model theory, model theory in computer science, higher order and other extensions of first order logic, and universal algebra.

1. Model theory and non-archimedean geometry
2. The valuation inequality for complex analytic structure
3. Cherlin's Conjecture and Generix's Adventures in Groupland
4. ω-stable semi-Abelian varieties
5. O-minimal triangulation respecting a standard part map
6. Some modest attempts at defining the notions of groups and fields of dimension one, and establishing their algebraic properties
7. Dependent theories: limit model existence and recounting the number of types
8. The non-elementary model theory of analytic Zariski structures
9. Difference fields, model theory and applications
10. Model Theory of the Adeles

The two currents of model theory that I want to contrast focus, broadly speaking, a) on the use of model theory in various parts of mathematics and b) on the development of an independent subject area of 'model theory'. In the early 70's these seemed wildly divergent subjects. But now, at least seven of the papers above, even those focused on algebraic notions such as non-archimedean geometry, semi-abelian varieties, or difference fields integrate the fundamental concepts introduced in the pure theory. For example, paper 3) concerns a conjecture of Cherlin which uses model theoretic concepts to lift the program of the classification of finite simple groups to the classification of simple groups of finite Morley rank. Even the statement of the problem is posed in model theoretic terms (that we discuss below). But this terminology provides a way to organize topics that are already in the mathematical air. The investigation involves significant techniques from model theory, finite group theory, and algebraic groups. Even the relatively few papers at this conference that were 'pure' developed concepts central to current research in e.g. the model theory of valued fields[3].

1 Historical Survey of model theory

The integral connections of model theory with modern mathematics as described in the introduction are often (and often correctly) seen as a falling away from philosophical concerns. But as we'll see below, many of these interactions do stem from concerns about explanation and coherence of mathematical ideas that have a philosophical basis. The divorce is from narrow concern with the formal justification of results. And there are natural philosophical issues that arise from more technical results. As noted below there are vast differences between the role of $\aleph_0$ and any uncountable cardinal in the study of categoricity. What is so different about countability?

After a survey of the history of model theory I expound the use of model theoretic concepts as a tool for such an analysis of the foundations of mathematics. We review this history from a standpoint similar to this paper but with an emphasis on the mathematical applications in [Bal0x].

We distinguish three types of analysis in first order model theory:

1. Properties of first order logic (1930-1965)
2. Properties of complete theories (1950-present)
3. Properties of classes of theories (1970-present)

1.1 Properties of first order logic (1930-1965)

The essence of model theory is a clear distinction between syntax and semantics. Sentences in a formal language for a vocabulary τ are true or false in structures for τ. While the full formal treatment of this notion first appears in [Tar35], the basic idea is essential for Gódel's completeness theorem [Göd29] to even make sense. While the completeness theorem plays a fundamental role in first order model theory, a formal proof system is *not* essential to formulating many of the crucial concepts.

The prehistory of model theory include the work before 1950 Löwenheim, Skolem, Gödel, Malcev, and Tarski. They isolated the fundamental properties of first order logic such as completeness, compactness, and the Lowenheim-Skolem-Tarski theorem. The prehistorical aspect is illustrated by references in logic courses to the 'Lowenheim-Skolem-Tarski theorem' and its proof by Malcev and Gödel.

The term model theory was popularised in the early 1950's, especially by Tarski and Robinson. Work in that decade provided syntactic characterization of preservation properties. E.g., The models of a first order theory are closed under unions of chains if and only the theory is axiomatized by 'for all, there exist' sentences. But what we might now call 'syntactic' and 'semantic' formulations are described as more of a contrast between 'logical' and 'mathematical'. In [Tar54], Tarski writes 'universal classes can be characterized in a purely mathematical terms'. The compactness theorem is given both 'logical' proofs from the completeness theorem and 'mathematical' proofs

[3]Shelah's concept of theories without the independence property (nip or dependent depending on the author) were expounded in the least-applied talk. Hrushovski's paper 'Stable groups and approximate group theory'[Hru09], which uses the model theoretic analysis of these theories as a tool for the study of groups, was the subject of semester-long seminars at UCLA, Berkeley, Urbana, and Leeds in the Fall of 2009. Fields medalist Terrence Tao discusses the progress of the UCLA seminar in the blog at `http://terrytao.wordpress.com/`.

via ultraproducts. Tarski and Vaught [TV56] define the notion of elementary extension and prove the union of elementary chain is an elementary extension of each member of the chain; this both refines the original theory and helps to develop the correct category for model theory.

Further general properties of first order logic developed in the 50's included interpolation theorems and the Robinson Consistency theorem. Much model theoretic work in the 60's and 70's extended these kinds of notions to logic with infinite conjunctions or with generalized quantifiers of various sorts. But we want to focus on a crucial idea that crystalized in the 1950's: a complete theory.

