The dividing line methodology: Model theory motivating set theory

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The 1960's produced technical revolutions in both set theory and model theory. Researchers such as Martin, Solovay, and Moschovakis kept the central philosophical importance of the set theoretic work for the foundations of mathematics in full view. In contrast the model theoretic shift is often seen as ' technical' or at least 'merely mathematical'. Although the shift is productive in multiple senses, is a rich mathematical subject that provides a metatheory in which to investigate many problems of traditional mathematics: the profound change in viewpoint of the nature of model theory is overlooked. We will discuss the effect of Shelah's *dividing line methodology* in shaping the last half century of model theory. This description will provide some background, definitions, and context for [She19].

In this introduction we briefly describe the *paradigm shift* in first order[1] model theory that is laid out in more detail in [Bal18]. We outline some of its philosophical consequences, in particular concerning the role of model theory in mathematics.

We expound in Section 1 the classification of theories which is the heart of the shift and the effect of this division of all theories into a finite number of classes on the development of first order model theory, its role in other areas of mathematics, and on its connections with set theory in the last third of the 20th century. We emphasize that for most practitioners of late 20th century model theory and especially for applications in traditional mathematics the effect of this shift was to lessen the links with set theory that had seemed evident in the 1960's. In Section 2 we explore how Shelah's underlying methodological precept of dividing lines led to the refinement of this classification to admit infinitely many classes, deeper connections with set theory, and to the current emphasis on 'neo-stability theory' in both pure and applied model theory. These developments undermine the impression from Section 1 that classification theory completely freed first order model theory from considering extensions of ZFC. In Section 3, we build on Maddy's [Mad19] discussion of the role of foundations to explore the ways in which model theory can provide *essential guidance* in both traditional mathematics and for axiomatic set theory.

What I call the paradigm shift in model theory [Bal18] took place in two phases. The *first phase* is a shift from the Russell-Hilbert-Gödel conception of higher-order logic as a general framework for all of mathematics to a Robinson-Tarski focus on first order theories to study distinct areas of mathematics. One key to this shift is the switch from a logic that allows quantification over predicate variables of all arities and all orders to the modern conception of fixing a vocabulary with a fixed set τ of relation symbols relevant to the area being studied and quantifying only over individuals. The focus becomes, not the study of logic(s),

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[1] While by first order logic we mean that formulas are only closed under finite Boolean operations, we will also touch on infinitary logic.

but theories, the consequences of a set of axioms. At the same time this transfers the focus from a single structure, e.g., *the natural numbers* to the collection of distinct (non-isomorphic) structures that satisfy (model) a set of axioms (e.g. *algebraically closed fields*). Unless all models are finite, there are such models of every infinite cardinality.

Henkin's 1948 proof of the completeness theorem enables this shift by carrying out the proof in an expansion of a given vocabulary[2] τ only by constants. In contrast, Gödel's proof requires an expansion of the vocabulary by additional relations and so moves outside the original context. This is not just a technical change [Bal17]. Gödel studies the completeness of the 'restricted[3] predicate calculus'; all of mathematics is analyzed in a global framework. Henkin's proof allows one to focus on particular topics formalized in a relevant vocabulary τ. The transition continued as a few specific properties of theories were investigated in the 1950's and 1960's. For example, Robinson [Rob56] not only introduced the notion of model completeness but developed the study of algebraically and, eventually, differentially closed fields. Morley [Mor65] proved a groundbreaking result: A countable theory is $\aleph_1$-categorical (all models of cardinality $\aleph_1$ are isomorphic) if and only if it is κ-categorical in every uncountable κ.

The *second phase* of the paradigm shift arises from Shelah's introduction in [She69] of the stability hierarchy and his classification program. Around 1970 there were two schools of model theory; both had adopted the topic-based notion of vocabulary, the study of theories. One can be thought of as 'internal'; determining the properties of theories and their models in the abstract. While this school is based on Tarski's semantics, the essential point is the study of all theories and developing useful properties for distinguishing among theories. The other is more 'external', 'applied' or 'algebraic' model theory. Here the focus is the study of theories of specific families of structures such as p-adic fields [AK65]. The eventual effect of Shelah's classification theory was a joining of those fields in the 1980's, when the usefulness of the stability classification for applications became evident. The classification has become an increasingly strong tool in applications to such diverse fields as combinatorics, number theory, and differential equations.

In accepting the 2013 Steele prize, Shelah wrote:

> I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones. It is expected that this will eventually help in understanding even specific classes and even specific structures. Some others see this as the aim of model theory, not so for me. Still I expect and welcome such applications and interactions. It is a happy day for me that this line of thought has received such honourable recognition. Thank you. [She13b]

Much of mathematics concerns only structures of cardinality at most the continuum (E.g., the reals are the only separable Dedekind-complete ordered field) or of statements whose truth in a structure is completely independent of the cardinality of the structure. (E.g., If every element a of a group G satisfies $a + a = 0$, then G is commutative.) Vaught [Vau61], focused on countable models of countable theories (i.e. $|\tau| = \aleph_0$); Morley [Mor65] showed the importance of uncountable structures in his epic treatment of categoricity in uncountable cardinalities. We explore below Shelah's demonstration that the properties of models of a theory can differ essentially depending on the cardinality of a model and cardinal arithmetic. His work provides

[2] A vocabulary is a list of relation, function, and constant symbols. It is a slightly less abstract notion than similarity type and more precise than the overloaded word 'language'.

[3] There are still predicate symbols of all orders, but quantification is restricted to individuals.

the first systematic exploration of Cantor's paradise in all cardinalities motivated by algebraic-structural (model-theoretic) rather than combinatorial or cardinal arithmetic considerations.

The use of uncountable cardinals in proving results about the model theory of countable theories initially led some to object to the set theoretic component of Shelah's model theory as distracting from core mathematical notions. Ironically, the actual effect of Shelah's classification theory (Section 1) was to free large portions of first order model theory from an apparent dependence on axiomatic set theory. Shelah reported, 'In '69, Morley and Keisler told me that model theory of first order logic is essentially done and the future is the development of infinitary logics' [She00]. The interaction with set theory remains central to the development of infinitary model theory. But, when it became clear that such issues as the existence of saturated models, two cardinal theorems, and the construction of indiscernible sequences[4] could be done in ZFC, by restricting to theories which behaved well in the stability classification, first order model theory flourished. And, when it turned out that many important areas of modern mathematics could be formalized in first order theories that behaved well in the stability classification; applications flourished as well.

But Shelah discovered that more subtle properties of first order theories and such fundamental properties as categoricicity in power for infinitary logic are much more closely entwined with set theory. Often, they require new techniques in set theory for their resolution. Such developments arising in first order logic are the main topics of this paper.

However, the more ambitious claim is that the 'method of dividing lines' is a useful technique in mathematics. There is no assertion that it is a universal methodology but only that it is not a one-off for the main gap (Section 1.1). The choice of classification or more precisely of dividing lines depends on the test problem. We study the stability classification aimed at the main gap in Section 1.1. Here are several further possibilities: saturation of ultrapowers and the Keisler order (Section 1.2), universality (Section 2.1), exact saturation ([She19]). These are all different ways of organizing the collection of first order theories. These frameworks provide tools to recognize connections across mathematics that are made evident by formalizing various topics. Much of Shelah's work in recent years attempts to apply this methodology to infinitary logic via studying abstract elementary classes, [She09, She10, Bal09, SV18]. We won't explore that topic in depth here; it concerns a semantic approach to infinitary logic and there are deep connections with axiomatic set theory.

The entire project raises questions about the nature of axiomatization; in Section 3 we discuss the effect on the axioms of set theory. The study of arbitrary theories in model theory reflects the view of axioms not as 'self-evident' or even 'well-established' fundamental principles but as tools for organizing mathematics. When dealing with specific examples, the standpoint is much like that of [Sch13], Russell [Rus73], and Detlefsen's notion of descriptive axiomatization [Det14]. For example, in [Rob59] Abraham Robinson formalized the framework that Ritt and Kolchin had developed for differential algebra, while keeping in mind his earlier work on Artin-Schrier and the Hilbert Nullstellensatz, so his theory yielded a differential Nullstellensatz. Schlimm [Sch85] explores the connections between axiomatizations of different but related fields. But Shelah's classification project takes this to a higher level of abstraction by providing general schemes for comparing theories. This raises new problems in the philosophy of mathematical practice. What are criteria for evaluating axiom systems? What are the connections among the justificatory and explanatory functions of axioms? E.g., are there criteria for choosing among first order, second-order, or infinitary logic? In what sense is second order logic simply a natural avatar for set theory [Vää12]? What principles underlie the development of a taxonomy of mathematics (or at least formal theories) such as the ones described here?

