Gerald Sacks, Model Theory and Differentially closed fields

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annotated bibliography, What I learned about model theory and geometry, number theory ...
great reliance on talks/slides/papers by Marker, Freitag, Nagloo

1. Axiomatic Analysis
2. Definable Analysis
3. Implicit Analysis

We don't discuss non-standard analysis as it deals directly with higher order objects.

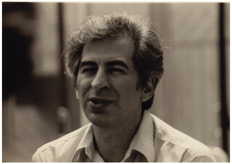
Saturated Model Theory

Saturated Model Theory

GERALD E. SACKS
Massachusetts Institute of Technology

1972
W. A. Benjamin, Inc.
ADVANCED BOOK PROGRAM
Reading, Massachusetts

Saturated Model Theory

Sacks 

The least misleading example of a totally transcendental theory is the theory of differentially closed fields of characteristic 0 (DCF_0).

[Sac72, p 5]

Axiomatic Analysis

Axiomatic Analysis

Axiomatic analysis

1. Study the behavior of fields of functions with operators
2. *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions.
3. The function symbols of the vocabulary act on the functions being studied;
4. the functions are elements of the domain of the model.

The principal example here is differentially closed fields.

But another outstanding exemplar is the theory of Transseries [AvdDvdH17],

Asymptotic Differential Algebra and Model Theory of Transseries by Aschenbrenner, van den Dries, and van der Hoeven.

Differential Fields

A differential field is a field K of characteristic 0 with a derivation $D : K \mapsto K$

$$D(a+b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a).$$

A differential polynomial in variables $X_1, ..., X_n$ over K is an element of the ring $K\{X_1, ..., X_n\}$ which is

$$K[X_1, ..., X_n, D(X_1), ..., D(X_n), ..., D^{(m)}(X_1), ..., D^{(m)}(X_n), ...].$$

The order of $f \in K\{X_1, ..., X_n\}$ is the largest m such that some $D^{(m)}$ occurs. The constant subfield of K is $C_K = \{x \in K : D(x) = 0\}$.

Differentially Closed Field

Robinson 

Abraham Robinson introduced the notion of a differentially closed field, analogous to an algebraically closed field in 1959.

A differential field K is existentially closed if for any finite system Σ of polynomial differential equations having a solution in some $L \supseteq K$ already has a solution in K.

Robinson [Rob59]

1. DCF_0 is the theory of existentially closed differential fields.
2. complicated axiomatization
3. DCF_0 is complete and model complete (Each formula is equivalent to an existential formula.)

Conceptual Axiomatization

Blum 

Axioms

$(K, D) \models \mathrm{DCF}_0$ if

1. K is an algebraically closed field of characteristic zero;
2. If $f(X), g(X) \in K\{X\}$ are nonzero and $\mathrm{order}(f) > \mathrm{order}(g)$, then there is an $x \in K$ such that $f(x) = 0 \wedge g(x) \neq 0$.

Theorem ([Blu68])

DCF_0 is a complete theory with quantifier elimination axiomatizing the theory of existentially closed differential fields.

Differential Closure

Theorem

1. *[Blu68]:* DCF_0 *is* ω*-stable and so has prime models over any subset A of any model.*
2. *By [She72], this model is unique over A.*
3. *But it is* not *minimal [Ros74, She73] and Kolchin.*

Vaught's Conjecture:

Theorem Shelah (1981)

A complete ω-stable theory has either $\leq \aleph_0$ or $2^{\aleph_0}$ countable models.

Theorem (Hrushovski-Sokolović 1992)

There are $2^{\aleph_0}$ countable differentially closed fields of characteristic zero.
(Because it has eni-dop.)

[Har84, HS93, Mar, Pil06, PZ03]

STRONGLY MINIMAL

Lachlan  Marsh 

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Definition

A definable set D is strongly minimal if and only if it is infinite and every definable subset of D is finite or cofinite.

This implies

1. algebraic closure induces a pregeometry on D;
2. any bijection between acl-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension on D
The complex field is strongly minimal.

Integrating in Finite Terms

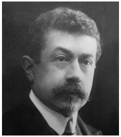
Painlevé termed 'irreducible' a differential equation that could not be solved as a combination of 'known' functions. (See [Ros72] for an introductory account.)