Before proceeding to complete theories we discuss a different notion with the same name: completeness of a logic. By a logic, we mean as in [BF85] a syntactical notion of a collection of sentences $\mathcal{L}(\tau)$ for a vocabulary τ and a satisfaction relation $\models_{\mathcal{L}}$ between sentences $\phi \in \mathcal{L}(\tau)$ and τ-structures M. The logic is complete if there is some proof system $\vdash_{\mathcal{L}}$ of $\mathcal{L}$ such that:

$$\vdash_{\mathcal{L}} \phi \text{ if and only if for every } M \models_{\mathcal{L}} \phi.$$

A theory T is a consistent set of sentences in a logic $\mathcal{L}$. (We will consider first order, second order, $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$.)

Our discussion of prehistoric times is not complete without mentioning the American Postulate Theorists [AR02a, AR02b]. Already in 1902, Huntington introduced the notion of an axiom system having exactly one model. By 1904 [Veb04], this notion had been christened 'categoricity' and Veblen proves the categoricity of a set of (second order) axioms for geometry. Following the terminology of [AR02a], which is reasonably standard, we say.

Definition 1 1. *A theory T is* semantically $\mathcal{L}$-complete *if for each $\mathcal{L}$-sentence ϕ and any pair of models M, N of T,*

$$M \models_{\mathcal{L}} \phi \text{ if and only if } N \models_{\mathcal{L}} \phi.$$

2. *A theory T is* deductively (or syntactically) $\mathcal{L}$-complete *if for each $\mathcal{L}$-sentence ϕ either $T \vdash_{\mathcal{L}} \phi$ or $T \vdash_{\mathcal{L}} \neg\phi$*

If $\mathcal{L}$ satisfies the (extended) completeness theorem then these notions are equivalent. Again as reported in [AR02a], Fraenkel [Fra28] had distinguished these notions without establishing that they are really distinct. In [Ken], Kennedy discusses the significance of the first paragraph of Gödel's thesis. She points out this distinction becomes clear only with Gödel's proof of the completeness theorem. Kennedy further notes that Gödel argues that categoricity and an effective proof theory implies syntactic completeness. Thus Gödel foreshadows the incompleteness theorem in his argument that a proof is needed for completeness (contrary to the view that 'consistency implies existence' is tautological). There is a categorical axiomatization of the real numbers with arithmetic in second order logic; this yields semantic, but not syntactic completeness of the second order theory. Vaught's proof [Vau54] of the Łos-Vaught test (a first order theory with no finite models that is categorical in some infinite power is complete) writes the argument in modern terms[4]: Categoricity plus upward and downward Löwenheim-Skolem implies semantic completeness; syntactic completeness follows by Gödel. What now seem obvious compactness arguments for the existence of non-standard models were clearly *not* in the air in 1930 [Ken, Vau86].

Note that for any structure M, $\mathrm{Th}(M) = \{\phi : M \models \phi\}$ is a *semantically* complete theory for every logic $\mathcal{L}$ is under consideration. This method of obtaining complete theories is fundamental.

[4]There is no indication of a connection with the Gödels argument cited above.

1.2 Properties of complete theories (1950-present)

The mathematical significance of the fundamental notion of a first order complete theory was stressed by Abraham Robinson [Rob56]. He provides a number of mathematically interesting examples of complete first order theories and shows common model theoretic characteristics involving the form of the axiomatization or quantifier elimination for a number of them.

Axiomatic theories arise from two distinct motivations. One is to understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$. The other is to find the common characteristics of a number of structures; theories of the second sort include groups, rings, fields etc. There are a number of *second order* theories of the first sort that are categorical.

Both of these motivations aim at studying fundamental properties which determine all properties of a structure or a group of structures. But the axiomatizations have quite different impact. The (usually) second order axioms characterizing a single important structure delineate exactly what makes that structure unique. These axioms illuminate a key feature of the structure: the reals are the unique complete ordered field with a countable dense subset. But this light is shed on the particular structure.

Bourbaki represents a triumph of axiomatization for the second reason. Large parts of mathematics were organized into coherent topics by providing informative axiomatizations.

Let us consider the relation with categoricity. To avoid trivialities, we deal only with infinite models. T is categorical if it has exactly one model (up to isomorphism). T is categorical in power κ if it has exactly one model in cardinality κ.

Note that under these definitions, every categorical first order theory is semantically complete. Further every theory in a logic which admits the upward and downward Löwenheim-Skolem theorem for *theories* that is categorical in some infinite cardinality is semantically complete. First order logic is the only one of our examples that satisfies this condition. Semantic (and indeed syntactic completeness) can be deduced from $\aleph_1$-categoricity for sentences of $L_{\omega_1,\omega}$ [She83a, She83b, Bal09]. At present the $\aleph_1$ plays an essential role in the proof.

