Some projective planes of Lenz-Barlotti class I

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In [1] we modified an idea of Hrushovski [4] to construct a family of almost strongly minimal nonDesarguesian projective planes. In this note we determine the position of these planes in the Lenz-Barlotti classification [7]. We further make a minor variant on the construction to build an almost strongly minimal projective plane whose automorphism group is isomorphic to the automorphism group of any line in the plane.

By a projective plane we mean a structure for a language with a unary predicate for lines, a unary predicate for points, and an incidence relation that satisfies the usual axioms for a projective plane.

We work with a collection $\mathbf{K}$ of finite graphs and an embedding relation $\leq$ (often called strong embedding) among these graphs. Each of these projective planes (M, P, L, I) is constructed from a $(\mathbf{K}, \leq)$-homogeneous universal graph (M^*, R) as in [1] and [4]. This paper depends heavily on the methods of those two papers and on considerable technical notation introduced in [1]. Each plane M is derived from a graph $M^* = (M^*, R)$. Formally this transformation sets $P = M^* \times \{0\}$, $L = M^* \times \{1\}$ and $\langle m, 0\rangle$ is on $\langle n, 1\rangle$ if and only if $R(m, n)$. Thus, if G^* denotes the automorphism group of M^*, there is a natural injection of G^* into the automorphism (collineation) group G of M. There is a natural polarity ρ of M ($\rho(\langle m, i\rangle) = \langle m, 1 - i\rangle$ for $i = 0, 1$). Then G^* is naturally regarded as the subgroup of G consisting of those collineations that are automorphisms of the structure (M, P, L, I, ρ).

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It should be noticed that this self-duality plays an important role in the construction of the homogenous-universal graph. If one attempts to avoid this by constructing a homogeneous universal bipartite graph, it is soon observed that the amalgamation property fails. (Construct B and C extending A and an element x of A such that B thinks x is a point and C thinks x is a line.)

The almost strong minimality of M is established in [1] by showing any line is strongly minimal and noting that the plane is in the algebraic closure of any single line and two points off the line. After making a minor variant in the construction we build here a plane which is in the definable closure of any line (without additional parameters!).

We can control certain aspects of the behavior of G^* by varying the class **K**. In order to actually determine e.g. the transitivity of G we must vary the construction further.

0.1 Notation. Let A be a subset of N.

i) We write $\mathrm{stb_N(A)}$ for the group of automorphisms of N that fix A pointwise.

ii) We write $\mathrm{sstb_N(A)}$ for the group of automorphisms of N that fix A as a set.

Note that if ℓ is a line, the notation $\mathrm{stb_M}(\ell)$ implicitly demands that we think of ℓ as a subset of P, not an element of L.

0.2 Definition. i) For any point p and line ℓ a (p, ℓ)-collineation is a collineation that fixes ℓ pointwise and p linewise.

ii) A projective plane M is (p, ℓ)-transitive if the subgroup of (p, ℓ)-collineations acts 1-transitively on $\ell' - \{p, q_{\ell'}\}$, where $q_{\ell'}$ is the intersection of ℓ and ℓ', for each line ℓ' through p (except ℓ if p is on ℓ).

Note that any collineation α that fixes a line ℓ pointwise also fixes a (unique) point p linewise [5, Theorem 4.9]. This point is called the *center* of the collineation. α is referred to as either a (p, ℓ)-*central collineation* or a (p, ℓ)-*perspectivity*. Then, $\mathrm{stb_M}(\ell)$ is the collection of all (p, ℓ)-perspectivities as p varies.

The Lenz-Barlotti classification describes a projective plane M by the set $F_M = \{(p, \ell) : M \text{ is } (p, \ell)\text{-transitive}\}$. Various conditions on the class

F_M correspond to properties of the ternary ring that coordinatizes P. While there are 16 different categories in the classification, we deal here with only class I_1: $F_M = \emptyset$.

0.3 Definition. i) A flag in the projective plane M is a pair (p, ℓ) with p on ℓ.

ii) The collineation group of M is *flag transitive* if it acts 1-transitively on flags.

In [1] each of a certain family of functions μ from ω to ω determines a different projective plane. All of these planes behave in the same way for the properties discussed in this paper so we do not keep track of the function μ. It isn't clear if the collineation group of any of the planes M_0 constructed in [1] is either flag transitive or acts transitively on pairs (p, ℓ) with p not on ℓ. However, by modifying the construction to define M_1 we can guarantee increased transitivity by ensuring that G_1^* acts with a sufficient required degree of transitivity on M_1^*.