Key Idea: (Nagloo-Pillay)

The solution set of an ODE is strongly minimal iff it is irreducible.

This is something of a 'Turing Thesis'. As the order of the equation goes up, what solutions of lower order equations should be considered classical?
See work of Unemura, [NP17].

Painlevè Property

Painlevè 

Definition

A differential equation has the Painlevè property if the movable singularities are ordinary poles.

Painlevé, Gambier, and Fuchs proved that any order 2 ODE with the Painlevè property can be mapped by fractional linear transformation to one of 50 normal forms. 44 of these are solvable in finite terms (previously known special functions).

Here are the remaining 6.

Painlevé equations

General problem: classify the (differential) algebraic relations over $\mathcal{C}(t)$ between solutions of Painlevé equations.

$$P_I: \quad \frac{d^2y}{dt^2} = 6y^2 + t$$

$$P_{II}(\alpha): \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$P_{III}(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_{IV}(\alpha,\beta): \quad \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

$$P_V(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Irreducibility of Painlevè equations

Painlevè believed that, at least for general values of the parameters, the set defined by the equations would be ‘irreducible’– not integrable in terms of ‘classical’ functions.

A series of papers by K. Okamoto, K. Nishioka, M. Noumi, H. Umemura, and H. Watanabe (survey in [Oka99]) resolved various parts of this intuition.

Identifying irreducibility with strong minimality powers further progress.

The trichotomy Theorem Hrushovski

Zilber conjectured the following trichotomy for the combinatorial geometry of a strongly minimal set.

Zilber Conjecture/Hrushovski-Sokolovič Theorem for DCF_0

Every strongly minimal geometry is one of

1. trivial: $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(x)$.
2. locally modular: X is essentially a vector space. Possibly after adding some constant symbols, X is nonorthogonal to the Manin kernel $A^\sharp$ of some simple abelian variety A that does not descend to the field of constants.
3. field-like: non-orthogonal to the field of constants (admits a differential Galois theory in the sense of Kolchin).

S.M. sets rule

Itai 

Definition

Two strongly minimal sets D_1 and D_2 are *non-orthogonal* ($D_1 \not\perp D_2$) if there is a definable *n*-*m*-ary binary relation definable on $D_1 \times D_2$.

Theorem (Hrushovski-Sokolovìc)

If $X \subseteq K^n$ is strongly minimal and non-locally modular, then $X \not\perp C$, the field of constants.

Theorem [HI03]

An *algebraically* closed differential field K is *differentially* closed if every strongly minimal formula over K has a solution in K.

Geometric Triviality

Geometric triviality: Y is geometrically trivial if for any $y_1, y_2, \ldots y_n \in Y$, if $y \in K\langle y_1, y_2, \ldots y_n\rangle^{\mathrm{alg}}$ there is an i with $1 \leq i \leq n$ such that $y \in K\langle y_i\rangle^{\mathrm{alg}}$.

The Painlevè conjectures Nagloo  Pillay 

Theorem [Pillay-Nagloo]

For generic values of the parameters, each of the equations of the Painlevé families defines a strongly minimal, geometrically trivial, and ω-categorical set.

E.g. in P_{II}, α is transcendental the solution set is strongly minimal and (geometrically) trivial.

K. Nishioka [Nis04] proved the strong minimality (irreducibility) for P_I using differential algebra.

Geometric triviality and even irreducibility when the coefficients are not generic has very different behavior depending on the constants. [FN22] for latest.

Geometric triviality and ω-categoricity

Theorem: [Hru95] If the solution set Y of an order 1 ODE is strongly minimal and geometrically trivial, then Y is ω-categorical.

Conjecture: If Y is the solution set of an ~~order 1~~ ODE that is strongly minimal and geometrically trivial, then Y is ω-categorical.

Theorem: [FS18] REFUTED

(Freitag-Scanlon) The solution set of the 3rd order differential equation satisfied by the j-function is strongly minimal and geometrically trivial, but not ω-categorical

Group Actions

Fact

$\mathrm{PSL}_2(\Re)$ acts on $\mathbb{H} = \{z \in \mathcal{C} : \mathrm{im}(z) > 0\}$ by fractional linear transformations: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$ $A(\tau) = \left(\frac{a\tau+b}{c\tau+d}\right)$.