Most people have an intuition for only a few infinite structures: arithmetic on the natural numbers, the rationals, and perhaps on the reals. Most mathematicians extend this to the complex numbers and then to a deeper understanding of various structures depending on their own specialization: ($SL_2(\Re)$, $\boldsymbol{P}_1$, initial segments of the ordinals [5]). But all these structures have cardinality at most the continuum. There are few strong intuitions of structures with cardinality greater than the continuum. However, there is a crucial exception to this remark. It is rather easy to visualize a model that consist of copies of single countable or finite object. Consider a vocabulary with a unary function f. Assert that $f(x)$ never equals x but $f^2(x) = x$. Then any model is a collection of 2-cycles. On the one hand we have the notion that there are models of arbitrarily large cardinality but I really have no really different vision distinguishing among the models of different large cardinality. This situation generalizes when the number of disjoint copies of the same structure is replaced by the dimension of a vector space or field. Thus we might consider the class of structures A_κ, a direct sum of κ copies of Z_2. The isomorphism type of the model depends solely on the number κ of copies (and not at all on the internal structure of the cardinal κ).

Many such visualizable structures arise in a standard way; they are the class of models of a first order theory that is categorical in all uncountable cardinalities. Categoricity is not a necessary condition for such a clear visualization: consider an equivalence relation with two infinite classes, fix a totally categorical theory and make each class a model of the given theory. Each model is determined by two cardinals– the cardinality (or more precisely the dimension) of each equivalence class. More sophisticated investigation and slightly relaxing the notion of 'visualize' shows that categoricity does provide a sufficient condition for such a visualization. And then interpreting 'visualise' as: admitting a structure theorem (in the sense of Subsection 2.1), we can obtain

[5] Recall Paul Cohen's intelligence test: for what ordinals can you visualize the descending chains witnessing well-foundedness?

exact conditions for being able to 'see' all models of a first order theory.

There is little general information to be discovered about a structure, just by the observation that it is the unique model of a second order sentence[6]. However, the situation for *categoricity in power* of a first order sentence is quite different. First order theories that are categorical in an uncountable power share a number of attributes that flow from that fact. Further their study stimulated a powerful unifying technique for the study of first order theories. Again, we contrast the two perspectives of investigating a particular structure and investigating a class of structures.

Starting with a single prototypical structure, such as the complex field, categoricity in power is the best approximation that first order logic can make to categoricity. But, it turns out to have far more profound implications for studying the original structure. If the axioms are universal existential then the theory is model complete (and under slightly more technical conditions admits elimination of quantifiers). Thus the complexity of definable sets is determined by global properties of the models. This general structural condition replaces what can be very technical proofs of quantifier elimination by induction on quantifiers that depend on the specific theory. We explore in Subsection 2.1 the fact that every model of an $\aleph_1$-categorical theory is 'determined' by a definable *strongly minimal* set which admits a dimension theory similar to that of vector spaces.

Work of e.g. Robinson, Tarski, Vaught, Łoś, Ehrenfeucht, Mostowski, Keisler, Morley, Shelah led to the understanding that complete first order theories admitting elimination of quantifiers provided the most fruitful field of study. Elimination of quantifiers can arise in two radically different ways. By fiat: Morley noticed that there is an extension by explicit definition of any complete first order theory which has elimination of quantifiers. Most studies in pure model theory adopt the convention that this has taken place. But this extension requires a large price; the vocabulary is no longer tied to the basic concepts of the area of mathematics. Thus algebraic model theorists work very hard to find the minimal extension by definitions that must be made to obtain quantifier elimination (or the weaker model completeness). But there is a clear understanding in either case that it is desirable to have a limited number (of alternations) of quantifiers available so that definable sets can be analyzed.

Starting from a class of structures, there is little gained simply from knowing a class is axiomatized by first order sentences. In general, the various completions of the theory simply provide too many alternatives. But for complete theories, the models are sufficiently similar so information can be transferred from one to another. One example is transfer from an analytic proof of the classification of finite dimensional algebras over the reals to classification of finite dimensional algebras over an arbitrary real closed field. The Lefschetz principle in algebraic geometry provides an interesting application by considering different completions of the theory of algebraically closed fields. Each completion is determined by specifying a characteristic and the informal Lefschetz principle of algebraic geometry can be formalized as any sentence true in an algebraically closed field of characteristic 0 is true in algebraically closed fields of characteristic p for almost all p.

Beeson [Bee] notes that the theory of 'constructible geometry' (i.e. the geometry of ruler and compass) is undecidable. This result is an application of Ziegler's proof [Zie82] that any *finitely axiomatizable theory* in the vocabulary $(+, \cdot, 0, 1)$ of which the real field is a model is undecidable. Thus the complete theory is tractable while none of its finitely axiomatized subtheories are.

Model theory is often characterized as the study of definability. But the deeper results, even in applications, are about uniform definability over all the models of a complete theory. This is evidence for our first thesis.

Thesis I: Studying the models of different (complete first order) theories provides a framework for understanding the foundations of *specific areas* of mathematics.

