Axiomatizing changing conceptions of the geometric continuum I: Euclid-Hilbert

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

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Abstract

We begin with a general account of the goals of axiomatization, introducing a variant (modest) on Detlefsen's notion of 'complete descriptive axiomatization'. We examine the distinctions between the Greek and modern view of number, magnitude and proportion and consider how this impacts the interpretation of Hilbert's axiomatization of geometry. We argue, as indeed did Hilbert, that Euclid's propositions concerning polygons, area, and similar triangles are derivable (in their modern interpretation in terms of number) from Hilbert's *first-order* axioms.

We argue that Hilbert's axioms including continuity show much more than Euclid's theorems on polygons and basic results in geometry and thus are an immodest complete descriptive axiomatization of that subject.

Consider the following two propositions.

(*) Euclid VI.1: Triangles and parallelograms which are under the same height are to one another as their bases.

Hilbert[1] gives the area of a triangle by the following formula.

(**) Hilbert: Consider a triangle ABC having a right angle at A. The measure of the area of this triangle is expressed by the formula

$$F(ABC) = \frac{1}{2} AB \cdot AC.$$

At first glance these statements seem to carry the same message, the familiar fact about computing the area of triangle. But clearly they are not identical. Euclid

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[1]Hilbert doesn't state this result as a theorem; I have excerpted the statement below from an application on page 66 of [Hilbert 1962].

tells us that the two-dimensional area of two triangles 'under the same height' is *proportional* to their 1-dimensional bases. Hilbert's result is not a statement of proportionality; it tells us the 2-dimensional measure of a triangle is computed from a product of the 1-dimensional measures of its base and height. Hilbert's rule looks like a rule of basic analytic geometry, but it isn't. He derived it from an axiomatic geometry similar to Euclid's, which in no way builds on Cartesian analytic geometry.

Although the subject, often called Euclidean geometry, seems the same, clearly much has changed. This paper and its sequel [Baldwin 2017a] aim to contribute to our understanding some important aspects of this change. One of these aspects is the different ways in which the *geometric continuum*, the line in the context of the plane, is perceived. Another is a different basis for the notion of proportion. The two papers deal with axiomatizations of the geometric line in logics of differing strengths. This analysis is integrated with a more general discussion of model theory and the philosophy of mathematical practice in [Baldwin 2017b]. Hilbert's axiomatization is central to our considerations. One can see several challenges that Hilbert faced in formulating a new axiom set in the late 19th century:

1. Delineate the relations among the principles underlying Euclidean geometry. In particular, identify and fill 'gaps' or remove 'extraneous hypotheses' in Euclid's reasoning.
2. Reformulate propositions such as VI.1 to reflect the 19th century understanding of real numbers as measuring both length and area.
3. Ground the geometry of Descartes, late nineteenth century analytic geometry, and mathematical analysis.

The third aspect of the third challenge is not obviously explicit in Hilbert. We will argue Hilbert's completeness axiom is unnecessary for the first two challenges and at least for the Cartesian aspect of the third. The gain is that it grounds mathematical analysis (provides a rigorous basis for calculus); that Hilbert desired this is more plausible than that he thoughtlessly assumed too much. For such a judgement we need some idea of the goals of axiomatization and when such goals are met or even exceeded. We frame this discussion in terms of the notion of *descriptive axiomatization* from [Detlefsen 2014], which is discussed in the first section of this paper.

The main modification to Detlefsen's framework addresses the concern that the axioms might be too strong and obscure the 'cause' for a proposition to hold. We introduce the term 'modest' descriptive axiomatization to denote one which avoids this defect. That is, one which meets a certain clear aim, but doesn't overshoot by too much. We describe several 'data sets', explicit lists of propositions from Euclid, and draw from [Hartshorne 2000], to link specified subsets of Hilbert's axioms that justify them.

Recall that Hilbert groups his axioms for geometry into 5 classes. The first four are first-order. Group V, Continuity, contains Archimedes axiom, which can be stated in the logic[2] $L_{\omega_1,\omega}$, and a second-order completeness axiom equivalent (over the other axioms) to Dedekind completeness[3] of each line in the plane.