This is precisely the group of orientation preserving isometries of $\mathbb{H}$.

Fact

For any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(\Re)$, the quotient $\Gamma \setminus \mathbb{H}^*$ is a compact Hausdorff space that can be given the structure of a Riemann surface. Therefore if Γ' is of finite index in Γ, the quotient $\Gamma' \setminus \mathbb{H}^*$ is a compact Riemann surface, and is therefore algebraic by the Riemann existence theorem.

Fuchsian groups

Definition: Fuchsian group

1. A subgroup G of $\mathrm{isom}(\mathbb{H})$ is discrete if it is discrete in the induced topology.
2. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\Re)$.

Definition

An automorphic function for a Fuchsian group Γ is a function such that for all $\tau \in \mathbb{H}$ and $g \in \Gamma$: $f(g\tau) = f(\tau)$.

The collection of all automorphic functions for Γ is a field generated (over $\mathcal{C}$) by an automorphic function j_Γ, called an hauptmodul or uniformizer for Γ.

[Kat92]

ODE return

Classically, each uniformizer j_Γ satisfies a third order ODE of Schwarzian type. Denoting the independent variable by t:

$$(*) \qquad S_{\frac{d}{dt}}(y) + \frac{1}{2}(y')^2 R_\Gamma(y) = 0,$$

where R_Γ is a rational function with coefficients in $\mathcal{C}$, $y' = \frac{dy}{dt}$, and $S_{\frac{d}{dt}}(y) = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)$ denotes the Schwarzian derivative and R_Γ is a rational function over $\mathcal{C}$.

Theorem [FS18]

The Schwarzian ODE satisfied by the classical j-function is strongly minimal and geometrically trivial but not ω-categorical.

This proof applied results of Pila's proved using o-minimality. (Generalization and DCF_0-proof below.)

19th to the 21st century:

Lindemann-Weierstrass Theorem:

If $\alpha_1, \ldots \alpha_n$ are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

Shanuel Conjecture:

If $\alpha_1, \ldots \alpha_n$ are linearly independent over $\mathbb{Q}$, then $\{\alpha_1, \ldots \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ has transcendence degree at least n over $\mathbb{Q}$.

Ax-Lindemann-Weierstrass Problem:

Let $t_1, \ldots t_n \in \mathcal{C}(X)$, X an algebraic variety. Classify when $(j_\Gamma(t_1), \ldots, j_\Gamma(t_n))$ algebraically dependent.

Ax-Lindemann-Weierstrass with derivatives:

Notation

Suppose $P(y, \delta y, \ldots, \delta^{(n)} y) = 0$, where $P \in k[X_0, ..., X_n]$ is irreducible.

For each $m \geq 1$, consider the condition:

(C_m): For any m distinct solutions $a_1, \ldots, a_m \notin k^{alg}$, the sequence of $n \times m$ functions

$\delta^i_j = \delta^i(a_j)$ for $i < n$, $j < m$
is algebraically independent over k.

Note C_{m+1} implies C_m.

Arithmetic Groups

Definition:

1. Groups A and B are commensurable, $A \sim B$, if $[A : A \cap B]$ and $[B : A \cap B]$ are both finite.
2. The commensurator $\mathrm{Comm}(\Gamma)$ of a Fuchsian group Γ is

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{PSL}_2(\Re) : g\Gamma g^{-1} \sim \Gamma\}$$

3. Γ is arithmetic if $[\mathrm{Comm}(\Gamma) : \Gamma]$ is finite (Margulis).

The connection of 'arithmetic' with number theory is beyond this talk.

ALW with derivatives I:

Casale

Blázquez-Sanz

Theorem: Casale, Freitag, Nagloo [CFN21]

Let Γ be a Fuchsian group of first kind and genus zero, and let X_Γ be the set defined by the Schwarzian equation (*). Then:

1. X_Γ is strongly minimal and (so) geometrically trivial.
2. X_Γ is ω-categorical if and only if X_Γ is nonarithmetic.

Theorem: Blázquez-Sanz, Casale, Freitag, Nagloo [BSCFN20]

This paper extends the above result by weakening the hypothesis on Γ.