[4]Here the improvement proves a theorem about simple without the axiom of replacement, obtaining indiscernibles by constructing non-forking sequences in stable theories instead of the original reliance on replacement to find $\beth_{\omega_1}$ instances of the Erdos-Rado theorem.

1 Classification Theory

Classification is one of the fundamental aims of mathematics. The description in [Gow08, 52-54] provides an overview. Usually the problem is seen as classifying the structures in a certain class: the finite simple groups, differentiable manifolds, finite dimensional vector spaces, etc. Shelah transformed model theory by proposing a two-step classification. The first step classifies complete first order theories. At first impression, this is just a routine 'divide and conquer' strategy. Divide a problem into cases that might require different kinds of arguments. *The method of dividing lines* makes this procedure more precise. Such a classification is aided by applying this method to a specific *test question*. Generalizing from the 'main gap' described in detail below, we think of the test question having either a 'wild' or 'tame' answer for each theory. The collection of all theories is successively divided into pairs of classes of theories. At each step, the models of theories in one class are *wild* because of a specific property; models of the other become explicitly more tame when this property fails. For the main gap, a second step classifies the models by assigning a system of invariants determining the models of those theories that are deemed *classifiable* at the first stage. After sketching the general strategy we will discuss several possible such classifications for first order theories. On the one hand, as we describe here and in Section 1.1, the particular case division given by the stability hierarchy has both led to important applications across mathematics and to the solution of problems unrelated to the specific test question. On the other hand, some new problems have led to new classifications (Sections 1.2 and 2.1).

Gregory Cherlin suggested that this strategy is a version for mathematics of Plato's strategy of definition, 'cutting through the middle' [Pla16]. We explored this analogy to Shelah's proof strategy in [Bal18, Chapter 13.4]. I say a property of a theory is virtuous if *the property has significant mathematical consequences for* any *theory satisfying it*. Then a *dividing line* is a property such that both it and its negation are virtuous. Some writers refer to any property of theories as a dividing line; this misses the core of the methodology. When investigating a property to determine if it an actual dividing line, one might call it a *candidate (or pre) dividing line.* We expound these distinctions in specific cases in Sections 1.1 and 1.2.

The crucial model-theoretic notion of *type*, arose in the 1950's. The *complete type* of an element a over a set B in a structure M is the collection of first order formulas $\phi(x, \mathbf{r})$ such that $M \models \phi(a, \mathbf{r})$ (with $\mathbf{r}$ in M). Thus, if M is the field of real numbers and $B = \{0, 1\}$, each rational number realizes a principal type (generated by one formula) over B ($m * x = n$ for some[5] m, n) while the $\sqrt{2}$ realizes a non-principal type ('in the cut'). (Further examples are in [Vää19]). Through the 50's and 60's it became clear that the number of complete types over each model M of a theory T was an important characteristic.

We sketch the stability hierarchy, the progenitor, though not the only example of applying the dividing line strategy. The notion of κ-stability is one of the key elements of the classification. The countable theory T is κ-stable if for every $M \models T$ with $|M| \leq \kappa$, there are only κ complete types over M. Morley proved ω-stability is equivalent to stability in every infinite cardinal. Shelah proves the equivalence of two ostensibly vastly different properties of a theory T: i) There is no formula $\phi(\mathbf{x}, \mathbf{y})$ which linearly orders an infinite subset of M^n where $M \models T$. ii) T is κ-stable for those κ such that $\kappa^{\aleph_0} = \kappa$. Such a T is called stable.

An implicit variation in the existential quantifier in this definition disguises some of the significance. A theory T is unstable if *there is* a formula with the order property. This formula may change from theory to theory. In a dense linear order one such is $x < y$; in a real closed field one is $(\exists z)(x + z^2 = y)$, in the theory of $(\mathbb{Z}, +, 0, \times)$ one is $(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$. In the theory[6] of $(\mathbb{C}, +, \times, \exp)$, one first notices that $\exp(u) = 0$ defines a substructure which is isomorphic to $(\mathbb{Z}, +, 0, \times)$ and uses the formula from arithmetic. It is this flexibility, grounded in the formal language, which underlies the wide applicability

[5]The mth successor of 1 is denoted m.

[6]Here exp denotes complex exponentiation, e^z, where z is a complex number.

of stability theory. In infinite boolean algebras an unstable formula is $x \neq y \;\&\; (x \wedge y) = x$; here the domain of the linear order is *not* definable.

Unstable theories are split into two classes by i) the *strict order property*[7]: for some formula $\phi(\mathbf{x}, \mathbf{y})$ and some (equivalently any) model of T for every m, there are sequences $\{\boldsymbol{a}_m : m < n\}$ such that for $m_1 < m_2 < n$, $\phi(\mathbf{y}, \boldsymbol{a}_{m_1}) \rightarrow \phi(\mathbf{y}, \boldsymbol{a}_{m_2} \wedge (\exists \mathbf{y})[\phi(\mathbf{y}, \boldsymbol{a}_{m_2}) \wedge \neg\phi(\mathbf{y}, \boldsymbol{a}_{m_1})]$ and ii) the *independence property*: for some formula $\phi(\mathbf{x}, \mathbf{y})$ and some (equivalently any) model M of T for every m, there are $\{\boldsymbol{a}_i : i < m\}$ and $\{\mathbf{b}_X : X \subseteq m\}$ such that $\phi(\boldsymbol{a}_i, \mathbf{b}_X)$ if and only $i \in X$. Naturally, linear orders satisfy SOP and the Rado random graph satisfies the independence property. Each of set theory and arithmetic satisfy both of these properties. We make the syntactic definitions of these properties explicit to emphasize that despite their consequences for the uncountable, no complicated 'foundation' is needed to define them; we give a more graphic definition of the strict order property in Section 2.1.

Such important mathematical theories as algebraically and differentially closed fields are ω-stable; all abelian groups are stable as, in a spectacular result of Sela, is each non-Abelian free group [Sel13]. The many applications of classification theory across algebra are described in such surveys as [Mar96, Pil95, Poi01, SP17, Sca01] as well as the book on Hrushovski's proof in all characteristics of the Manin-Mumford conjecture [Bou99]. Another line of results stem not directly from the stability hierarchy but from another example of the method of characterizing a class of theories. The notion of o-minimality, which provides a general setting for studying expansions of the real field [Dri99], underlies Karp prizes awarded in 2013 and 2018 for contributions to number theory and to analysis. All of these works exemplify the paradigm shift.

Shelah suggests in [She19, Section 1.2] three adjectives to describe a dividing line program. Here, he assumes that there is a guiding question that the strategy aims to answer. A program is *internally successful* if there is a serious structure theory on the positive side, and *externally successful* if it implies a negative answer to the guiding question (e.g., for the main gap question, the theory has many models) and *fruitful* if that structure theory has an impact in other areas of mathematics; *robust* is discussed in the next paragraph. Shelah's discussion there refines his earlier writings and conflicts in some ways with the account in [Bal18, Chapter 13]. In particular, fruitful has been specialized to the effect of the positive structure theory and *versatile* now describes the wider impacts of the theory, earlier called fruitful. The stability classification is successful: models of a stable theory admit a kind of 'local' dimension generalizing the notion of dimension in vector spaces or, 'more roughly', of geometric space. Further, it is a dividing line as the unstable theories have the maximal number of models in every uncountable cardinality. Finally, the many ways in which the stability hierarchy has been applied to solve problems ([She09, Introduction], [Bal18, Chapters 5, 6, 13]) other than those originally targeted illustrate its fruitfullness and versatility.

Shelah [She19, §1] calls a dividing line 'robust' if it has both an internal definition (i) in terms of first order definability and ii) an external one in terms of properties of the class of models. Thus an internal definition is robust when it is absolute – as are the notions of the stability hierarchy. On the other hand, external conditions such as counting the number of models may be subject to the vagaries of cardinal arithmetic. Thus, an external condition is more robust if it is less susceptible to deformation by forcing.

Thus, κ-stability is disqualified as implying robust because it involves types and so is not external – it refers to more than models. But, Shelah views as an external characterization of stability, the fact that a theory T is stable in exactly those cardinals where it has a unique resplendent[8] model (Compare [She00, 5.2, 5.3] and [Sheb].).

[7] Abbreviated SOP or StOP.