The study of complete theories has become the basic framework for model theoretic investigations. We discuss

[6] Jouko Vaananen has pointed out: If $V = L$, then a structure is a model of a second order categorical sentence if and only if it has a second order complete characterization by a single sentence. When $V \neq L$, it is possible that some structure has a second order complete characterization by a single sentence but no second order sentence characterizes the structure up to isomorphism.

in Section 2 the classification of theories according to structural properties. Over the twentieth century there has been an important shift in the choice of which logic to use for formalization and in choice of which mathematical topic to investigate. Early work focused on 'foundational theories' and the line between the various logics had not yet been clarified. The introduction to Gödel's thesis [Ken] implicitly assumed that any system (at least of the real numbers) will include an axiomatization of arithmetic. But work of Gödel and Tarski shortly after that thesis established that arithmetic is undecidable while the theory of the real field is decidable. Most current model theoretic research into specific theories focuses on theories are both mathematically important and tractable. Model theory has given tools for discovering which theories are tractable. The gain is that many theories of general mathematical interest are tractable. But the cost is that tractable theories are *not* foundational in the traditional sense; both ZFC and PA suffer from the Gödel phenomena and are not susceptible to the general model theoretic techniques discussed here. By the Gödel phenomena, we mean the existence of a pairing function and sufficient strength to encode syntax. A theory displaying the Gödel phenomena will be undecidable for intrinsic reasons. (It is perfectly possible to code undecidability into the axioms of extremely well-behaved theories.) There is in fact another area of model theory which specifically studies models of arithmetic. There is some overlap of techniques but there is a different viewpoint [KS06]. Because of the foundational significance the interplay between PA and true arithmetic is an important theme.

As one example of the use of complete theories to provide a foundation for a specific area of mathematics, we consider algebraic geometry. A long standing model theoretic aphorism asserts: Algebraic geometry is the study of definable subsets of algebraically closed fields. There is much truth in this. Algebraic geometry studies the solution in fields of systems of equations. And the requisite unity of studying solution sets in different fields is provided by using the complex numbers as a universal domain and interpreting the same equation in different subfields. Even more, the notion of a generic point on a variety which is a bit 'squishy' in e.g., [Lan64], becomes clear under the Morley analysis: a generic point of a variety is a realization in an extension field of a type of maximum Morley rank in the variety. The Weil-Hrushovski theorem, every constructible group is definably isomorphic to an algebraic group (Theorem 4.13 of [Poi87]), is a further example of definability providing a different conceptual foundation for a fundamental mathematical idea.

But the aphorism fails in two ways. The most obvious is that algebraists are concerned with systems of equations. This seems to be a great deal more restrictive than arbitrary first order definability. After all neither logical connectives nor quantifiers are involved. But the quantifiers are illusory. A fundamental result goes by two names with rather different connotations: Chevalley-Tarski Theorem:

1. Chevalley: The projection of a constructible set is constructible.
2. Tarski: The theory of algebraicaly closed fields admits elimination of quantifiers.

The connection between the two versions is the observation that projection of defined by $\phi(x_1, \ldots x_n)$ in n-space to $n-1$-space is the solution set of $\exists(x_n)\phi(x_1, \ldots x_n)$. This theorem shows that any first order definable subset in an algebraically closed field is definable by a Boolean combination of equations. But the algebraic geometer really distinguishes the case where there are no negations (a conjunction of equations - a trick makes disjunctions disappear.) However in the early 90's Hrushovski and Zilber [HZ93] introduced the notion of a Zariski geometry, which via the use of a topology provides an abstract basis for being able to distinguish sets definable by positive formulas.

The second drawback is that, more precisely, this approach describes 'Weil' style algebraic geometry of the 1950's and does not directly interpret the more modern 'Grothendieck' style. There is disagreement about the significance of this alleged weakness in the usual model theoretic approach [Mac03, Hru02].

There are a number of important theorems that invoke model theoretic ideas to attain more traditional mathematical results.

1. Artin-Schreier theorem (A. Robinson)
2. Decidability and quantifier eliminability of the real field (Tarski)
3. Decidability and quantifier eliminability of the complex field (Tarski/Robinson)
4. Decidability and model completeness of valued fields (Ax-Kochen-Ershov)
5. Quantifier elimination for p-adic fields (Macintyre)
6. O-minimality of the real exponential field (Wilkie)

Many of these results seem to be 'logical' as they concern such notions as decidability and quantifier elimination. In fact, as noted in the case of the Chevalley-Tarski theorem, the notion of 'quantifier elimination' answered fundamental mathematical questions. Even more, each of the last three results leads to the solution of long-standing mathematical problems.

We have argued that the notion of a complete theory provides an appropriate unit of analysis for distinguishing an area of mathematics. In the examples so far the main model theoretic idea is definability and the main tools are compactness and elimination of quantifiers. In the next section we will discuss more sophisticated model theoretic tools and their mathematical role. These techniques also illustrate the more important philosophical contribution of model theory: providing tools for understanding the connections across areas.