ALW with derivatives II: Jaoui  Moosa 

When any 3 solutions are independent (Freitag, Jaoui, Moosa)

[FJM18] For an ODE of order $n > 1$, $C_3 \rightarrow C_m$ for all m.

3-independence implies independence. (A natural way to classify sm sets considered but discarded in the 80's.)

The results on the last two slides generalize [FS18]. They use the methods described here and no longer rely on o-minimality.

Generic ODE are strongly minimal Devilbiss 

The solution set X of an algebraic differential polynomial f over K with order h is strongly minimal iff
f is irreducible as a (multivariate) polynomial over $\mathrm{acl}K$ and given any $a \in \mathcal{U} \supseteq K$ with $f(a) = 0$, and any differential field K_1 with $K \leq K_1$ the transcendence degree of $K_1(a)$ over K_1 is either 0 or h.

Theorem (Devilbiss-Freitag)

If the coefficients of f are algebraically independent over K, X is strongly minimal.

Definable Analysis

Definable Analysis

1. The domain of the functions is the universe of the model.
2. The functions being studied are the compositions of the functions named in the vocabulary; one cannot quantify over them.

The main model theoretic tool is *o*-minimality.
The study of linearly ordered structures such that **Every definable subset is a Boolean combination of intervals**.
Sample theorems involving *o*-minimality

1. $(\Re, +, \times, \exp)$ is o-minimal [Wil96]
2. Pila-Wilkie theorem [PW06]
3. Andrè-Oort [Pil11]
4. Ax-Lindemann-Weierstrass using o-minimality [PT14]

o-minimal approach to transcendence theory

Recall is an o-minimal structure in which the definable sets are given by inequalities built from the algebraic functions, the exponential function, and real analytic functions restricted to bounded sets
The analytic function under consideration (restricted to an appropriate fundamental domain) is interpretable in $\Re_{an,exp}$. (See [CFN21].)

Key tool:

Pila-Wilkie theorem

There are few rational points on the transcendental part of a set definable in an o-minimal expansion of the real field.
The number of such points of height T grows more slowly that T^ϵ

(See [CFN21] for connections to Axiomatic analysis approach.)

Implicit Analysis

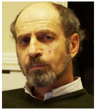
implicit analysis

Goal

Provide 'algebraic' characterizations of important mathematical structures by axiomatizations in infinitary logic that are categorical in power.

Goal: Zilber's $L_{\omega_1,\omega}(Q)$ theory of psuedo-exponentiation – conjecturally an axiomatization $\mathrm{Th}(\mathcal{C}, +, \times, e^x)$ – that is categorial in all uncountable powers.

Zilber's program

Zilber 

The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture), was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics.
([Zil05a])

The geometric value of the project is perhaps in the fact that the formulation of the categorical theory of the universal cover of a variety X (essentially the description of $\mathbb{U}$) is essentially a formulation of a complete formal invariant of X. [ZD21, 2]

Categoricity of Covering Spaces

topic	paper	method
Complex exponentiation	[Zil05b]	quasiminimality
cover of mult group	[Zil06]	quasiminimality
	[BZ11]	quasiminimality
j-function	[Har14]	quasiminimality
Modular/Shimura Curves	[DH17]	quasiminimality
	[ZD21]	quasiminimality
Abelian Varieties & Kummer	[BGH14]	finite Morley rank groups
	[BHP20]	fmr & notop
Shimura *varieties*	[Ete22]	notop

The simplest example of such a cover is given by the short exact sequence:

$$0 \to \ker(\exp) \to (\mathcal{C}, +) \xrightarrow{q} (\mathcal{C}, \cdot) \to 1. \qquad (1)$$

The logic $L_{\omega_1,\omega}$ is used to guarantee the kernel is 'standard' – isomorphic to $(\mathbb{Z}, +)$.

More recent works extends to algebraic geometry.

$$X \xrightarrow{q} \mathbb{V} \to 1. \qquad (2)$$

The map remains a projection but there is no longer a kernel when $\mathbb{V}$ is not a group. So the infinitary description needs at least a 'standard fibre' rather than a 'standard kernel'.