[8] The structure M is resplendent whenever M can be expanded to $\hat{M} = (M, \mathbf{c})$ by naming $< |T|$ individual constants and $\hat{M}$ has an elementary extension M' that is expandable to be a model of T where $\mathrm{Th}(\hat{M})T'$ with $|T| < |T|$ then already $\hat{M}$ can be expanded to a model of T. Every saturated model is resplendent but not conversely.

1.1 Morley's conjecture and the main gap: Stability classification

The main gap theorem, described in [Vää19], arose to answer Morley's conjecture. For every first order theory T and every infinite cardinal κ, the *spectrum function* of T, $I(T,\kappa)$, counts the number of non-isomorphic models of T with cardinality κ. Morley conjectured that if κ, λ are uncountable and $\lambda \geq \kappa$ then $I(T,\lambda) \geq I(T,\kappa)$. Shelah's amazing solution to this problem arose from his new strategy. Rather than proving the result by arguing directly that the function $I(T,\lambda)$ is non-decreasing on uncountable cardinals[9], he rephrased it as an apparently much harder problem. Find all possible spectrum functions (as T varies) and observe that each is non-decreasing.

Shelah proposed solving this problem by finding a series of dividing lines. The first dividing line is stability. For every uncountable κ, any unstable theory T satisfies $I(T,\kappa) = 2^\kappa$, the maximal possible value. For stable theories the local dimension described just after Theorem 1.1.1 is a step toward finding invariants. For the Morley conjecture, at each step the 'wild' side will imply the theory has the maximal number of models in every uncountable cardinal and the 'tame side' will provide more tools for eventually assigning (trees of) cardinal invariants to determine each model of a classifiable theory up to isomorphism. Eventually Shelah finds a finite number (Section 2.3) of classes of theories such that all theories in the same class have the same (modulo parameters) spectrum function [She90, HL97, HHL00]. And, these functions are all non-decreasing. Moreover[10], the *main gap* separates the growth rate of spectra into two classes.

$$I(T,\aleph_\alpha) \begin{cases} = 2^{\aleph_\alpha} & \text{or} \\ \leq \beth_{\omega_1}(|\alpha| + \omega) \end{cases} \tag{1}$$

Although there is a detailed study of the slow growing spectra functions, which yields much more detailed structural information, the main gap appears to say the number is maximal or well below the maximal. Section 2.3 explores the extent to which the malleability of cardinal arithmetic undermines 'well below'.

By coding with stationary sets[11], first, certain linear orders and then their Skolem hulls, Shelah established that each unstable or even unsuperstable theory (replacing linear order by trees of height ω) has the maximal number of models in each uncountable cardinal. Thus stable/unstable is the first dividing line for the classification.

Theorem 1.1.1 (The Stability Hierarchy:). *Every countable complete first order theory lies in exactly one of the following classes.*

1. *(unstable) T is stable in no λ.*
2. *(strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$*
3. *(strictly superstable) T is stable in exactly those $\lambda \geq 2^{\aleph_0}$.*
4. *(ω-stable) T is stable in all infinite λ.*

Superstability is a dividing line as it entails a number of structural tools which are essential for describing the next more technical tools that are explicitly for counting the number of models. While it negation implies T has the maximal number of models.

The test question for the stability classification was Morley's conjecture. But the stability hierarchy is both fruitful and versatile. The tools developed to solve it had far wider consequences. As noted in Section 1,

[9] Hart [Har89] proves the result in this form but only by resorting to cases depending on the classification.

[10] The $\beth$-cardinals are defined by induction $\beth_0 = \aleph_0$ but $\beth_{\gamma+1} = 2^{\beth_\gamma}$ while limits are taken as sups.

[11] A *closed unbounded set* (club) of an uncountable cardinal is one that is unbounded and closed in the order topology. A *stationary set* is one that intersects each club. Roughly, stationary sets are those which are not small (analagous to a set of positive measure).

a key consequence of stability is the existence of a notion of dependence generalizing that in vector spaces. This relation is called 'forking'; in many cases it induces a combinatorial geometry which allows one to assign a dimension to certain (type)-definable sets. In algebraically closed fields, this dimension is the same as the Krull/Weil dimension in algebraic geometry, illustrating the versatility of Morley rank. The notion of *orthogonality* allows one to describe the relations among the dimensions of various sets. Orthogonality is an important component in one of Shelah most important innovations from an algebraic standpoint. Rather than characterize a structure by a dimension of a family of subsets, these dimensions are arranged on a tree, but one with countable height, regardless of the cardinality of the model. These dimensions are the basis of the invariants which describe the models to establish the main gap. But the geometries are central to many results across mathematics. One example is the recent resolution using classification theory [FS18, NP16] of transcendence problems arising in Painlevé's study of partial differential equations at the turn of the 20th century.

Shelah introduced the notion of a simple theory in [She80b]. One test problem was to characterize spectra for a theory T of pairs (λ, κ) such that every model in λ extends to a κ-saturated[12] model also of cardinality κ. See [She80b, She96, She00]. Although simple theories were defined to study the saturation spectrum, Pillay [BKPS01] summarizes some of the work on simple theories, as an 'amazing journey from "finite fields" to the "independence theorem" '. Building on the Ax proof of the decidability of the theory of finite fields, numerous authors [CvdDM92, Hru93, KP97] built up a general account which both showed that pseudo-finite fields and ACFA[13] are simple and that simple theories are characterized by the property: the dependence relation of forking satisfies the independence theorem. This kind of interplay is central to modern model theory.

In one sense Shelah's classification theory broke the tight connection between model theory and set theory that seemed natural in the 1960's [Bal18, Chapter 8]. The classification is given by absolute properties; the definition is in ZFC and is impervious to extensions by forcing. A certain collection of tools from combinatorial set theory (Ramsey theorem, Erdős-Rado theorem, and stationary sets) or at least certain key consequences of them are used to establish the classification and basic properties thereof. But for most practicing model theorists, set theory faded into the background. We see below that this vanishing was ephemeral.

1.2 From Saturation of Ultrapowers to Cardinal Invariants of the Continuum: Keisler order

We provide here some background [She19, I.5] on the Keisler order; [Kei67] showed the following order on theories is well-defined.

Definition 1.2.1 (Keisler order). *For complete countable first order theories T_1, T_2, we write $T_1 \trianglelefteq T_2$ if for any set I $A_1 \models T_1, A_2 \models T_2$, and regular*[14] *ultrafilter D on I, if A_2^I/D is I^+-saturated then A_1^I/D is I^+-saturated.*

That is, T_2 is more complex than T_1 if it is harder for ultrafilters to saturate models of T_2 than models of T_1.

Keisler's order preceded Shelah's classification theory. But, partly because of its clear syntactic content, Shelah's stability classification became the central model theoretic tool. After the early work, Shelah [She78] showed that all countable stable theories fell into two classes under the Keisler order, that these two classes

[12] Defined in Section 2.1.

[13] The theory of algebraically closed fields with a generic automorphism.

[14] 'Regular' is a technical condition on ultrafilters; the relevance here is that regularity guarantees that if an ultrafilter saturates one model of a complete theory T, it satisfies all models of T. So the Keisler order is on theories.

were the minimal class and its successor, and found three additional classes. The subject then languished for decades until Malliaris [Mal09] showed that, as for stability, the Keisler order reduces to syntactic properties of single formulas. This work unleashed a renaissance. The Keisler order really establishes a correspondence between syntactic properties of theories and the fine structure of ultrafilters. Malliaris and Shelah [MS13, MS16] established the first dividing line for the Keisler order within simple theories. In the other direction, they improved Shelah's [She96] result that every SOP_3 theory is maximal in the Keisler order with the striking result that SOP_2, (the 2-strong order property[15] is a sufficient condition for a theory to be maximal in the Keisler order. Still more striking they showed [MS15], contrary to expectation, that there is a decreasing chain of distinct classes in the Keisler order. Thus, while the original stability hierarchy had only finitely many classes the finer investigation of such areas as the spectra of universal models discussed in Section 2.1 and the Keisler order led to refining the stability classification and eventually to infinitely many classes. Indeed, [MS19] shows there are the maximal number ($2^{\aleph_0}$) of classes for the Keisler order.

This line of work had a remarkable corollary. In early twentieth century, topologists and set theorists discovered a large family of *cardinal invariants* of sets of subsets of the natural numbers. Here are two examples. We write $A, B, ...$ for sets of integers and $\mathbb{F}$ for families of such sets. We say A is almost-contained in B if $A - B$ is finite. A family $\mathbb{F}$ is a *tower* if 'almost-contained in' linearly orders $\mathbb{F}$ and we say $\mathbb{F}$ has the *strong finite intersection property* if the intersection of any finite collection of sets from $\mathbb{F}$ is infinite. $\mathbb{F}$ has an *infinite pseudo-intersection* if there is an infinite set A that is almost-contained in every $F \in \mathbb{F}$.