To see that G_0^* does not act flag transitively on M_0 note that by slightly extending the construction in [1] (or see [2]) of a 14 element graph A with $y(A) = 3$, it is possible to construct a square-free graph B with 17 elements and $y(B) = 2$. Now if $C = \langle a, b \rangle$ is contained in B with aRb and B is strongly embedded in M_0^*, there is no automorphism of M_0^* taking C to a $C' \leq M_0^*$ since C is not a strong submodel of M_0^*. But C and C' both represent flags of M. But this leaves open the possibility that G_0 could still act flag transitively.

As in [1] we fix the dimension function $y(A) = 2|A| - e(A)$ where $e(A)$ is number of edges of the graph A.

0.4 Definition. $\mathbf{K}$ is the collection of all finite graphs B such that

i) For every nonempty $B' \subseteq B$, $y(B') > 1$.

ii) There is no square (4-cycle) embedded in B.

iii) The adjacency relation is symmetric and irreflexive.

Fix a function μ as in [1] and recall from there such notions as minimally 0-simply algebraic.

0.5 Definition. For $n = 2$ or 3, let $\mathbf{K}_n^*$ be the class of finite structures $M \in \mathbf{K}$ satisfying the following additional conditions.

i) If $A \subseteq M$ and $|A| > n$ then $y(A) > n$.

ii) For each pair $\langle A, B\rangle$ with B minimally 0-simply algebraic over A there are at most $\mu(A, B)$ disjoint copies B_i of B over A in M.

Now slightly varying the proof in [1] yields the following

0.6 Lemma. *For $n = 2$ or 3, the class $\mathbf{K}_n^*$ has the amalgamation property.*

Proof. We do in detail the case of $\mathbf{K}_3^*$. Recall from [1] that the proof that there is an amalgam of the strong extensions B_1 and B_2 over B_0 proceeds by induction on $|\hat{B}_1| + |\hat{B}_2|$. ($\hat{B}_i$ denotes $B_i - B_0$.) The proof reduces to checking two cases. In the first $B_1 = B_0 b$ and there is one relation between b and B_0. In the second B_1 is 0-simply algebraic over B_0. In the first case $B_1 \otimes_{B_0} B_2$ is the amalgam. In the second case, either B_1 can be embedded in B_2 over B_0 or $B_1 \otimes_{B_0} B_2$ is the amalgam. Thus, to extend the result to our situation we simply have to show that $B_1 \otimes_{B_0} B_2$ does not contain a D with $|D| > 3$ and $y(D) \leq 3$ if neither B_1 nor B_2 does. For this, note that $3 \leq y(D) = y(\hat{D}_2/D_1) + y(D_1)$ and $y(\hat{D}_2/D_1) \geq y(\hat{D}_2/B_0) \geq 0$ so $y(D_1) \leq 3$. Since $B_1 \in \mathbf{K}_3$ this implies $|D_1| \leq 3$. Similarly, $|D_2| \leq 3$. Thus, $|D| \leq 6$. It can easily be checked by inspection that there are no square free graphs D of cardinality 4, 5 or 6 with $y(D) \leq 3$. (See [2] for more general results as well as this fact.) The possibility that $|D| \leq 3$ is easily eliminated to complete the proof.

The case of $\mathbf{K}_2^*$ is even easier as the reduction is to graphs of size 3 or 4.

0.7 Theorem. *i) The almost strongly minimal plane associated with $\mathbf{K}_2^*$ is flag transitive.*

ii) The almost strongly minimal plane M_3 associated with $\mathbf{K}_3^*$ is flag transitive. There are at most two orbits of pairs (p, ℓ) where p does not lie on ℓ. For each ℓ, $\mathrm{sstb}_{\mathrm{M_3}}(\ell)$ is infinite.

Proof. We write out only the more complicated second argument. The choice of K_3 shows that any embedding of a pair into the homogeneous-universal model M_3^* is actually a strong embedding. If p lies on ℓ, then $p = \langle a, 0\rangle$ and

$\ell = \langle b, 1\rangle$ for some a, b in M_3^* with aRb. All such pairs lie in the same orbit under G^*; this establishes flag transitivity. Now suppose $p = \langle a, 0\rangle$ $\ell = \langle b, 1\rangle$ for a, b in M_3^* but p does not lie on ℓ. There are two cases. In the first, $a \neq b$ and point line pairs of this kind are automorphic since G^* acts transitively on pairs a, b with $\neg aRb$. In the second, $a = b$ and point line pairs of this kind are automorphic since G^* acts 1-transitively on M_3^*.

Now fix any ℓ. There is a point p not on ℓ such that $\langle p, \ell\rangle$ is of the second type. For any other p', p'' $\langle p', \ell\rangle$ and $\langle p'', \ell\rangle$ are of the first type so there is a collineation fixing the line ℓ (setwise) and mapping p' to p''. Thus $\mathrm{sstb}_{\mathrm{M}_3}(\ell)$ is infinite.