The two-sorted structure

We want to give an 'algebraic' axiomatization of a classical universal cover.

$$X^+ \stackrel{p}{\to} \mathbb{V} \to 1. \tag{3}$$

Situation:

Let the variety $\mathbb{V}$ be given as the quotient of the action a discrete group on $\mathbb{H}$ (hyperbolic space) or a more generally a hermitian symmetric domain X^+ (Shimura varieties) and suppose $T := \mathrm{Th}(\mathbb{V})$ in a large enough countable language that T has quantifier elimination. (Obtained because $\mathbb{V}$ is definable in $(\mathcal{C}, +, \times)$.)

The two-sorted structure

A structure will be of the form $\langle D, S(F), q\rangle$ where

1. **covering sort** The structure on D will include function symbols f_g for $g \in G$, and perhaps more complicated relations $\mathcal{R}_D$.
2. **field sort** The set $S(F)$ be the realizations of $\mathbb{V}$ (i.e. the Shimura variety) over a field F with a specified named set of constants (of geometric significance). The relations $\mathcal{R}_S$ will be imposed from an underlying field.
3. $q : D \to S(F)$ is a function.

A Classical Example

$$\langle \mathbb{H}, \{f_g : g \in \underset{2}{\mathrm{PSL}}(\mathbb{Z})\}, \mathcal{C}, j\rangle$$

The classical j function gives a complex analytic isomorphism

$$j : \mathrm{PSL}_2(\mathbb{Z}) \setminus \mathbb{H} \to \mathcal{C}$$

realizing $\mathbb{A}^1(\mathcal{C})$ as the moduli space of elliptic curves.

1. $j(\tau) = j(\tau')$
2. $\tau = g\tau'$ for some $g \in \Gamma$
3. τ and τ' generate isomorphic elliptic curves.

The 'etale' cover

Fact

For any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(\Re)$, the quotient $\Gamma \setminus \mathbb{H}^*$ is a compact Hausdorff space that can be given the structure of a Riemann surface. Therefore if Γ' is of finite index in Γ, the quotient $\Gamma' \setminus \mathbb{H}^*$ is a compact Riemann surface, and is therefore algebraic by the Riemann existence theorem.

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $G = \mathrm{GL}_2^+(\mathbb{Q})$ modulo its center, and

$$\Gamma_N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : \; b \equiv c \equiv 0, a \equiv d \equiv 1 \mod N \right\}.$$

Note that each Γ_N has finite index in Γ and if $N|M$ then $\Gamma_M \subseteq \Gamma_N$. Let Z_N be subvariety of $\mathrm{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}$ given by $\Gamma_N \setminus \mathbb{H}$.

Example: A non-classical cover

Central to each proof is the construction of an 'etale' or 'profinite' cover $\tilde{M}$ that is an inverse limit of groups (varieties) that are substructures of the covered object M in the field sort such that the 'algebraic properties' of the triple $(\tilde{M}, M, q)$ where q is a universal cover allow for a characterization (perhaps categorical in uncountable cardinals for an $L_{\omega_1,\omega}$ sentence.

The etale cover $\tilde{M}$ is $\varprojlim Z_N$ with maps $\rho_{M,N} : Z_N \twoheadrightarrow Z_M$ if $M|N$.

Proof Strategy

General strategy

1. Axiomatize $L_{\omega,\omega}$-theory of $(\tilde{M}, M, q)$ by a superstable, quantifier-eliminable theory $\tilde{T}$.
2. Show the classical cover $(\mathbb{H}, \mathbb{V}, p)$ satisfies T.
3. Show for any $(\tilde{M}, M, q)$, the isomorphism type of the triple is determined by the isomorphism type (cardinality) of M and of the kernel
4. $L_{\omega_1,\omega}$-axioms gain ω-stability and control the kernel.