The *side-splitter theorem* asserts that a line parallel to the base cuts the sides of a triangle proportionately. To prove it Euclid passes through a theory of area that is based on a theory of proportion that implicitly uses a form of Archimedes Axiom. Bolzano called this path an atrocious detour. Hilbert met what we call *Bolzano's challenge* by developing proportionality, similar triangles, and then area on the basis of his first order axioms.

We conclude that Hilbert's first-order axioms provide a modest complete descriptive axiomatization for most of Euclid's geometry. In the sequel we argue that the second-order axioms aim at results that are beyond (and even in some cases antithetical to) the Greek and even the Cartesian view of geometry. So Hilbert's axioms are immodest as an axiomatization of *traditional* geometry. This conclusion is no surprise to Hilbert[4] although it may be to many readers[5]. In the preface to [Hilbert 1962] the translator Townsend writes, 'it is shown that the whole of the Euclidean geometry may be developed without the use of the axiom of continuity.' Hilbert concludes his introduction of the continuity axioms with:

> From a theoretical point of view, the value of this axiom is that it leads indirectly to the introduction of limiting points, and, hence, renders it possible to establish a one-to-one correspondence between the points of segment and the system of real numbers. However, in what is to follow, no use will be made of the 'axiom of completeness. ([Hilbert 1962], page 26)

How should one compare such statements as (*) and (**)? We lay out the relations among three perspectives on a mathematical topic. After clarifying the notion of data set in the next section, the two papers focus on aligning the latter two perspectives for various data sets.

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[2] In the logic, $L_{\omega_1,\omega}$, quantification is still over individuals but now countable conjunctions are permitted so it is easy to formulate Archimedes Axiom: $\forall x, y(\bigvee_{m\in\omega} mx > y)$. By switching the roles of x and y we see each is reached by a finite multiple of the other.

[3] Dedekind defines the notion of a cut in a linearly ordered set I (a partition of $\mathbb{Q}$ into two intervals (L, U) with all elements of U less than all elements of U). He postulates that each cut has unique realization, a point above all members of L and below all members U -it may be in either L or U (page 20 of [Dedekind 1963]). If either the L contains a least upper bound or the upper interval U contains a greatest lower bound, the cut is called 'rational' and no new element is introduced. Each of the other (irrational) cuts introduces a new number. It is easy to see that the *unique* realization requirement implies the Archimedes axiom. By Dedekind completeness of a line, I mean the Dedekind postulate holds for the linear ordering of that line. See the sequel.

[4] Hilbert lectured on geometry several summers in the 1890's and his notes (German) with extremely helpful introductions (English) appear in [Hallett & Majer 2004]. The first *Festschrift* version of the *Grundlagen* does not contain the continuity axioms. I draw primarily on the (2nd (Townsend edition) of Hilbert and on the 7th [Hilbert 1971].

[5] The first 10 urls from a google search for 'Hilbert's axioms for Euclidean geometry' contained 8 with no clear distinction between the geometries of Hilbert and Euclid and two links to Hartshorne, who distinguishes.

1. *A data set* [Detlefsen 2014], a collection of propositions about the topic.
2. A system of axioms and theorems for the topic.
3. The different conceptions of various terms used in the area at various times.

In Section 1, we consider several accounts of the purpose of axiomatization. We adjust Detlefsen's definition of complete descriptive axiomatization to guarantee some 'minimality' of the axioms by fixing on a framework for discussing the various axiom systems: a *modest descriptively complete axiomatization*. One of our principal tools is Detlefsen's notion of 'data set', a collection of sentences to be accounted for by an axiomatization. 'The data set for area X' is time dependent; new sentences are added; old ones are reinterpreted. Section 2 lists data sets (collections of mathematical 'facts'), then specific axiom systems, and asserts the correlation. In Section 3, we consider the changes in conception of the continuum, magnitude, and number. In particular, we analyze the impact of the distinction between ratios in the language of Euclid and segment multiplication in [Hilbert 1962] or multiplication[6] of 'numbers'. With this background, we sketch in Section 4 Hilbert's theory of proportions and area, focusing on Euclidean propositions that might appear to depend on continuity axioms. In particular, we outline Hilbert's definition of the field in a plane and how this leads to a good theory of area and proportion, while avoiding the Axiom of Archimedes.