1.3 Properties of classes of theories (1970-present)

The development of Shelah's stability theory could be (and indeed was) misperceived as mere technical mathematics concerned with abstruse cardinalities. As we'll see it provides both a mathematically powerful classification of areas of mathematics and tools for methodological investigations.

Thesis II: Studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics.

The second current of model theory revolves around properties of classes of theories. The key to this analysis is Shelah's concept of the Stability Hierarchy.

Theorem 2 (Shelah) *Every complete first order theory T falls into one of the following 4 classes.*

1. *ω-stable*
2. *superstable but not ω-stable*
3. *stable but not superstable*
4. *unstable*

Moving down this list in general reflects decreasing structure of the models of T. Note that the hierarchy provides an organization of various areas of mathematics that illuminates connections that are not apparent from the usual mathematical standpoints. We list a number of different algebraic examples at various levels in the hierarchy. Some ω-stable theories are: algebraically closed fields (of any fixed characteristic) and algebraic groups over algebraically closed fields , differentially closed fields (of characteristic 0), compact complex manifolds. Some strictly superstable theories are: $(\mathbb{Z}, +)$, $(Z_2^{\omega}, H_i)_{i<\omega}$ (where H_i is a subgroup of finite index). Some strictly stable theories are: $(\mathbb{Z}, +)^{\omega}$ and separably closed fields of characteristic p. Unstable theories include Arithmetic, Real closed fields, complex exponentiation, and the theory of the random graph. Recent model theoretic work in two directions provide systematic tools to distinguish and analyze theories with intractible Gödel phenomena from those more susceptible to model theoretic analysis. The two directions are dependent theories (theories without the independence property)[HPP08, She] and infinitary logic [Zil04].

The basic idea of type, already central in many of the investigations discussed in Subsection 1.2 is essential to understand the notion of stability.

Definition 3 *Let M be a structure and write $F^n(M)$ for the Boolean algebra of all formulas with n free variables and constants for elements of M. $F^n(T)$ is the Boolean algebra of formulas with no parameters.*

1. (syntactic:) *A* complete n-type *over M is an ultrafilter in the Boolean algebra $F^n(M)$.*
2. (semantic:) *Let $N \succ M$ and $a \in N$.*

$$\mathrm{tp}(\boldsymbol{a}/M) = \{\phi(\mathbf{x}, \mathbf{m}) : \phi(\mathbf{x}, \mathbf{m}) \in F(M) \text{ and } N \models \phi(\boldsymbol{a}, \mathbf{m})\}.$$

3. *$S(M)$ denotes the set of types over M.*

Just as above we defined a complete theory both syntactically and semantically, we now have similar dual definition for a type[7].

Perhaps the most basic feature is that this classification provides a totally new way of organizing mathematical discourse. The underlying invariant is the cardinality of the Stone space of the Boolean algebra of formulas over a model of T. That is, if we say T [8] is stable in λ if for every M with $|M| = \lambda$, $|S(M)| = \lambda$, ω-stable implies stable in all λ, superstable means stable above the continuum; stable means stable in some λ and unstable means stable in no λ. But this purely model theoretic and apparently combinatorial notion imposes important structural conditions on the models of the theory that we discuss in Section 2.

Shelah's techniques (that we sketch below) for analysis of models of stable theories and his more complex notions such as: orthogonality, canonical bases, regular types, etc. have many applications. In particular, Hrushovski combined these methods and those of 'geometric stability theory' with a deep understanding of Diophantine geometry to provide fundamental advances related to the Mordell-Lang conjecture [Bou99, Hru96]. Notably, although the application is to an ω-stable theory of algebraically closed fields; the analysis (for the characteristic p-case) involves strictly stable theories of separably closed fields. We have noted first that both basic model theoretic ideas of definability and compactness and later the more sophisticated model theoretic methods have been used to solve problems of core mathematics. Just this fact is important from the standpoint of any analysis of mathematical methodology. But these model theoretic tools themselves provide tools for analysis. On their face they illustrate distinctions and similarities across different areas. In the next section of the paper we show that these tools allow us to analyze some mathematical notions as they span areas of mathematics.

The simplest notion of type is when the domain is the empty set. And the 1950's already provided a characterization of countable categoricity in terms of the type space over the empty set.

Theorem 4 *[Eng59, RN59, Sve59] A first order theory is $\aleph_0$-categorical if and only if $F^n(T)$ is finite for each n.*

The property of $\aleph_0$ categoricity is virtually orthogonal to the stability hierarchy. There are examples that are ω-stable (vector spaces over a finite field) and examples that are unstable: dense linear order without

[7] In fact it is the same; we could think of p as the complete theory of the structure $M \cup \{a\}$

[8] We restrict to countable theories for simplicity.

endpoints and the random graph. As noted above the unstable theories have wild uncountable models. Thus $\aleph_0$ categoricity does not have strong implications for a theory to be well-behaved. In contrast, categoricity in uncountable cardinalities has deep structural consequences that reflect a fundamental mathematical notion: dimension. That is the subject of our next section. It is crucial to analyze not just types over the empty set but types over arbitrary models.