0.8 Corollary. *M_3 satisfies one of the following two conditions.*

i) G acts transitively on pairs (p, ℓ) with p not on ℓ.

ii) ρ is definable in (M_3, P, L, I).

Proof. We know from Theorem 0.7 ii) that there are at most two orbits of non-incident point-line pairs. One of them contains all pairs of the form $(\langle a, 0\rangle, \langle a, 1\rangle)$, which is the graph of ρ. All pairs not in this orbit realize the same type. If there are really two distinct orbits, since M_3 is homogeneous, there must be a formula which holds exactly of the graph of ρ as required.

The following result is just a special case of Zilber's general definition of a linking group as reported in Theorem 2.20 of [6].

0.9 Lemma. *Let M be an $\aleph_1$-categorical projective plane.*

i) For any line ℓ, $\mathrm{stb}_{\mathrm{M}}(\ell)$ is a definable group and its action on M is definable.

ii) For any line ℓ and point p the group of (p, ℓ)-collineations is definable.

Proof. Here is a sketch of the proof of i). This situation is somewhat simpler than the general case. Fix two points p_1, p_2 that are not on ℓ. As ℓ is an infinite definable subset of M and Th(M) admits no two cardinal models, M is prime over ℓ so the orbit X of $\langle p_1, p_2\rangle$ under $G = \mathrm{stb}_{\mathrm{M}}(\ell)$ is definable by some formula γ. If $\alpha \in G$, α is determined by $\langle \alpha p_1, \alpha p_2\rangle$ and this determines a $1-1$ correspondence beween G and X. Thus we regard X as the universe of our definable copy of G. For any $x \in M$, that is not on p_1p_2, call the intersection of xp_1 and xp_2 with ℓ the $\langle p_1, p_2\rangle$ coordinate of x. Suppose $\langle q_1, q_2\rangle$,and $\langle q_1', q_2'\rangle$, and $\langle q_1'', q_2''\rangle$ are in X. The product of $\langle q_1, q_2\rangle$ and $\langle q_1', q_2'\rangle$ is $\langle q_1'', q_2''\rangle$ just if the $\langle q_1'', q_2''\rangle$ coordinates of $\alpha' \circ \alpha(x)$ are the $\langle p_1, p_2\rangle$ coordinates of x. We have shown how to define the value of the composition on any point x not on p_1p_2. To extend this to points on p_1p_2 note that the entire argument can be repeated replacing p_1 by any p_1' not on p_1p_2 to determine the action on x. Moreover, this construction does not depend on the choice of p_1'. Since $\alpha' \circ \alpha(x)$ is definably computed from x, $\langle q_1, q_2\rangle$ and $\langle q_1', q_2'\rangle$ the multiplication is definable. In particular we have defined the action of each member of $\mathrm{stb}_\mathrm{M}(\ell)$.

ii) If p is not on ℓ the (p, ℓ)-collineations are just the subgroup of $\mathrm{stb}_\mathrm{M}(\ell)$ that fix p. If p is on ℓ, it is the subgroup of those elements of $\mathrm{stb}_\mathrm{M}(\ell)$ that fix (setwise) each line ℓ' through p.

0.10 Lemma. *If there is no infinite group definable in the projective plane M and lines in M are infinite then $F_M = \emptyset$. Thus all the planes constructed in [1] are of Lenz-Barlotti class I_1.*

Proof. By 0.9 ii), for any p and ℓ the group of (p, ℓ) collineations is definable. If M is (p, ℓ) transitive, it is infinite. But this is impossible so $F_M = \emptyset$ as required.

Note that applying this line of reasoning in the other direction and using Theorem 0.7 ii) we conclude that for each line ℓ, $sstb_{M_3}(\ell)$ is not definable.

A ternary ring is definable in any projective plane. In any ternary ring addition and multiplication are defined by $a + b = T(a, 1, b)$ and $a \cdot b = T(a, b, 0)$. The ring is *linear* if $T(x, a, b) = x \cdot a + b$. The basic properties of the Lenz-Barlotti classification as expounded in e.g. [3] show that for each plane of Lenz-Barlotti class I_1 the associated planar ternary ring is not even linear. So not only is neither the additive nor the multiplicative structure of the ternary ring defined in our projective plane associative but it is actually impossible to split the ternary operation into two binary operations.

We can extend this 'rigidity' even further.