This paper expounds the consequences of Hilbert's first-order axioms and argues they form a modest descriptive axiomatization of Euclidean geometry. The sequel extends the historical analysis from Euclid and Hilbert to Descartes and Tarski, explores several variants on Dedekind's axiom and the role of first-order, infinitary, and second-order logic. Further, it expounds a first-order basis for the formulas for area and circumference of a circle.

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1 The Goals of Axiomatization

In this section, we place our analysis in the context of recent philosophical work on the purposes of axiomatization. We explicate Detlefsen's notion of 'data set' and investigate the connection between axiom sets and data sets of sentences for an area of mathematics. Hilbert begins the *Grundlagen* [Hilbert 1971] with:

[6] That is, a multiplication on points rather than segments. See Heyting [Heyting 1963]; the most thorough treatment is in [Artin 1957].

> The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent axioms* and to deduce from them the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the groups of axioms and the scope of the conclusions to be derived from the individual axioms.

Hallett ([Hallett 2008], 204) delineates the meaning of facts in this context, 'simply what over time has come to be accepted, for example, from an accumulation of proofs or observations. Geometry, of course, is the central example ....' Hallett ([Hallett & Majer 2004], 434) presaged the emphasis here on 'data sets'.

> Thus completeness appears to mean [for Hilbert] 'deductive completeness with respect to the geometrical facts'. ... In the case of Euclidean geometry there are various ways in which 'the facts before us' can be presented. If interpreted as 'the facts presented in school geometry' (or the initial stages of Euclid's geometry), then arguably the system of the original Festschrift [i.e. 1899 French version] is adequate. If, however, the facts are those given by geometrical intuition, then matters are less clear.

Hilbert described the general axiomatization project in 1918.

> When we assemble the facts of a definite, more or less comprehensive field of knowledge, we soon notice these facts are capable of being ordered. This ordering always comes about with the help of a certain *framework of concepts [Fachwerk von Begriffen]* in the following way: a concept of this framework corresponds to each individual object of the field of knowledge, a logical relation between concepts corresponds to every fact within the field of knowledge. The framework of concepts is nothing other than the *theory* of the field of knowledge. ([Hilbert 1918], 1107)

Detlefson [Detlefsen 2014] describes such a project as a *descriptive axiomatization*. He motivates the notion with this remark by Huntington (Huntington's emphasis):

> [A] miscellaneous collection of facts ... does not constitute a *science*. In order to reduce it to a science the first step is to do what *Euclid* did in geometry, namely, to *select a small number of the given facts as axioms and then to show that all other facts can be deduced from these axioms by the methods of formal logic*. [Huntington 1911]

Detlefsen uses the term *data set* (i.e. facts[7]) to describe a local descriptive axiomatization as an attempt to deductively organize a data set of commonly accepted sentences.

[7] There is an interesting subtlety here (perhaps analogous to the Shapiro's algebraic and non-algebraic theories). The data set for group theory could be interpreted as a) formal consequences of the axioms for groups in the vocabulary $(+, 0)$, or b) theorem proved by group theorists. For Euclid-Hilbert geometry these meanings coalesce.

The axioms are *descriptively complete* if all elements of the data set are deducible from them. This raises two questions. What is a sentence? Who commonly accepts?

From the standpoint of modern logic, a natural answer to the first question would be to specify a logic and a vocabulary and consider all sentences in that language. Detlefsen argues (pages 5-7 of [Detlefsen 2014]) that this is the wrong answer. He thinks Gödel errs in seeing the problem as completeness in the now standard sense of a first-order theory[8]. Rather, Detlefsen presents us with an *empirical* question. We (at some point in time) look at the received mathematical knowledge in some area and want to construct a set of axioms from which it can all be deduced. In our case we want to compare the commonly accepted sentences from 300 BC with a twentieth century axiomatization. As we see below, new interpretations for terms arise. Nevertheless, a specific data set delineates a certain area of inquiry. The data set is inherently flexible; conjectures are proven (or refuted) from time to time. The analysis in this paper tries to distinguish propositions that are simply later deductions about the same intuitions and those which invoke radically different assumptions. By analyzing the interpretations in particular cases, we can specify a data set. Comparing geometry at various times opens a deep question worthy of more serious exploration than there is space for here. In what sense do (*) and (**) opening this paper express the same thought, concept etc.? Rather than address the issue of what is expressed, we will simply show how to interpret (*) (and other propositions of Euclid) as propositions in Hilbert's system.