2 Concept Analysis: Dimension

Our general claim is that the techniques and concepts developed in stability theory can be useful for a philosophical investigation of the methodology of mathematics. We illustrate this claim by studying the notion of dimension. Other notions that could be given a similar analysis include: chain conditions, notions of finiteness, 'genericity', group actions (E.g., what are sufficient conditions for the development of Galois Theory [MTB0x]?).

In this section we develop two themes. The notion of dimension is a basic mathematical idea and model theory provides a unifying approach among various avatars of this notion. Moreover, the stability hierarchy provides a way to compare different areas of mathematics in terms of the strength of their dimension notion.

The article on Dimension in [Gow08] suggests five notions of dimension that occur in such fields as real or complex geometry, differential geometry, topology and algebra. We discuss two formulations of the notion here. For purposes of this essay we call one of them size and the other rank; each would normally just be called dimension. No such rigid distinction between the notion exists in mathematics and we will see why. They reflect an 'algebraic' and a 'geometric' perspective on the notion.

I. Size The reals have uncountable dimension as a $\mathbb{Q}$-vector space.

II. Rank A surface is a two-dimensional set.

Size is a measure of a particular mathematical structure. Rank is a measure of a definable set and in interesting cases will make sense (and be invariant) over different models of the same theory. As we noted, the 'dimension' of a vector space is a natural example of size. We explore that notion in more generality in the next subsection. Consider the solution set of the equation $x^2 + y^2 = 1$ in an algebraically closed field. In the natural model of the complex numbers, it is a circle and so geometrically has dimension 1. The formula has Morley rank 1 and we can assign this dimension to the solution set in each algebraically closed field.

2.1 Size

The notion of a combinatorial geometry arises in a number of areas of mathematics. In particular, Van Der Waerden [VdW49] gave a unified treatment of vector space and transcendence degree. The notions of vector space dimension and transcendence degree permeate much of mathematics. Combinatorial geometries play a fundamental role in modern modern theory. In particular, they allow us to describe the notion of dimension which appears in any theory categorical in power. We introduce a couple of technical notions to give a background for the analysis.

A pregeometry is a specific sort of closure system (Axioms A1-A3) which also satisfies the exchange axiom A4. It generalizes the notion of the closure of a subset of a vector space and allows the assigning of a dimension to a set – the cardinality of a maximal independent subset, where X is independent if each $x \in X$ satisfies $x \notin \mathrm{cl}(X - \{x\})$.

Definition 5 *A* pregeometry *is a set G together with a relation*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$

A2. $X \subseteq cl(X)$

A3. *If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.*

A4. $cl(cl(X)) = cl(X)$

If points are closed the structure is called a geometry.

The connection with model theory appears first in the notion of a strongly minimal set [BL71, Mar66]. Model theorists generalize the notion of algebraic closure in field theory (the finitely many solutions of a polynomial equation are 'algebraic') to (if a first order formula with one free variable has only finitely many solutions, each of them is 'algebraic').

Definition 6 *Let $a \in M$ and $B \subset M$.*

1. *a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.*
2. *A complete theory T is strongly minimal if and only if it has infinite models and*
   (a) *algebraic closure induces a pre-geometry on models of T;*
   (b) *any bijection between acl-bases for models of T extends to an isomorphism of the models*

The complex field is strongly minimal. Strongly minimal sets are the building blocks of structures whose *first order* theories are categorical in uncountable power. That is, all models of the same uncountable cardinality are isomorphic. But this notion can (and for some purposes) must be extended beyond the first order context. In particular Zilber introduced the following notion in studying the theory of complex exponentiation. This study must go beyond first order model theory. The kernel of the exponential function is the ring of integers. So the so-called Gödel phenomena, described after Thesis I, make first order model theoretic analysis chaotic here. Zilber's solution [Zil05, Zil04] is to use infinitary logic to insist that the kernel of exponentiation is *exactly* $Z\eta$ for some transcendental η.

Definition 7 *A class $(\boldsymbol{K}, \mathrm{cl})$ is* quasiminimal excellent *if* cl *is a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$:*

1. *Any pair of maximal independent subsets can be mapped from one to the other by an automorphism of the model,*
2. *a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.*
3. *the closure of a countable set is countable*
4. *A more complicated condition on the amalgamation of finite independent configurations of countable models, called excellence.*

Excellence is immediate in the first order context; it is both essential and difficult to obtain in the infinitary context.

Theorem 8 (Zilber) *A quasiminimal excellent class is categorical in every uncountable power.*

Zilber[Zil04] conjectures that the complex exponential field is quasiminimal excellent. More specifically this conjecture provides a set of axioms for the complex exponential field. But this is not sterile axiom chopping. It is not known if the 'axioms' are in fact true of the complex exponential field. Rather, they are a collection of properties that powerfully describe a definite mathematical object. And perhaps they describe one of the most important structures of twentieth century mathematics: the complex field with exponentiation.