The invariant $\mathfrak{t}$ is the cardinality of the smallest tower with no infinite pseudo-intersection, and $\mathfrak{p}$ is the cardinality of the smallest F with the strong finite intersection property but no infinite pseudo-intersection. A number of such characteristics or invariants were defined. Van Douwen [vD84] introduced the now standard convention of naming these cardinals by lower case fraktur letters, following $\mathfrak{c}$ for the cardinality of the continuum. In particular $\mathfrak{b}, \mathfrak{p}, \mathfrak{t}$ had been isolated by Rothberger [Rot39, Rot48].

Under the continuum hypothesis, all these invariants are equal to $\aleph_1$. If the continuum hypothesis is false these numbers may be different. So before Gödel proved the consistency of the continuum hypothesis, attempts to establish inequalites among cardinal invariants were attacks on the continuum hypothesis. Afterwards, the alternatives were equality and consistent inequality. For example, Van Douwen [vD84] gives six equivalents to $\mathfrak{b}$; specific equivalences among other invariants had been established by various authors from Rothberger to Van Douwen. Once the independence of the continuum hypothesis was established, forcing established the consistency of inequality of most pairs in the last 50 years. So it was surprising when [MS13] proved in ZFC that $\mathfrak{p} = \mathfrak{t}$.

The connections between model theory and invariants of the continuum were explored at least as early as [She04], where Shelah proved the consistency of $\mathfrak{a} > \mathfrak{d}$. He distinguishes the property $\mathfrak{a}$, which 'speak of sets', from $\mathfrak{d}$, which deals with cofinality[16]. He writes in [She04, page 188], 'This manifests itself by using ultrapowers for some x-complete ultrafilter (in model-theoretic outlook), and by using "convergent sequences" (see [She87]], or the existence of Av, the average, in [She78]) in §7 and 3, respectively. The meaning of "model-theoretic outlook" is that by experience set theorists starting to hear an explanation of the forcing tend to think of an elementary embedding $j : V \to M$, and then the limit practically does not make sense (though of course we can translate).' The work with Malliaris emphasizes that the underlying issue is to control the cofinality of various cuts and the fine structure of ultrafilters is a powerful tool for this

[15]For $n \geq 3$, $\phi(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}, \mathbf{y}$ of the same length has SOP_n if there is no n-cycle, but there are arbitrarily long finite chains. SOP_2 has a more technical definition. See Section 2.1.

[16]The *cofinality* of a cardinal λ is the least cardinal κ such that there is an increasing function f from κ into λ such that the supremum of the range of f is κ, i.e., $\lambda = \mathrm{cf}(\lambda)$; otherwise λ is singular. In the context of this section this notion is extended to the cofinality of a partial order.

purpose. But it takes the interaction between model theory and set theory to a different level. Rather than just applying cofinality in two related areas, they define [MS13] the notion of a *CSP, Cofinality spectrum property,* and prove that a specific such property, $\mathcal{C}(\mathfrak{s}, \mathfrak{t}_\mathfrak{s}) = \emptyset$, implies both a) (model theory) every SOP_2 theory is maximal in the Keisler order and b) (set theory) $\mathfrak{p} = \mathfrak{t}$. They deduce their two goals by showing that for any CSP, $\mathfrak{s}$, $\mathcal{C}(\mathfrak{s}, \mathfrak{t}_\mathfrak{s}) = \emptyset$.

How does the dividing line strategy fare as a method for investigating the Keisler order? It is robust in the sense of Section 1; the ultrapower definition is external and Malliaris [Mal09] gives the internal characterization mentioned above. In [She19, 9E], Shelah points out a recent proof that SOP_2 is a robust dividing line; it defines the class of $\trianglelefteq$ (and $\trianglelefteq^*$)-maximal theories in the Keisler order. Thus among the SOP_n candidate dividing lines, SOP_2 is now identified as a dividing line. Successful is not so clear. The minimal and near minimal class are identified with low classes in the stability classification and so share their success. The case for internal success is boosted by recent advances obtaining a useful notion of independence for $NSOP_1$ theories[CR16, KR19] and positive structural consequences from $NSOP_2$ [MS17]. Instances of versatility include both the $\mathfrak{p} = \mathfrak{t}$ problem and applications to study of the Szemerédi's Regularity Lemma in combinatorics [MS14, MP16].

In his joint review of several Malliaris-Shelah papers [Kei17], Keisler wrote, 'The methods developed in these papers are likely to stimulate more research in model theory and set theory. An enticing possibility is that the general results on cofinality spectrum will have broader applications.'

2 Model Theoretic problems motivate Set Theoretic results

We noted in Secion 1 an important effect of the stability classification was to reduce for many model theorists the needed familiarity with ZFC. They just employ ZFC as an implicit background foundation and employ the stability classification to organize their work. But this attitude really brackets foundations rather than discarding them. Shelah wrote,

> 'My feeling is that ZFC exhausts our intuition except for things like consistency statements, so a proof means a proof in ZFC. This is of course a strong justification for B1.'[the position[17] that ZFC should be *the* basis for set theory]
>
> Position B.2 [the forcing position] in its strong form in essence tells us that all universes are equally valid, and hence in fact we should be interested in extreme universes. In particular L has no special status, and proving a theorem in ZFC or assuming GCH is not a big deal. This is the strong defense, but I suspect that it has few adherents in this sense.
>
> But in the moderate sense, this position is quite complementary to the ZFC position - one approach gives the negative results for the other, so being really interested in one forces you to have some interest in the other. In fact, I have been forced to really deal with forcing ([She77], [BD78]) was too "soft" in forcing for my taste) because I wanted to prove that I was right to use $\diamondsuit$ on "every stationary subset of $\aleph_1$" rather than CH in solving the Whitehead problem[18] for abelian groups of cardinality $\aleph_1$.
>
> Shelah [She02].

A close entanglement of model theory and set theory appears in works concerning the set theoretic definability of logics [KMV16, Vää85]. But we are thinking here of a less close entanglement. Shelah's

[17] Shelah is considering positions he labels B1-B5.
[18] See Section 2.2.

investigation into model theoretic questions led him to issues that were not resolvable in ZFC. In this section, we recount several examples. We assess the motivation into one of three (possibly overlapping) categories: i) addressing a new problem, ii) clarifying hypotheses, iii) increasing robustness.

I now summarise in very general terms the impact of the set theoretic revolution unleashed by Cohen's method of forcing. It allows one to prove the independence of propositions from the axioms of ZFC by constructing models of those basic axioms where a proposition is, say, false and others where it is true. The first use of the forcing method complemented Gödel's earlier construction of an *inner model* ($V = L$) that showed the consistency of the continuum hypothesis ($2^{\aleph_0} = \aleph_1$) by constructing an *outer model(s)* with $2^{\aleph_0} \neq \aleph_1$. Much of the vast development using this tool focuses on cardinal arithmetic and topology. We describe here some of Shelah's efforts that arise from more model theoretic issues. Large cardinal axioms are another genre of extensions of ZFC that try to extend the ability of ZFC to found mathematics.

2.1 Universality order

This section provides an introduction to Shelah's work [She19, Part II] on obtaining a classification for a different ordering on first order theory; now, complexity is measured by allowing fewer universal models. A major issue is to find robust dividing lines for this problem.

Already with Pythagoras, we realize that the basic systems of numbers need to be extended to account for all phenomena. These extensions become more clear with finding all solutions of higher degree equations. In the late 19th century as precise notions of structures and classes of structures arose so did the idea of a universal structure for a particular class. In the 20th century the notion of a universal domain for such classes arose in such diverse fields as linear orders [Hau14], topology [Ury27], algebraic geometry [Wei62] (originally 1946), and logic [Fra54, Jón56, Jón60]. In these papers M is universal means every structure N (in a given class) with $|N| \leq |M|$ can be isomorphically embedded in M.

Hausdorff ([Hau14] and the paper [Hau05, H 1908]) proved that assuming $2^{\aleph_n} = \aleph_{n+1}$, there is a universal linear order of cardinality $\aleph_n$ for each finite n. Twenty years later Tarski dubbed the extension of this principle from $\aleph_n$ to arbitrary $\aleph_\alpha$ the generalized continuum hypothesis (GCH).