Working in a framework of formal logic, we return to our question, 'What is a sentence?' The first four groups of Hilbert's axioms are sentences of first-order logic: quantification is over individuals and only finite conjunctions are allowed. As noted in Footnote 2, Archimedes axiom can be formulated in $L_{\omega_1,\omega}$. But the Dedekind postulate in any of its variants is a sentence of a kind of second-order logic[9]. All three logics have deductive systems and the first and second order systems allow only finite proofs so the set of provable sentences is recursively enumerable. Second-order logic (in the standard semantics) fails the completeness theorem but, by the Gödel and Karp [Karp 1959] completeness theorems, every valid sentence of $L_{\omega,\omega}$ or $L_{\omega_1,\omega}$ is provable.

Adopting this syntactic view, there is a striking contrast between the data set of geometry in earlier generations and the axiom systems advanced near the turn of the twentieth century. Except for the Archimedean axiom, the earlier data sets are expressed in first-order logic.

Geometry is an example of what Detlefsen calls a *local* as opposed to a *foundational* (global) descriptive axiomatization. Beyond the obvious difference in scope, Detlefsen points out several other distinctions. In particular, ([Detlefsen 2014], 5) asserts axioms of a local axiomatization are generally among the given facts while those of a foundational axiomatization are found by (paraphrasing Detlefsen) tracing each truth in a data set back to the deepest level where it can be properly traced. Hilbert's geometric axioms have a hybrid flavor. Through the analysis of the concepts involved,

[8] We dispute some of his points just before Remark 2.1 in the sequel.
[9] See the caveats on 'second-order' (e.g. sortal) in the sequel.

Dedekind arrived at a second-order axiom[10] that is not in the data set but formed the capstone of the axiomatization: Dedekind completeness for geometry.

An aspect of choosing axioms seems to be missing from the account so far. Hilbert [Hilbert 1918] provides the following insight into how axioms are chosen:

> If we consider a particular theory more closely, we always see that a few[11] distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework. ...
>
> These underlying propositions may from an initial point of view be regarded as the axioms of the respective field of knowledge ...

By a *modest* axiomatization of a given data set[12], we mean one that implies all the data and not too much more[13]. Of course, 'not too much more' is a rather imprecise term. One cannot expect a list of known mathematical propositions to be deductively complete. By too much more, we mean the *axioms* introduce essentially new concepts and concerns or add additional hypotheses proving a result that contradicts the explicit understandings of the authors of the data set. (Section 3 in the sequel).

In this paper, we are investigating modern axiomatizations for an ancient data set. As we'll see below, using Notation 2.2, Hilbert's first-order axioms (HP5) are a modest axiomatization of the data (Euclid I): the theorems in Euclid about polygons (not circles) in the plane. We give an example later showing that HP5 + CCP (circle-circle intersection), while modest for Euclid II, is an immodest first-order axiomatization of polygonal geometry. In the twentieth century the process of formalization is usually attentive to modesty so it is a bit hard to find non-artificial examples. However, to study complex exponentiation, [Zilber 2005] defined a *quasi-minimal excellent class*. The axioms asserted models were combinatorial geometries satisfying extra conditions, most importantly *excellence*. Although his axioms were informal they can be formalized[14] in $L_{\omega_1,\omega}(Q)$. Quite unexpectedly, [Bays et al. 2014] showed 'excellence' was not needed for the main result. Thus, the original axiomatization was immodest.

No single axiom is modest or immodest; the relation has two arguments: a set

[10]We discuss the 'equivalence' of Dedekinds and Hilbert's formulation of completeness in the sequel.

[11]Often, few is interpreted as finite. Whatever Hilbert meant, we should now be satisfied with a small finite number of axioms and axiom schemes. At the beginning of the Grundlagen, Hilbert adds 'simple, independent, and complete'. Such a list including schemes is simple.

[12]We considered replacing 'modest' by 'precise or'safe' or 'adequate'. We chose 'modest' rather than one of the other words to stress that we want a sufficient set and one that is as necessary as possible. As the examples show, 'necessary' is too strong. Later work finds consequences of the original data set undreamed by the earlier mathematicians. Thus just as, 'descriptively complete', 'modest' is a description, not a formal definition.