Quasiminimal sets are the building blocks of structures whose $L_{\omega_1,\omega}$-theories are categorical in uncountable power. However, Shelah's proof [She83a, She83b, Bal09] of this profound result uses the very weak generalized continuum hypothesis: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ if $n < \omega$.

We have discussed the role of dimension in theories that are categorical in power. These theories are *unidimensional*; one cardinality determines the model. Shelah's stability hierarchy provides a more general machine to study theories where different sets may have different dimensions. The key is to define a family of 'almost combinatorial geometries': $a \in \mathrm{cl}_A(B)$. We say family because for each A, $\mathrm{cl}_A(B)$ (closure of B over A) is almost a 'geometry'.

Definition 9 *A dependence notion is a relation $a \in \mathrm{cl}_A(B)$ such that for each A, $\mathrm{cl}_A(*)$ satisfies the first three conditions of a combinatorial geometry (and coherence conditions among the cl_A's that we don't spell out here [Bal88, She78, Adl08]).*

But the, previously thought crucial, requirement A4, that cl be idempotent, is not required. In the study of vector spaces, the exchange axiom is usually seen as the guarantee of the uniqueness of dimension of a structure. This is an accident of studying notions where closure is easy to obtain and exchange is difficult. Shelah's innovation here is to recognize that there are deeper and more general reasons to obtain exchange (model theorists call it symmetry) than idempotence.

❚ The next page is a target for shortening for this paper.

Even with this weaker notion, dimension can be assigned to uncountable structures. One assigns dimension (size) to types in uncountable models by replacing $cl(cl(X)) = cl(X)$ with: for every B, there is a finite B_0 such that a is independent from B over B_0. This provides an approximate dimension. Two bases for the same formula may differ by a finite number. But if the cardinality of the base is uncountable this is inconsequential. Let us summarize the situation in a theorem.

Theorem 10
1. *In every stable theory, there is a dependence notion.*
2. *In any superstable theory this dependence relation assigns a size to type-definable subsets of* uncountable *models.*

To regain a precise notion of dimension and in particular to study the dimension of countable structures a further notion is needed. One must consider relations on and among regular types. Regularity is a weakening of the axiom A4 (idempotence) which allows again the assignment of a dimension as the cardinality of a maximal independent set.

Definition 11 *$p \in S(A)$ is regular if on realizations of p, if $a \in \mathrm{cl}_A(BC)$ and each $c \in C$ satisfies $c \in \mathrm{cl}_A(B)$ then $a \in \mathrm{cl}_A(B)$.*

On the realizations of a regular type closure gives a combinatorial geometry. Now one can give an exact dimension to models (by means of the relations between various regular types) of superstable theories satisfying certain additional conditions. This leads to a fundamental theorem: the Main Gap [She91].

Theorem 12 (Shelah's Main Gap) *For every first order theory T, either*

1. *Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or*
2. *The theory T has the maximal number of models in all uncountable cardinalities.*

The impact of this theorem is to divide first order theories into two classes. The models (of any cardinality) of a classifiable theory can be decomposed in a uniform way from countable models. The models of unclassifiable theories are creative; new patterns continually emerge.

A fundamental idea, that appears only technical, is to decompose into trees of models. This decomposition was a tool for counting the number of models in each cardinality of a theory. But *systematic representation of a model as prime over a tree of (independent) submodels is a fundamentally new mathematical notion.*

2.2 Rank

The geometrical notion of rank has many exemplars across mathematics. Two of the notions expounded in [Gow08] are closely connected to that expounded here. One intuition from [Gow08] is 'the number of coordinates you need to specify a point'. This corresponds to the notion of size discussed in the previous subsection. Another provides a notion of topological dimension that is reminiscent of the model theoretic ranks that are defined by partitioning into *definable* sets which are disjoint (or at least have small intersection). In [Gow08] topological notions play the role definability plays in stable theories. This interaction among definability and topology is seen in three examples of rank functions. We have discussed the role of independence notions determined by definability in stable theories and have briefly mentioned the role of Zariski geometries which combine topological and definability notions. Here is another example of such a combination.

Definition 13 *An ordered structure is o-minimal if every definable set is a Boolean Combination of intervals.*

o-minimal structures are a natural solution to Grothendieck's request to isolate 'tame' topologies. The prototypic *o*-minimal structure is the real field. Key to the applicability of the idea is the fact that *o*-minimality is a property of a complete theory; *o*-minimality is preserved by elementary equivalence. There has been vast work in the last twenty years on *o*-minimality. Most strikingly, Wilkie[Wil96] showed that the theory of the real exponential field is *o*-minimal. The fundamental notions are summarised in [dD99]. The field is now well integrated with classical real algebraic geometry. And it provides a method to study problems of analysis. For example, Macintyre, Marker, and Van den Dries solved a half-century old problem of Hardy [vdDMM97].