Fraïssé [Fra54] studied a class $\boldsymbol{K}$ of finite relational structures closed under substructure and provided conditions that guaranteed an $\aleph_0$-universal and homogeneous (any isomorphism between two finite substructures extends to an automorphism) model for the class axiomatized by the universal sentences satisfied by all structures in $\boldsymbol{K}$.

In [Jón56] Jónsson introduced what is often called a *Jónsson class*. He generalized Fraïssé's notion to consider a collection of structures of arbitrary cardinality, closed under isomorphism with the joint embedding and amalgamation property, closed under unions of chains, and with downward Löwenheim Skolem to some fixed cardinal. Examples included the class of linear orders and Boolean algebras. Generalizing Hausdorff, he proved that under GCH a Jónsson class has a universal model in every $\aleph_\alpha$. In [Jón60], he noticed that the construction of a κ-universal model also yielded a κ-homogeneous-universal model M. Namely, if N_0, N_1 are isomorphic substructures of M of smaller cardinality, that isomorphism extends to an automorphism. He proves such a *homogeneous-universal* model exists in λ if λ is regular and $2^{<\lambda} = \lambda$.

Morley and Vaught [MV62] changed the context by requiring the class to be the models of a complete theory and the embedding to be *elementary* (preserve the truth values of each first order formula). They discovered that this requirement is equivalent to saying every type over a subset of cardinality less than κ is realized in M; such a model is called *saturated*[19]. But this discovery also demonstrates an obstruction to the existence of saturated structures. If a theory is unstable it cannot have a saturated model in λ if $\lambda^{\aleph_0} > \lambda$. Universality is more subtle. For example, the theory of dense linear order is unstable in every cardinality but

[19]More precisely, A is λ-saturated if A realizes all complete types over $X \subseteq A$ when $|X| < \lambda$ and *saturated* if it is $|A|$-saturated.

the rational numbers are a universal countable model. We write $D(T)$ to denote the collection of n-types over the empty set for $n < \omega$ for a theory T. Morley and Vaught prove that if $\lambda \geq |D(T)|$ and $\lambda^{<\lambda} = \lambda$, i.e. $\mu < \lambda$ implies $\lambda^\mu \leq \lambda$, there is a saturated model in $|T|$.

With the use of saturated models we can give a more global picture of the strict order property promised in Section 1. A theory T has the strict order property if there is a formula $\phi(\mathbf{x}, \mathbf{y})$ such that on every $\aleph_0$-saturated model of T, $\phi(\mathbf{x}, \mathbf{y})$ defines a partial order of M^n which contains an infinite chain. This version of the definition suggests the sequence of properties *n-strong order property* introduced in [She96]: For $n \geq 3$, $\phi(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}, \mathbf{y}$ of the same length has SOP_n if there is no n-cycle, but there are arbitrarily long finite chains. Now the world of theories has been complicated to allow infinitely many classes. All of these properties are implied by the strict order property; SOP_n implies SOP_{n-1}; T has SOP_3 implies T is not simple. In Shelah's taxonomy [She19, Figure 1, §9E], SOP_n for larger n properties are pre-dividing lines at best; they arise in the discussion but have no known strong consequences in either direction. Conant's map `http://www.forkinganddividing.com/` showing the geography of the stability hierarchy is a widely use resource among model theorists.

The stability classification[20] entirely determines the possible spectra for uncountable saturated models: i) (ω-stable) all cardinals, ii) (superstable) all cardinals λ with $\lambda \geq 2^{\aleph_0}$, iii) (strictly stable) all λ with $\lambda^{\aleph_0} = \lambda$ and iv) (unstable) no uncountable λ satisfying $\lambda > \lambda^{<\lambda}$. There is some, but clearly demarcated, variance among uncountable cardinals.

The situation is quite different if we replace saturated by universal. Since a saturated model is universal, the first two classes are unchanged. But the same arguments as in [MV62] show that if $\lambda = 2^{<\lambda}$ (i.e. $\kappa < \lambda$ implies $2^\kappa < \lambda$) there is a special[21] and thus a universal model in λ. Thus for any first order theory, the existence of a universal model is open only when $\lambda < 2^{<\lambda}$.

Shelah now aims to classify first order theories[22] with respect to a new test problem: universality. The intuition is that if there are 'fewer' cardinals in which T_2 has universal models, then a theory T_2 is more complicated than T_1. Thus, he defines an ordering on theories that is analogous to the Keisler order.

Definition 2.1.1. *Let T be a complete first order theory.*

$M \models T$ is universal[23] in λ if $N \models T$ and $|N| = \lambda$ implies N is elementarily embeddable in M. M is universal if it is universal for all models with cardinality $\leq |M|$.

The universality spectrum *of $\boldsymbol{K}$, $\mathrm{univ}(T)$ is the class of uncountable cardinals λ such that there is a universal model for $\boldsymbol{K}$ with cardinality λ.*

We define $T_1 \leq^0_{\mathrm{univ}} T_2$ if $\mathrm{univ}(T_1) \supseteq \mathrm{univ}(T_2)$.

Since saturated models are universal, stability theory clearly delineates the initial classes for $\leq^0_{univ}$, those which have universal models in the 'most' cardinals. If a countable theory T is ω-stable (superstable) it has saturated — and hence universal—models in all uncountable cardinals (all $\lambda \geq 2^{\aleph_0}$).

Unfortunately, the ordering $\leq^0_{\mathrm{univ}}$ may be very uninteresting. Under the generalized continuum hypothesis (GCH), the [MV62] argument described for $\lambda = \lambda^{<\lambda}$ shows every theory has a universal model in every uncountable cardinal. Given the independence of GCH, the degree of robustness must be investigated.

[20] For convenience we restrict the analysis here to countable vocabularies.

[21] A *special model* is one that is a union of β^+-saturated model A_β for infinite cardinals $\beta < |A|$. The construction of special models allows one to delete the hypothesis of regularity from the result of [Jón60] quoted above at the cost of weakening saturated (homogeneous-universal) to universal in the conclusion.

[22] The notion of universality here specifies elementarily embeddible, as we are studying first order theories. The general setting in [She19, Part II] is for classes $\boldsymbol{K}$ and the notion of embedding depends on the class. We explore this distinction in Section 2.4.

[23] Earlier work, e.g. [MV62] require $|N| \leq \lambda$; this is the same as requiring $|N| = \lambda$ for saturation but different for universality. The homogeneity implied by saturation guarantees embedding over smaller models.

Kojman and Shelah in [KS92a] give an example of a theory T that has a universal model in $\aleph_1$ if and only if $\aleph_1 = 2^{\aleph_0}$. They[24] further show in ZFC, using the method of guessing clubs, which is discussed in Section 2.4, that there is no universal linear order in a regular cardinal λ with $\aleph_1 < \lambda < 2^{\aleph_0}$. To address this sensitivity to extensions of set theory, we build the set-theoretic options into a revised definition of the order on theories. Given the examples of theories with the existence of a universal model below the continuum being equivalent to the CH, Definition 2.1.2 restricts to cardinals above the continuum.

Definition 2.1.2. *Define $T_1 \leq_{\mathrm{univ}} T_2$ if and only if $\lambda \in \mathrm{univ}\ (T_2)$ implies $\lambda \in \mathrm{univ}\ (T_1)$ in* every forcing *extension where $\lambda \geq 2^{\aleph_0}$.*

We describe below some precise results or conjectures for the minimal and maximal classes with respect to $\leq$-univ. The phrase 'almost maximal (minimal)' describes classes that are expected to be near the top (bottom) of the order but which may fragment after further investigation. Here are the relevant classes.

Remark 2.1.3.
1. *The $\leq$-univ maximal class is conjectured to be those T such that for every forcing extension V_1, T has a universal model in λ if and only if*[25] $\lambda^{V_1} = 2^{<\lambda^{V_1}}$.
2. *T is almost $\leq$-univ maximal if for* every forcing *extension V_1, there is a μ with $\mu^{++} = \lambda = \lambda^{\aleph_0} < 2^{\mu}$ then T has no universal model in λ. This class includes linear orders and any first order theory satisfying the strict order property or even SOP_4. (Sections 3 and 5 of [KS92a]). But it also includes the class of groups which is $NSOP_4$ [She16].*

   The classes SOP_n are refinements, introduced in [She96], of the stability hierarchy which have consequences for both the Keisler order and the universality order.[26].
3. *The class of superstable (ω-stable if we include cardinals below the continuum) theories is the $\leq$-univ minimal class.*
4. *The almost $\leq$-univ minimal class is conjectured to be those theories, if any, such that for every forcing extension V_1, $\{\lambda : \lambda = \lambda^{\omega}\}$ is the class of λ in which T has a universal model.*

Further refinements of this general picture are discussed in [She19, Part II] and here in Section 2.4. Model theory continues to address algebraic problems in this area. In [Fuc70, Problem 5.1] Fuchs asked about the existence of universal groups for the classes of torsion and torsion-free groups under the relation of pure embedding[27]. By considering the class of models of certain theories of Abelian groups, partially ordered by the relation of pure subgroup, as an Abstract Elementary Class [MA, KMA] make further advances on this problem.