[13]This concept describes normal work for a mathematician. "I have a proof; what are the actual hypotheses so I can convert it to a theorem."

[14]The Q is the quantifier, 'there exist uncountably many'.

of sentences is a modest axiomatization of a given data set. If the axioms are contained in the data set the axiomatization is manifestly modest and this is just a mathematical fact that can be clarified by formalization. But some other subset might later be taken as axioms that imply the whole set. This might just happen by a clever proof. But, the cases studied here are more subtle. New interpretations of the basic concepts developed over time (of multiplication and number) so that the sentences attained essentially new meanings. As (**) illustrates, such is the case with Euclid's VI.1. Distinct philosophical issues arise in checking modesty and immodesty. For modesty, an historical investigation, as in this paper, can explore how changing conceptions are reflected in new proofs and whether such arguments are modest. For, immodesty, the conceptual content of the new axioms must be compared with that of the data set.

In our view, *modesty* and *purity* are distinct, though related, notions. We just saw that formalization is useful to check modesty. [Hallett 2008], [Arana & Mancosu 2012, Detlefsen & Arana 2011, Baldwin 2013a] argue that purity asks about specific *arguments* for a proposition. [Baldwin 2013a] emphasized that the same theorem can have *both pure and impure proofs* and [Baldwin 2017b] extends the analysis of modesty versus purity. In contrast, modesty concerns the existence of proofs and appropriateness of hypotheses.

In the quotation above, Hilbert takes the axioms to come from the data set. But this raises a subtle issue about what comprises the data set. For examples such as geometry and number theory, it was taken for granted that there was a unique model. Even Hilbert adds his completeness axiom to guarantee categoricity and to connect with the real numbers. So one could argue that the early 20th century axiomatizers took categoricity as part of the data[15]. But such an intuition is inherently *metatheoretic* and so dissimilar to the other data. It is certainly not in the data set of the Greeks for whom 'categoricity' is meaningless.

2 Some geometric Data sets and Axiom Systems

This section is intended to lay out several topics in plane geometry that represent distinct data sets in Detlefsen's sense[16]. In cases where *certain axioms are explicit, they are included in the data set*. Although we describe five sets here, only polygonal geometry and circle geometry are considered in this paper; the others are treated in the sequel. Each set includes its predecessors; the description is of the added propositions.

Notation 2.1. *(5 data sets of geometry)*

Euclid I, polygonal geometry: Book I (except I.1, I.22), Book II.1-II.13, Book III (except III.1 and III.17), Book VI.)

[15]Huntington invokes Dedekind's postulate when axiomatizing the complex field in [Huntington 1911].

[16]In the first instance we draw from Euclid: Books I-IV, VI and XII.1, 2 clearly concern plane geometry; XI, the rest of XII and XIII deal with solid geometry; V and X deal with a general notion proportion and with incommensurability. Thus, below we put each proposition Books I-IV, VI, XII.1,2 in a group and consider certain geometrical aspects of Books V and X.

Euclid II, circle geometry: CCP, I.1, I.22, II.14, III.1, III.17 and Book IV.

Archimedes, arc length and π: XII.2 (area of circle proportional to square of the diameter), approximation of π, circumference of circle proportional to radius, Archimedes' axiom.

Descartes, higher degree polynomials: nth roots; coordinate geometry

Hilbert, continuity: The Dedekind plane

Our division of the data sets is somewhat arbitrary and is made with the subsequent axiomatizations in mind.

The importance of Euclid II appears already in Proposition I of Euclid where Euclid makes the standard construction of an equilateral triangle on a given base[17]. Why do the two circles intersect? While some[18] regard the absence of an axiom guaranteeing such intersections as a gap in Euclid, Manders (page 66 of [Manders 2008]) asserts: 'Already the simplest observation on what the texts do infer from diagrams and do not suffices to show the intersection of two circles is completely safe[19].'

The circle-circle intersection axiom resolves those continuity issues involving intersections of circles and lines[20]. As noted, this proposition is in the data set; so adding this one axiom in our axioms EG (below), which does not appear explicitly in either the *Grundlagen* or Euclid, does not detract from the modesty of our axioms. It is a first-order consequence of the Dedekind postulate which plays an essential role in Euclidean geometry. Hilbert is aware of that fact; he chooses to resolve the issue (implicitly) by his completeness axiom.