Here are three model theoretic uses of rank.

1. Zariski dimension in algebraic geometry is a special case; it is Morley rank for algebraically closed fields.
2. Notion 1) is generalized to the Hrushovski-Zilber [HZ93] notion of Zariski Geometries.

3. In the class of o-minimal theories there is a rank on definable sets.

A 'Tame Theory' is one which admits a rank on definable sets. By 'admits' we mean that dimension can be assigned on n-tuples of any length meeting certain regularity properties connecting the different dimensions. Roughly, a definable set ϕ has rank n if there is a definable bijection between M^n and $\phi(M)$.

The connection between 'size' and 'rank' is given by the following fact.

Fact 14 *Suppose T has a good notion of independence. Define $R(\phi(\mathbf{x})) = n$ iff the maximal size of a solution $\boldsymbol{a}$ if ϕ is n. This notion of rank has the good properties of a tame theory.*

Many notions of the rank of a definable set are developed in stable theories [She78]. But they connect to notions of independence by arguments like those for Fact 14. Examples of areas of mathematics where this identification applies include real and complex algebraic geometry and indeed any o-minimal theory.

There are crucial limitations on when a sense II notion of dimension (rank) (and thus by Fact 14 notions in sense I) can exist. Arithmetic is the paradigmatic example where no notion of dimension makes sense.

Theorem 15 *1. If a model admits a pairing function, it has no rank (dimension in sense II).*

2. If T admits a pairing function then T is not superstable.

The first of these observations is folklore. Much more well-behaved (from a model-theoretic standpoint) theories can have pairing functions. Lachlan (reported in [BM82]) showed the second result in Theorem 15, which shows that pairing does force a theory to the non-structure side. In particular, any theory with a pairing function has many models in all uncountable cardinalities. Thus the stability hierarchy becomes a tool for determining which theories admit good notions of dimension. Note that the coding of these tame theories into foundational theories such as ZFC or arithmetic completely destroys these salient tame properties of mathematical notion under study.

But there are also important structures, most notably the real field, which do not have pairing functions, which do admit rank functions, but are not stable. These led to the study of o-minimality discussed above.

3 Conclusion

We have discussed three issues concerning the relationship of contemporary model theory, mathematics, and philosophy. The first observation is that model theory is a vigorous part of mathematics that uses tools that were invented for 'logical analysis' to solve problems arising in more traditional mathematics. In this respect model theory differs only in degree from logic in general. Ideas stemming from computability and relative computability permeate computer science and model theoretic ideas arise in many aspects of computer science. Such notions as the Curry-Howard isomorphism and the analysis of weak theories of arithmetic to study computational complexity show the influence of proof theory across mathematical disciplines. Set theory has a similar interaction with mathematics both by the discovery that certain mathematical problems depend on set theoretic principles[9] and by the integration of set theoretic methods with those from dynamical systems in studying the Borel classification of problems [KM04].

The identifying characteristic of *logic* in these mathematical examples is not an 'analysis of reasoning' but an explicit attention to means of definability. The intricate history of the relationship between 'core mathematics' and 'logic' is certainly a fit topic for study in the practice-based philosophy of logic.

[9] This is most common in set theoretic topology; but the Whitehead problem is a notable model theoretic example.

Secondly we made the argument that the notion of a complete theory provides a unit of analysis for examining different areas of mathematics. We both examined the abstract reasons that it is a suitable unit of analysis and examined one case, algebraic geometry, in a bit more detail.

And thirdly, recall Thesis II: studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics. We have illustrated this thesis by connecting the notion of dimension in the study of general model theory and seen how these notions connect with those in algebraic geometry, complex exponentiation and tame topology.

But model theory also provides entirely new areas of mathematics for study. It provides two new general notions of how mathematical properties might change as the cardinality of the structures involved change. Eventual behavior: what happens on all sufficiently large cardinals. Initial behavior: what can we say about the 'lower infinite', cardinals below say, $\beth_{\omega_1}$. Much of core mathematics is much coarser: it studies either properties of particular structures of size at most the continuum or makes assertions that are totally cardinal independent. E.g., if every element of a group has order two then the group is abelian. Model theory allows a more sophisticated analysis in two directions; determination of properties that hold only eventually rather than everywhere and study of classes that are well-behaved on small cardinals to determine whether this behavior propagates to the entire universe. Certain properties allow us to chart the infinite. Some properties (e.g. categoricity for certain classes of models) are now known to be eventual; but major questions remain about from what level they propagate. But other properties (amalgamation, tameness) may not propagate; there is a real difference between large and small models for such properties. Still other properties, e.g. saturation, occur cofinally but not eventually for interesting classes of models. Thus, model theory begins to explore the paradise of the infinite, conceived by Grosseteste and delivered by Cantor. But with Shelah's classification theory the study of infinity moves into adolescence– it moves beyond combinatorial analysis into structural and algebraic investigations.

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