0.11 Proposition. *If the automorphism group of the plane M acts both flag transitively and transitively on pairs (p, ℓ) with p not on ℓ and* $\mathrm{stb}_\mathrm{M}(\ell)$ *is nontrivial then* $\mathrm{stb}_\mathrm{M}(\ell)$ *is infinite.*

Proof. Choose $\beta \in \mathrm{stb_M}(\ell)$ which is not the identity. Let p_β be the center of β. If p_β is on ℓ then by flag transitivity for each point p_i on ℓ there is an automorphism α_i mapping p_β to $p - i$. If p_β is not on ℓ let $\{p_i : i < \omega\}$ be an infinite set of points that are not on ℓ. Choose for each i, by the second transitivity hypothesis an automorphism $\alpha_i \in \mathrm{sstb_M}(\ell)$ with $\alpha_i p_\beta = p_i$. In either case β^{α_i} is a perspectivity with axis ℓ and center $\alpha_i p_\beta = p_i$ so $\mathrm{stb_M}(\ell)$ is infinite.

0.12 Theorem. *The almost strongly minimal plane M_3 associated with $\mathbf{K}_3^*$ is in the definable closure of any line. That is, M_3 admits no perspectivities.*

Proof. By Corollary 0.8 either ρ is definable or G acts transtively on nonincident point-line pairs. In the first case let a_0, a_1, a_2 lie on ℓ. Then at least two of the three pairwise intersections of the $\rho(a_i)$ are not on ℓ. Thus two points off ℓ and *a fortiori* all points in P are in $\mathrm{dcl}(\ell)$. In the second case, it suffices to show that $\mathrm{stb_M}(\ell)$ is the identity for any ℓ. By Lemma 0.9 $\mathrm{stb_M}(\ell)$ is definable and by Proposition 0.11 and Theorem 0.7, if $\mathrm{stb_M}(\ell)$ is nontrivial then it is infinite. Thus, for any ℓ, $\mathrm{stb_M}(\ell)$ is trivial.

Three questions arise.

i) What is a geometric explanation of the phenomenon of Theorem 0.12?

ii) Is ρ always definable?

iii) Does $\mathrm{Aut}(M_3)$ contains any involutions?

We can essentially rephrase ii) by asking whether M admits any automorphisms that are not induced in the obvious way by automorphisms of M^*.

We show now that $\mathrm{Aut}(M_3)$ has no definable involutions. By a subplane of M we mean a subset M_0 of points and lines that, with respect to the same incidence relation, form a projective plane. M_0 is a Baer subplane if each point (line) in M lies on (contains) a line (point) from M_0.

0.13 Corollary. *Any involution of M_3 fixes a subplane pointwise.*

Proof. By Theorems 4.3 and 4.4 of [5], the fixed set of any involution of a projective plane is a Baer subplane unless the involution is a perspectivity but we know there are no perspectivities.

0.14 Lemma. *If α is an involution of M_3 and ℓ is any line fixed by α,* $\mathrm{stb_M}(\alpha) \cap \ell$ *is infinite and coinfinite.*

Proof. Let $F = \mathrm{stb_{M_3}}(\alpha)$, F_P the points fixed by α, F_L the lines fixed by α, and $X = F_P \cap \ell$. Clearly the intersection of F_P with any line in F_L is infinite. Choose $\ell_1 \in F_L$ with $\ell_1 \cap \ell = b \in F_P$. Let $a \in \ell_1 - F_P$. Fix $l_2 \in F_L$ but not equal to ℓ or ℓ_1. Now, for any of the infinitely many points y on $\ell_2 \cap F_P$, the line determined by a and y intersects ℓ in a point not in F_P. So $\ell - F_P$ is infinite.

0.15 Corollary. *M_3 has no definable involutions.*

Proof. If α were a definable automorphism of M, its stabilizer F would be defined and for any ℓ in F, $F \cap \ell$ would be definable. But then Lemma 0.14 contradicts the strong minimality of each line in M.

This leaves open the question of whether this structure admits any involutions that are not definable.

References

[1] John T. Baldwin. An almost strongly minimal non-Desarguesian projective plane. *Transactions of the American Mathematical Society*, 342:695–711, 1994.

[2] C.R.J. Clapham, A. Flockhart, and J. Sheehan. Graphs without four-cycles. *Journal of Graph Theory*, 13:29–47, 1987.

[3] Peter Dembowski. *Finite Geometries*. Springer-Verlag, 1977.

[4] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[5] D.R. Hughes and F.C. Piper. *Projective Planes*. Springer-Verlag, 1973.

[6] Bruno Poizat. Sous-groupes définissables d'un groupe stable. *The Journal of Symbolic Logic*, 46:137–146, 1981.

[7] J. Yaqub. The Lenz-Barlotti classification. In R. Sandler, editor, *Proceedings of the Projective Geometry Conference, 1967*, pages 129– 163. University of Illinois at Chicago Circle, 1967.