Circle-Circle Intersection Postulate (CCP): If from distinct points A and B, circles with radius AC and BD are drawn such that one circle contains points both in the interior of one and in the exterior of the other, then they intersect in two points, on opposite sides of AB.

[17] Proposition I.1 is omitted from Euclid I, because it is not possible to construct an equilateral triangle with side length 1 in HP5. Proposition I.2 (transferring a length) is an axiom for Hilbert and of course follows from CCP.

[18] E.g. (Veblen [Veblen 1914], 4)

[19] Manders emphasized the use of diagrams as a coherent mathematical practice. Properties that are *not* changed by minor variations in the diagram such as subsegment, inclusion of one figure in another, the intersection of two lines, betweenness are termed *coexact*. Properties that *can be* changed by minor variations in the diagram, such as whether a curve is a straight line, congruence, a point is on a line, are termed *exact*. We can rely on reading coexact properties from the diagram. The difficulty in turning this insight into a formal deductive system is that, depending on the particular diagram drawn, after a construction, the diagram may have different coexact properties.

Independently, Miller [Miller 2007] and Avigad et al. [Avigad et al. 2009] formulated deductive systems whose rules are intended to capture properties that formerly were read off from diagrams. Miller treats diagrams as topological objects; this leads to difficulties bounding the number of cases. The system in [Avigad et al. 2009] incorporates metric information and provides a complete and sound deductive system for a collection of $\forall\exists$ sentences on models of EG; these include Euclid I-IV, but do not appear to include the theorems relating area and proportionality that are central here.

[20] Circle-circle intersection implies line-circle intersection. Hilbert already in [Hilbert 1971] shows (page 204-206 of [Hallett & Majer 2004]) that circle-circle intersection holds in a Euclidean plane. See Section 4.3.

We have placed Euclid XII.2 (area of a circle is proportional to the square of the diameter) with Archimedes rather than Euclid's other theorems on circles. The crux is the different resources needed to prove VI.1 (area of a parallelogram) and XII.2; the first is provable in EG; The sequel contains a first-order extension of EG in which XII.2 is provable. Note also that we consider only a fraction of Archimedes, his work on the circle. We analyze the connections among Archimedes, Descartes and Tarski in the sequel.

Showing a particular set of axioms is descriptively complete is inherently empirical. One must check whether each of a certain set of results is derivable from a given set of axioms. Hartshorne [Hartshorne 2000] carried out this project independently from Detelfsen's analysis; we organize his results at the end of this section.

We identify two levels of formalization in mathematics. By the Euclid-Hilbert style we mean the axiomatic approach of Euclid along with the Hilbert insight that postulates are implicit definitions of classes of models[21]. By the Hilbert-Gödel-Tarski style, we mean that that syntax and semantics have been identified as mathematical objects; Gödel's completeness theorem is a standard tool, so methods of modern model theory can be applied[22]. We will give our arguments in English; but we will be careful to specify the vocabulary and the postulates in a way that the translation to a first-order theory is transparent.

We will frequently switch from syntactic to semantic discussions so we stipulate precisely the vocabulary in which we take the axioms above to be formalized. We freely use defined terms such as collinear, segment, and angle in giving the reading of the relevant relation symbols. The fundamental relations of plane geometry make up the following vocabulary τ.

1. two-sorted universe: points (P) and lines (L).
2. Binary relation $I(A, \ell)$: Read: a point is incident on a line;
3. Ternary relation $B(A, B, C)$: Read: B is between A and C (and A, B, C are collinear).
4. quaternary relation, $C(A, B, C, D)$: Read: two segments are congruent, in symbols $\overline{AB} \cong \overline{CD}$.
5. 6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \cong \angle A'B'C'$.

Notation 2.2. Consider the following axiom sets[23].

[21]The priority for this insight is assigned to such slightly earlier authors as Pasch, Peano, Fano, in works such as [Freudenthal 1957] as commented on in [Bos 1993] and chapter 24 of [Gray 2011].

[22]See [Baldwin 2014] and the sequel for further explication of this method.

[23]The names HP, HP5, and EG come from [Hartshorne 2000] and $\mathcal{E}^2$ from [Tarski 1959]. In fact, Tarski also studies EG under the name $\mathcal{E}_2''$.