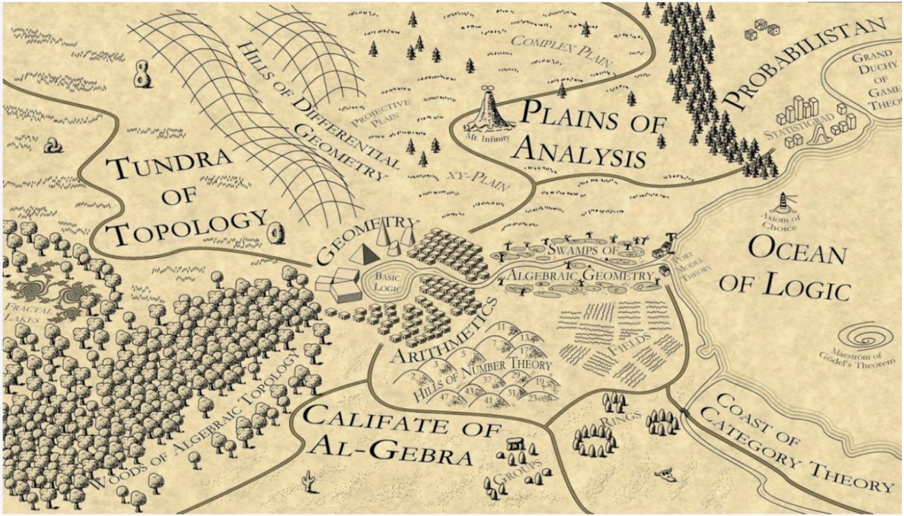
A fork in the road

1. $\mathbb{V}$ is a (modular or Shimura) curve or an Abelian variety.
2. $\mathbb{V}$ is a Shimura variety or higher dimensional Abelian Variety

A closer look at the forks: Shelah's analysis returns

1. Method 1: quasiminimal excellence:
   The setting is an aec with a global combinatorial geometry and n-dimensional amalgamation for all *n* (excellence).
   The result is categoricity in all uncountable powers.
2. Method: first order excellence: $\tilde{T}$ is superstable. There are excellent (n-dimensional amalgamation) I-independent systems of 'constructible' models.
   At least each model in $\aleph_1$ can be characterized.
   Are the characterizations 'fully formalized'? [BHP20, BGH14]

Geography of Mathematics: Deep not Basic Logic



References I

Mathias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven.
Asymptotic Differential Algebra and Model Theory of Transseries.
Annals of Math Studies. Princeton University Press, 2017.

M. Bays, M. Gavrilovich, and M. Hils.
Some definability results in abstract Kummer theory.
International Mathematics Research Notices, 43:3975–4000, 2014.
https://doi.org/10.1093/imrn/rnt057.

M. Bays, B. Hart, and A. Pillay.
Universal covers of commutative finite Morley rank groups.
Journal of the Institute of Mathematics of Jussieu, 19:767–799, 2020.
https://www3.nd.edu/~apillay/papers/universalcovers-BHP.pdf.

References II

L. Blum.
Generalized Algebraic Structures: A Model Theoretical Approach.
PhD thesis, MIT, 1968.

D. Blzquez-Sanz, C. Casale, J. Freitag, and J. Nagloo.
Some functional transcendence results around the schwarzian differential equation.
Annales de la Facult des sciences de Toulouse : Mathmatiques,, 29, 2020.
https://arxiv.org/abs/1912.09963.

M Bays and B Zilber.
Covers of multiplicative groups of algebraically closed fields of arbitrary characteristic.
Bull. Lond. Math. Soc., 43:689–702, 2011.

References III

Guy Casale, James Freitag, and Joel Nagloo.
Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups.
Annals of Mathematics, 2021.
arXiv:1811.06583.

Christopher Daw and Adam Harris.
Categoricity of modular and Shimura curves.
Journal of the Institute of Mathematics of Jussieu, 66:1075–1101, 2017.
math arxiv https://arxiv.org/pdf/1304.4797.pdf.

Sebastian Eterević.
Categoricity of Shimura varieties.
2022 version, 2022.

References IV

J. Freitag, R. Jaoui, and R. Moosa.
When any three solutions are independent.
Inventiones, 230:1249–1265, 2018.
https://doi.org/10.1007/s00222-022-01143-8.

J. Freitag and J. Nagloo.
Algebraic relation between solutions of painlev''e equations.
math arxiv: https://arxiv.org/pdf/1710.03304.pdf, 2022.

J. Freitag and T. Scanlon.
Strong minimality and the j-function.
Journal of the European Mathematical Society, 20:119–136, 2018.