2.2 PC-classes and the Whitehead Problem: Proper Forcing

In this subsection we discuss two problems, one algebraic and one model theoretic, which led Shelah to a fundamental set theoretic idea: the proper forcing axiom [She19, 12A]. In both cases, clarifying the necessary hypothesis for a result is central. In the model theoretic case, this clarifies the robustness of the stability classification.

[24] See the nice account in [DŎ5].

[25] Here we write the superscript V_1 to emphasize that the cardinal equality in computed in the forcing extension; we omit the superscript V_1 below.

[26] See `http://www.forkinganddividing.com/` for an overview of the geography.

[27] S is a pure subgroup of an abelian group G if whenever an element of S has an n^{th} root in G, it necessarily has an n^{th} root in S.

In [She74] Shelah proved that the topologist J.H.C. Whitehead's famous conjecture[28] about Abelian groups was independent from ZFC. This conjecture asserts that a 'Whitehead group' is necessarily a free group. That conjecture is true for a countable group G. But Shelah used established set theoretic [29] to falsify it when $|G| = \aleph_1$; the conjecture is true if $V = L$ and false under Martin's axiom[30] and $\neg CH$.

The consistency of each of these axioms was known. But the possibility remained (and was actively pursued by some group theorists) that the Whitehead conjecture was decidable from the continuum hypothesis. In [She77] he introduced a new principal of set theory, that was consistent with the continuum hypothesis, implied Martin's axiom, and implied the failure of the Whitehead conjecture. This axiom eventually developed into the *proper forcing axiom* (PFA), which, in particular, implies $2^{\aleph_0} = \aleph_2$. See [Mek82]. Perhaps the most important lesson for mathematics at large was that a statement purely about isomorphism types of Abelian groups was solved by resort to a specific set theoretic construction of an object, a Whitehead group, which is not free [EM02].

The model theoretic example concerns classes of models that go slightly beyond first order. While there is a vast model theory for first order logic, there has been no such development for second order logic. There is a fragment of second-order logic, *pseudo-elementary classes*, that is susceptible to model theoretic treatment. These are classes axiomatized by formulas of the form

$$(\exists X)\phi(X, \overline{S})$$

where X is a new relation variable and $\overline{S}$ lists the formal symbols in the vocabulary τ. For, example the class of non-well-orders is axiomatized in this way by adding a function that is required to list a decreasing sequence. More generally, we can think of theories in the vocabularies τ, $\tau_1 = \tau \cup \{X\}$, and ask, for theories T in τ and T_1 in τ_1, what do we know about the class of reducts of models of T_1 to T. Such a class is designated $PC(T_1, T)$ (pseudo-elementary class) and $I(\lambda, T_1, T)$ denotes the number of non-isomorphic models in $PC(T_1, T)$ that have cardinality λ.

One of Shelah's major innovations in set theory arose from the attempt to understand the effect of moving a result about PC-classes from a countable to an uncountable vocabulary. In [She78] it is shown that:

(*) If $|T_1| = \aleph_0$, $2^{\aleph_0} < 2^\lambda$, and T is complete but not ω-stable then

$$I(\lambda, T_1, T) = \min(2^\lambda, 2^{2^{\aleph_0}}).$$

While most of first order model theory focuses on theories T in a countable vocabulary (written $|T| = \aleph_0$), there are natural examples of situations where an uncountable vocabulary is needed. The usual formalism [Pre88] for studying vector spaces and, more generally, modules is to consider a vocabulary with a unary function symbol λ for each λ in the field of scalars, denoting the scalar multiplication of a vector by λ. Thus, in order to formalize such basic mathematical structures as a real vector space, an uncountable vocabulary is needed. But Shelah's motivation is more basic. What happens when we move from a countable to an uncountable language? And this curiosity led to an enormously important new technique in set theory.

It follows (by naming constants) from the work in [She78, Chapter VII] that $(*)$ can be improved to requiring only $|T_1| \leq \lambda$, provided that $2^{\aleph_0} < 2^\lambda$. In [She80a], he shows that this result is not provable in ZFC. Thus, this external characterization of ω-instability requires (in particular) the weak continuum hypothesis ($2^{\aleph_0} < 2^{\aleph_0}$). So it is less robust than the external characterization of instability in Section 2.3.

[28] An Abelian group is *Whitehead* if every short exact sequence $0 \to \mathbb{Z} \to B \to A$ satisfies $B = \mathbb{Z} \oplus A$. While details are not relevant here, [Ekl76] has an introductory account.

[29] Furthermore, by focusing on the combinatorial essence of the conjecture he was able to show the independence of a conjecture concerning the chromatic number of graphs with cardinality $\aleph_1$.

[30] Martin's axiom is both a consequence of the CH and consistent with its negation. It arose in the study of the Suslin conjecture.

The significance of [She80a] is not only in the result but in the method. In it, Shelah introduced the notion of *oracle forcing* and what, in retrospect, is a progenitor of *proper* forcing. The immense significance of these techniques is explained in [Vää19]. The paper is also an early contribution to the study of universal models as it contains a proof of the consistency of $ZFC + 2^{\aleph_0} = \aleph_2$ and 'there is a universal linear order in $\aleph_1$'.

2.3 Beautiful Cardinals

In this section, we consider Shelah's introduction of a large cardinal axiom in order to clarify a model theoretic result concerning the *uncountable* spectrum, the number of models with each uncountable cardinality. The actual form of the main gap theorem attaches to each of a small finite number of (parameterized) families of classes of theories a formula for $I(T, \kappa)$. While the classification is absolute (i.e. does not depend on extensions of ZFC), the evaluation of the formula is not; it depends on cardinal arithmetic. But, the variability of cardinal arithmetic in various extensions of ZFC was the original point of forcing. Thus, it is consistent that there is a theory which is very classifiable, there is a clear way to assign cardinal invariants (so, intuitively there are 'few' models), and nevertheless in some models of set theory it has 2^κ models of cardinality κ for arbitrarily large[31] κ. In [She00, 5.2], Shelah refers to this situation as a *semi-ZFC* result on the structure side of the main gap. This is one, though far from the only, of the motivations for the study of $IE(T, \lambda)$, the maximal cardinality of a set of mutually non-embeddible models (IE, isomorphically embeddible), of cardinality λ.

The relation 'A is (elementarily) embedded in B' (but not necessarily vice versa) satisfies transitivity $A \precneqq B$ and $B \precneqq C$ implies $A \precneqq C$. Such relations are called a *quasiorder* and they are well-studied in combinatorics. Fraïssé conjectured: if $\{(A_i, \leq) : i < \omega\}$ is a countable collection of countable ordered sets, then for some $i < j$ $(A_i, \leq)$ isomorphically embeds into $(A_j, \leq)$. That is, embedding is a *well quasi-ordering*[32] on countable linear orders. Nash-Williams [NW65] introduced the more technical but ultimately more malleable notion of better-quasi-order and proved that the embeddability order on countable trees is 'better' and thus 'well' quasi-ordered. Laver [Lav71] adapted this strategy and proved the Fraïssé conjecture by showing countable linear orders are better-quasi-ordered. To investigate the elementary embedding on structures of uncountable cardinalities, Shelah [She82] had to generalize the methods of Nash-William and Laver. He analyses the class $\boldsymbol{K}_\lambda$ of colored trees $\lambda^{<\omega}$ with the usual partial order of initial sequence and requires that nodes with same color are on the same level. Considering level preserving embeddings between such trees. Shelah [She82] proves:

Theorem 2.3.1. *For any cardinal $\lambda \geq \lambda_{\mathrm{beaut}}$, any family of pairwise non-embeddible colored trees in $\boldsymbol{K}_\lambda$ has cardinality less than the first beautiful cardinal λ_{beaut}.*

To describe the model-theoretic application fully, we need a more detailed description of the proof of the main gap theorem. Recall that if a theory is unstable then it has the maximal number of models in every uncountable cardinality. In fact, this holds if T is not superstable. But for full control of the spectrum three further properties are needed. We say a theory is *classifiable* if it is superstable and ndop — doesn't have the Dimensional Order Property— and notop — doesn't have the Omitting Types Order Property. Now the crucial ingredient of the main gap theorem is that any model of a classifiable theory is decomposed into a family of countable models indexed by a tree in $\boldsymbol{K}_\lambda$. There is a finer treatment of the shallow classifiable case in [She90] and even finer in [HHL00], where five separate classes with some finer partitions of two of them and the respective spectrum functions are described. By studying IE, the number of cases are almost

[31] See details in [She19, page 3]. The ambiguity is resolved in the other direction in [Bal18, 192] by making every spectrum $< 2^{\aleph_\alpha}$ on sufficiently large cardinals.