1. First-order axioms:

HP, HP5: We write HP for Hilbert's incidence, betweenness[24], and congruence axioms. We write HP5 for HP plus the parallel postulate.

EG: The *axioms for Euclidean geometry*, denoted EG[25], consist of HP5 and in addition the circle-circle intersection postulate CCP.

$\mathcal{E}^2$**:** Tarski's axiom system for a plane over a real closed field (RCF[26]).

EG_π **and** $\mathcal{E}_\pi$**:** Two new systems, which extend EG and $\mathcal{E}^2$, will be described and analyzed in the sequel.

2. Hilbert's continuity axioms, infinitary and second-order, will also be examined in detail in the sequel.

Archimedes: The sentence in the logic $L_{\omega_1,\omega}$ expressing the Archimedean axiom.

Dedekind: Dedekind's *second-order* axiom[27] that there is a point in each irrational cut in the line.

With these definitions we align various subsystems of Hilbert's geometry with certain collections of propositions in Euclidean geometry as spelled out in Hartshorne[28]. With our grouping, Hartshorne shows the following results. **They imply the statements in HP5 or EG are either theorems or implicit (CCP) in Euclid, they are a modest axiomatization of Euclid I and II.**

First-order Axiomatizations

1. The sentences of Euclid I are provable in HP5.

2. The additional sentences of Euclid II are provable in EG.

In this framework we discuss the changing conceptions of the continuum, ratio, and number from the Greeks to modern times and sketch some highlights of the proof this

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[24]These include Pasch's axiom (B4 of [Hartshorne 2000]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[25]In the vocabulary here, there is a natural translation of 'Euclid's axioms' into first-order statements. The construction axioms have to be viewed as 'for all – there exist' sentences. The axiom of Archimedes is of course not first-order. We write Euclid's axioms for those in the original as opposed to modernized (first-order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [Avigad et al. 2009].

[26]A field is real closed if it is formally real (-1 is not a sum of squares) and every odd degree polynomial has a solution.

[27]Hilbert added his *Vollstandigkeitsaxiom* to the French translation of the 1st edition and it appears from then on. In Section 4.2 of the sequel we explore the connections between various formulations of completeness. We take Dedekind's formulation (Footnote 3) as emblematic.

[28]See Theorems 10.4, 12.3-12.5 in Section 12 and Sections 20-23 of [Hartshorne 2000].

result to demonstrate the *philosophical* modesty of the axiomatization. The sequel contains modest descriptive axiom systems[29] for the data sets of Archimedes and Descartes and argues that the full Hilbert axiom set is immodest for any of these data sets.

3 Changing conceptions of the continuum, magnitude, and number

In the Section 3.1, we distinguish the geometric continuum from the set-theoretic continuum. In Section 3.2 we sketch the background shift from the study of various types of magnitudes by the Greeks, to the modern notion of a collection of real numbers which can measure any sort of magnitude. In Section 3.4 we contrast the current goal of an independent basis for geometry with the 19th century arithmetization project.

3.1 Conceptions of the continuum

In this section, we motivate our restriction to the *geometric* continuum; we defined it as a linearly ordered structure that is situated in a plane. Sylvester[30] describes the three divisions of mathematics:

> There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order.

This is a slightly unfamiliar trio. We are all accustomed to the opposition between arithmetic and geometry. While Newton famously founded the calculus on geometry ([Detlefsen & Arana 2011]) the 'arithmetization of analysis' in the late 19th century reversed the priority. From the natural numbers the rational numbers are built by taking quotients and the reals by some notion of completion. And this remains the normal approach today. We want here to consider reversing the direction again: building a firm grounding for geometry and then finding first the field and then some completion and considering incidentially the role of the natural numbers. In this process, Sylvester's

[29] In [Baldwin & Mueller 2012] and [Baldwin 2013b] we give an equivalent set of postulates to EG, which returns to Euclid's construction postulates and stress the role of Euclid's axioms (Common Notions) in interpreting the geometric postulates. While not spelled out rigorously, our aim is to consider the diagram as part of the argument. For pedagogical reasons the system used SSS rather than SAS as the basic congruence postulate, as it more easily justifies the common core approach to similarity through dilations and makes clear that the equality axioms in logic, as in Euclid's Common Notions, apply to both algebra and arithmetic. This eliminates the silly 6 step arguments in high school texts reducing subtraction of segments to the axioms of the real numbers.