Harrington, L. and Makkai,M. and Shelah, S. .
A proof of Vaught's conjecture for ω-stable theories.
ISRAEL JOURNAL OF MATHEMATICS, 49:259–280, 1984.

References V

Adam Harris.
Categoricity and covering spaces.
PhD thesis, Oxford, 2014.
https://arxiv.org/pdf/1412.3484.pdf.

E. Hrushovski and M. Itai.
Strongly minimal sets in differentially closed fields.
Transactions of the American Mathematical Society,
355:4267–4296, 2003.

E. Hrushovski.
Odes of order 1 and a generalization of a theorem of Jouanolou.
unpublished preprint, 1995.

E. Hrushovski and Sokolović.
Minimal subsets of differentially closed fields.
submitted approximately 1993; accepted but did not appear, 1993.

References VI

S Katok.
Fuchsian Groups.
University of Chicago Press, Chicago, 1992.

D. Marker.
Vaught's conjecture for differentially closed fields.
Notes from 2005 Vaught Conjecture conference; http://homepages.math.uic.edu/~marker/vcdcf-slides1.pdf, http://homepages.math.uic.edu/ marker/vcdcf-slides2.pdf.

J. Nagloo and A. Pillay.
On algebraic relations between solutions of a generic Painlevè equation.
Journal für die reine und angewandete Mathematik (Crelle's journal), 726:1–27, 2017.
DOI: 10.1515/crelle-2014-0082.

References VII

A. Pillay.
Differential algebraic groups and the number of countable differentially closed fields.
In *Model Theory of Fields, 2nd edition*, Lecture Notes in Logic 5. ASL-CUP, 2006.

J. Pila.
O-minimality and the Andre-Oort conjecture for C^n.
Annals Math., 173:1779–1840, 2011.

J. Pila and J. Tsimerman.
Ax-lindemann for a_g.
Annals Math., 179:659–681, 2014.

J. Pila and A. J. Wilkie.
The rational points of a definable set.
Duke Math. J., 133:591–616, 2006.

References VIII

A. Pillay and M. Ziegler.
Jet spaces of varieties over differential and difference fields.
Selecta Math. New series, 9:579–599, 2003.

A. Robinson.
On the concept of a differentially closed field.
Bulletin of the Research Council of Israel, 8F:113–128, 1959.

Maxwell Rosenlicht.
Integration in finite terms.
American Mathematical Monthly, 79:963–972, 1972.

Maxwell Rosenlicht.
The non-minimality of differential closure.
Pacific Journal of Mathematics, 52:253–274, 1974.

References IX

G. Sacks.
Saturated Model Theory.
Benjamin, Reading, Mass., 1972.

S. Shelah.
Uniqueness and characterization of prime models over sets for totally transcendental first-order theories.
J. Symbolic Logic, 37:107–113, 1972.
paper 15.

S. Shelah.
Differentially closed fields.
Israel Journal of Mathematics, 16:314–328, 1973.

References X

A. Wilkie.
Model completeness results for expansions of the real field by restricted Pfaffian functions and exponentiation.
Journal of American Mathematical Society, pages 1051–1094, 1996.

B.I. Zilber and C. Daw.
Modular curves and their pseudo-analytic cover.
math arxiv: `https://arxiv.org/pdf/2107.11110.pdf`, 2021.

B.I. Zilber.
The structure of models of uncountably categorical theories.
In *Proceedings of the International Congress of Mathematicians August 16-23, 1983, Warszawa*, pages 359–68. Polish Scientific Publishers, Warszawa, 1984.

References XI

B.I. Zilber.
Analytic and pseudo-analytic structures.
In Rene Cori, Alexander Razborov, Stevo Todorcevic, and Carol Wood, editors, *Logic Colloquium 2000; Paris, France July 23-31, 2000*, number 19 in Lecture Notes in Logic. Association of Symbolic Logic, 2005.

B.I. Zilber.
A categoricity theorem for quasiminimal excellent classes.
In A. Blass and Y. Zhang, editors, *Logic and its Applications*, volume 380 of *Contemporary Mathematics*, pages 297–306. American Mathematical Society, Providence, RI, 2005.

B.I. Zilber.
Covers of the multiplicative group of an algebraically closed field of characteristic 0.
Journal of the London Mathematical Society, pages 41–58, 2006.