[32] Well-quasi ordering also implies there are no uncountable descending chains but that is not important here.

reduced to classifiable or not. classifiable case is further split according to the third property: whether the decomposition tree is *shallow* (well-founded[33]) where the number $IE(\lambda, T)$ is bounded by the depth of the tree, or *deep* (non-well-founded) where we apply Theorem 2.3.1.

Theorem 2.3.2. *Every countable first order theory satisfies one of*

1. *If T is not classifiable, $IE(\lambda, T) = 2^\lambda$ for all $\lambda \geq \aleph_\omega$.*
2. *If T is classifiable, then*
   (a) *If T is shallow then $IE(\lambda, T) \leq \beth_{\mathrm{depth(T)}} < \beth_{\omega_1}$ for all λ.*
   (b) *If T is deep then*
      i. *If $\lambda < \lambda_{\mathrm{beaut}}$ then $IE(\lambda, T) = 2^\lambda$.*
      ii. *If $\lambda \geq \lambda_{\mathrm{beaut}}$ then for every $\rho < \lambda_{\mathrm{beaut}}$, there is family of pairwise non-mutually embeddible models of cardinality ρ. But there is no such family of cardinality λ_{beaut}.*

Here $\beth_\alpha$ is the analog to $\aleph_\alpha$ where, inductively, the cardinal successor of $\aleph_\alpha$ is replaced by the cardinality of the power set of $\beth_\alpha$. But what is λ_{beaut}? Shelah's notion of a *beautiful cardinal* was characterized (earlier) by [Sil70] as the least ω-Erdős cardinal. That is the least κ such that[34] $\kappa \rightarrow (\omega)^\omega_2$. This is a rather small large cardinal (strongly inaccessible but not weakly compact[35]). Beautiful cardinals have been applied to the study of Abelian groups [ES99]. Shelah makes the following remark about his use of large cardinals in this connection.

> But if we want to go any further, we have to consider some mildly large cardinal, but don't be afraid if you don't believe in them. The theorems do not say 'If some large cardinal exists then...'. But, rather 'the well-ordering cardinal of some naturally defined Q is a specific large cardinal'; so our results are meaningful even if no such cardinal exists.
>
> [She82, page 179]

That is, if there are no 'beautiful cardinals', for every λ, either $IE(\lambda, T)$ is very small if T is classifiable and shallow or it is maximal. And if there is a beautiful cardinal then the relatively small least such cardinal bounds the cardinality of maximal pairwise non-embeddable families if the theory is deep. Thus, by choosing a finer measure, IE – non-mutually embeddible, the vulnerability of the original main gap for non-isomorphism (I) to variations in cardinal arithmetic has been substantially reduced (The vulnerability can be eliminated by fixing a small initial segment of the universe, $V_{\beth_{\omega_1}}$.). In contrast to the lower bound in the main gap, $\beth_{\omega_1}(|\alpha| + \omega)$, the number of non-mutually embeddible models in $\aleph_\alpha$ of classifiable theories is uniformly bounded by $\beth_{\omega_1}$ rather than a cardinal depending on α.

2.4 Club Guessing

The method of club guessing allows more detailed analysis of the problems described in the earlier subsections of Section 2. We first explore more refined results on the universality spectrum. Then we briefly

[33] Some authors reserve the term classifiable for this case.

[34] This is a simpler statement than the equivalent principle used in Shelah's proof [She82, Definition 2.3]. In the Erdős notation for Ramsey style theorems $\kappa \rightarrow (\omega)^\omega_2$ means: If f is a coloring of the ω-element subsets of a set of cardinality κ, with 2 colors, then there is a homogeneous set of cardinality ω (a set, all whose countable-element subsets get the same f-value).

[35] For an overview of the large cardinal hierarchy see `http://cantorsattic.info/Upper_attic`. See [Sil70], who proved that a beautiful cardinal 'lives in L'.

recount the use of club guessing in Sections 2.2 and 2.3. The club guessing method is expounded in [She00, Ch.III], [She93], [She13a], and [Jec06, Theorem 23.3]. Unlike, e.g. Jensen's diamond, cases of this principle are provable in ZFC.

Definition 2.4.1. *We say λ has club guessing for $\kappa < \lambda$ when some $\overline{C}$ witnesses it, which means:*

(a) S is a stationary subset of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

(b) $\overline{C} = \langle C_\delta : \delta \in S\rangle$, where C_δ is a club of δ of order type $\mathrm{cf}(\delta)$.

(c) if E is a club of λ, then for stationarily many $\delta \in S$ we have $C_\delta \subseteq E$.

Theorem 2.4.2. *If $\lambda > \kappa$ are regular cardinals and $\kappa^+ < \lambda$ then λ has club guessing for κ.*

We describe two hypotheses on λ that, using club-guessing, imply there are no universal models in λ. We will sketch the proof of the first. Then we describe the classes in the second and indicate the variation in the proof.

Theorem 2.4.3. *1. If there exists μ and regular λ such that $2^{\aleph_0} \leq \mu^+ < \lambda < 2^\lambda$ then for any theory of linear order, with the strict order property, or even SOP_4 has no universal model in λ (Sections 3 and 5 of [KS92a]). The argument also applies to class of groups although they have $NSOP_4$ [She16].*

2. If there exists μ, $2^{\aleph_0} \leq \mu^+ < \lambda < \mu^{\aleph_0}$, then $\mathfrak{K}^{\mathrm{tr}}$ [KS92b], reduced torsion free groups $\mathfrak{K}^{\mathrm{rtf}}$ and $\mathfrak{K}^{\mathrm{rs(p)}}$ do not have universal models (under either pure or arbitrary embeddings).

Proof Sketch. Suppose for some μ, $\mu^+ < \lambda = \mathrm{cf}(\lambda) < 2^\mu$. Apply Theorem 2.4.2 for $\kappa = \mu$. In case 1), we are able to assign to each model M of size λ a set of invariants I_M consisting of $\leq \lambda$ subsets of μ in such a way that modulo an ideal in $\mathcal{P}(\lambda)$, using club guessing, the set I_M determines M up to isomorphism. Further if M can be embedded in N, $I_M \subseteq I_N$. Now fix an N that pretends to be universal. Choose a model M' whose set of invariants contains a subset not contained in any invariant of N. This is possible since $2^\mu > \lambda$. Then M' cannot be embedded in N and the pretence fails. Thus, there is no universal model in λ.

We turn to the applications. Shelah's analysis of superstability revolves around the Shelah tree, a tree of formulas of width λ and height $\omega + 1$ such that each path is consistent and the successors of each node are pairwise inconsistent. The existence of such a tree implies T is non-superstable. So the investigation of strictly stable theories begins with the study of the *class* $\mathfrak{K}^{\mathrm{tr}}$: trees T with $(\omega + 1)$-levels ordered by initial segment, i.e. $T \subseteq {}^{\omega \geq}\alpha$ for some α, with the relations $\eta E^0_n \nu := \eta \mid n = \nu \mid n$. For $\mathfrak{K}^{\mathrm{tr}}$, by varying the sketched proof for Theorem 2.4.3 we get $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ implies there is no universal for $\mathfrak{K}^{\mathrm{tr}}_\lambda$ (by [KS92b]). We need only $\lambda < \mu^{\aleph_0}$, since the invariants can be taken as sets of countable sequences from μ.

The class $\mathfrak{K}^{\mathrm{tr}}$ is closely related to certain classes of abelian groups. An abelian group is torsion free if every element has infinite order. It is *reduced* if there is no divisible subgroup. The analysis of cases for the existence of universal models in λ of Abelian groups depends on several parameters: a) the specific class of groups: torsion free vs. torsion and in the reduced torsion case, whether the group is required to be separable; b) whether the embedding is required to be pure c) various restrictions on the cardinal λ. We now elaborate on the fourth row in the chart in [She19, §10B], where the other cardinal possibilities are listed.

We denote by $\mathfrak{K}^{rtf}$ the class of *reduced torsion free abelian groups*. Every torsion free abelian group is a direct sum of a divisible group and a member of $\mathfrak{K}^{rtf}$. Since every divisible abelian group is a direct sum of copies of $\mathbb{Q}$, in every cardinality there is a universal group, universal p-group (for every prime p), universal torsion group and universal torsion-free group. This follows from the fact that every abelian group (p-group, torsion group, torsion-free group) is embeddable in a divisible abelian group (p-group, torsion group, torsion-free group) of the same cardinality [Fuc70, 23 and 24] and [KS95, Theorem 3.1]. Thus, the universality question is only interesting when we restrict to reduced groups.

In particular, $\mathfrak{K}^{rtf}$ is an interesting class. Each such group interprets a member of $\mathfrak{K}^{tr}$, by defining the equivalence relation $E_n(x, y)$ if and only if $x - y$ is divisible by $n!$. However, this class of groups does not behave well with respect to the ordinary notion of substructure. In an extension an element may be become more divisible. But, an embedding is *pure* if this doesn't happen so as in the case of $\mathfrak{K}^{tr}$, $\mathfrak{K}^{rtf}$ has no universal model for pure embeddings (Lemma 2.4.3.2) if for some μ, $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \lambda^{\aleph_0}$ [KS95, 3.8]. For arbitrary embeddings the same result is obtained in [She01, 1.6]. The new step is a refinement in the Abelian group construction; but, one that depends on an explicit construction of a structure on λ.

If we turn to torsion groups, note that each one decomposes in a unique way into direct sums of p-groups for different primes p; so we can fix p. Finally, we may restrict to p-groups that are separable (no elements of infinite height). Thus we study $\mathfrak{K}^{rs(p)}$ called *reduced separable p-groups*. The simplest example is the group of p^n roots of unity for each n. While, in general, there are no elements of infinite height, there is a topological closure so that the tree controls, without exhausting, the model. Again the pure case is in [KS95, 3.3] and arbitrary embeddings in [She01, 2.7]. But this last case requires that $\lambda > \beth_\omega$.

In [She88, Shea], Shelah engages the topics raised in Section 2.2 and 2.3. As discussed in Section 2.3, using beautiful cardinals, he counted the number of non-mutually embeddible models to separate classifiable from non-classifiable classes by a calculation which was not susceptible to the vagaries of cardinal arithmetic. Here the same theme is pursued but within ZFC. As in Section 2.2, [She88] deals with pseudo-elementary classes, but now with respect to IE; the goal is the following result.

Theorem 2.4.4. *([She88, Conclusion 3.1]) If T is a complete first order theory, which is not superstable, not only does $|PC(T_1, T, \lambda)| = 2^\lambda$ but for $\lambda > |T_1|$, $IE(T_1, T, \lambda) = 2^\lambda$.*

In [She78, Chapter VIII] this result is proved with the *additional hypothesis that λ is regular*[36]. Using club guessing, Shelah extends the result to singular cardinals. This makes the external definitions of the stability classification via spectrum functions more robust.

The moral here is that specific combinatorial analysis of the tree λ^ω and the development of important club guessing techniques arose from extending a result from many non-isomorphic model in λ for regular λ to many non-mutually embeddible models of a PC-class in λ for *singular* and thus arbitrary λ. The first facet represents a fundamental model theoretic contribution; the second introduces the new line of club guessing in set theory [CFM04, D0̌5, S.I92].

3 The role of set theoretic axioms

In [Mad19], Maddy writes

> So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides *Risk Assessment* for mathematical theories, a *Generous Arena* where the branches of mathematics can be pursued in a unified setting with a *Shared Standard of Proof*, and a *Meta-mathematical Corral* so that formal techniques can be applied to all of mathematics at once.

[36]The *cofinality* of a cardinal λ is the least cardinal κ such that there is an increasing function f from κ into λ such that the supremum of the range of f is κ; otherwise λ is singular.

I think one can distinguish Generous Arena from Meta-mathematical Corral as follows. Generous Arena refers to the role of ZFC as establishing a framework for traditional mathematics. This is the sense of set theory employed by Bourbaki. The meta-mathematical corral is the collection of extensions of ZFC that provide different and perhaps contradictory arenas (Witness V=L, Martin's axiom, and the Whitehead problem.).

Maddy includes another criteria, *essential guidance*, described as follows [Mad19, page 300], 'such a foundation is to reveal the fundamental features – the essence, in practice – of the mathematics being founded, without irrelevant distractions; and it's to guide the progress of mathematics along the lines of those fundamental features and away from false alleyways.'

I will argue in [Bal20] that inserting model theory as an intermediate step between the (in principle) reduction of arguments in, say, algebraic geometry to set theory has several virtues. First we preserve the role of set theory for risk assessment, shared standard of proof, and the Meta-mathematical Corral. And model theory makes this more convenient for most of mathematical practice than the implied but not carried out full coding of traditional mathematics into set theoretic foundations. For algebra (category theory excepted), only formulating the notion of structure is 'set theoretic'. The description of operations for combining structures is part of traditional practice. But model theory also provides *essential guidance* for traditional mathematical research in two ways. By providing formal frameworks aligned to each subject area it helps to clarify arguments within the area and by exposing, through notions such as stability, combinatorial principles that hold in several areas it helps to build connections among areas. Until very recently, this support seemed to be primarily for algebraic topics, notably real algebraic geometry and Diophantine geometry. But recent work in differentially closed fields on the partial differential equations of Painlevé [NP16] show that the stability hierarchy and geometric stability can have significant applications in analysis. Further we noted above the role of o-minimality on the frontiers of number theory [Pil11, PW06] and in asymptotic analysis [AvdDvdH17]. And in the last few years there have been deeper connections with combinatorics and even learning theory [CF19, CS18, LT19].

Essential guidance is a purported advantage of univalent foundations according to supporters. But Maddy argues, I think correctly, that this claim holds only for certain restricted, but highly influential, areas of mathematics that depend heavily on category theoretic methods. My first point is that model theory, via classification theory, provides such essential guidance in a wider range of mathematics because the formalization for a particular area takes its concerns into account and easily accommodates the constructions of the area. But the examples discussed in this paper support a widening of that claim; the examples here illustrate that model theory (and algebra) have a role in guiding set theoretic research. And this leads to an effect of model theory on the meta-mathematical corral.

We have described above various instances of model theoretic problems engendering new animals in the metamathematical corral. But they may also serve a normative function in evaluating the alternatives. Shelah raised such a possibility by making a distinction between axioms (apparently those with 'internal[37]' justification) for set theory and the vast majority of extensions, 'semi-axioms' which are justified primarily by their consequences.

> What are our criterions for semi-axioms? First of all, having many consequences, rich, deep beautiful theory is important. Second, it is preferable that it is reasonable and 'has positive measure'. Third, it is preferred to be sure it leads to no contradiction (so lower consistency strength is better).
>
> Shelah [She03]

[37] Here we mean Gödel's distinction between internal justifications that 'follow from the notion of set' and external or pragmatic justifications.

While the first and third criteria are inexact, the general intent is clear. But, 'reasonable and *has positive measure*' need some explanation. In [She03], 'has positive measure' is an impressionistic phrase; roughly the more conflicting sentences a semi-axiom ϕ permits, the greater the measure of ϕ. The lack of independence over $V = L$ (given its canonical model) is the justification for saying $V = L$ has measure zero. Reasonability is a judgement based on the plausibility of the consequences of the semi-axiom.

The work of Shelah discussed here emphasizes a subtlety in the nature of dividing lines. Shelah's Steele prize acceptance, quoted in the introduction, asserted, 'this means meaningful things are to be said on both sides of the dividing line'. In the introduction I interpreted 'to be said' as 'consequence'. But the examples here, particularly Section 2.1 weaken this condition to 'consistent consequence'. Shelah makes a similar comment in [She00, 5.2] ('poor man ZFC answer'). And this weakening allows model theory to both motivate and arbitrate amongst semi-axioms.

The entire discussion here depends on a fundamental contribution of modern logic; it enables a new (20th century) tool in mathematics: Formalize a particular area of mathematics as a (usually) first order theory. Study the models of a complete first order theory by the model theoretic methods discussed above. Or, for the incomplete theory ZFC, study its models and the extensions of the theory using, in particular, the method of forcing. Shelah has brilliantly integrated these two projects.

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