[30] As quoted in [Mathias 1992].

third cardinal notion, order, will play a crucial role. The notion that one point lies between two others will be fundamental and an order relation will naturally follow; the properties of space will generate an ordered field and the elements of that field will be numbers albeit not numbers in the Greek conception.

There are different conceptions of the continuum (the line); hence different axiomatizations may be necessary to reflect these different conceptions. These conceptions are witnessed by such collections as [Ehrlich 1994, Salanskis & Sincaceur 1992] and further publications concerned with the constructive continuum and various non-Archimedean notions of the continuum.

In [Feferman 2008], Feferman lists six[31] different conceptions of the continuum: (i) the Euclidean continuum, (ii) Cantor's continuum, (iii) Dedekind's continuum, (iv) the Hilbertian continuum, (v) the set of all paths in the full binary tree, and (vi) the set of all subsets of the natural numbers. For our purposes, we will identify ii), v), and vi) as essentially cardinality based as they have lost the order type imposed by the geometry; so, they are not in our purview. We want to contrast two essentially geometrically based notions of the continuum: those of Euclid and Hilbert/Dedekind. Hilbert's continuum differs from Dedekind's as it has the field structure derived from the geometric structure of the plane, while Dedekind's field is determined by continuity from known field operations on the rationals. Nevertheless they are isomorphic as ordered fields.

We stipulated that '*geometric* continuum' means 'the line situated in the plane'. One of the fundamental results of 20th century geometry is that *any* (projective[32] for convenience) plane can be coordinatized by a 'ternary field'. A ternary field is a structure with one ternary function $f(x, y, z)$ such that f has the properties that $f(x, y, z) = xy + z$ would have if the right hand side were interpreted in a field. In dealing with Euclidean geometry here, we assume the axioms of congruence and the parallel postulate; this implies that the ternary field is actually a field. But these geometric hypotheses are necessary. In [Baldwin 1994], I constructed an $\aleph_1$-categorical projective plane where the ternary field is as wild as possible (in the precise sense of the Lenz-Barlotti classification in [Yaqub 1967]: the ternary function cannot be decomposed into an addition and a multiplication).

3.2 Ratio, magnitude, and number

In this section we give a short review of Greek attitudes toward magnitude and ratio. Fuller accounts of the transition to modern attitudes appear in such sources as [Mueller 2006, Euclid 1956, Stein 1990, Menn 2017]. We by no means follow the 'geometric algebra' interpretation decried in [Grattan-Guinness 2009]. Rather, we attempt

[31] Smorynski [Smorynski 2008] notes that Bradwardine already reported five in the 14th century. Reeder [Reeder 2018] analyzes five different philosophical approaches to the notion of 'infinite divisibility.

[32] That is, any system of points and lines such that two points determine a line, any two lines intersect in a point, and there are 4 non-collinear points.

to contrast the Greek meanings of propositions with Hilbert's understanding. When we rephrase a sentence in algebraic notation we try to make clear that this is a modern formulation and often does not express the intent of Euclid.

Euclid develops arithmetic in Books VII-IX. What we think of as the 'number one', was 'the unit'; a number (Definition VII.2) was a multitude of units. These are counting numbers. So from our standpoint (considering the unit as the number 1) Euclid's numbers (in the arithmetic) can be thought of as the 'natural numbers'. The numbers[33] are a discretely ordered collection of objects.

Following Mueller we work from the interpretation of magnitudes in the Elements as 'abstractions from geometric objects which leave out of account all properties of those objects except quantity: length for lines, area of plane figures, volume of solid figures etc.' ([Mueller 2006], page 21) Mueller emphasizes the distinction between the properties of proportions of magnitudes developed in Book V and those of number in Book VII. The most easily stated is implicit in Euclid's proof of Theorem V.5; for every m, every magnitude can be divided in m equal parts. This is of course, false for the (natural) numbers.

There is a second use of 'number' in Euclid. It is possible to count unit magnitudes, to speak of, e.g. four copies of a unit magnitude. So (in modern language) Euclid speaks of multiples of magnitudes by positive integers.

Magnitudes of the same type are also linearly ordered and between any two there is a third[34]. Multiplication of line segments yields rectangles. Ratios are not objects; equality of ratios is a 4-ary relation between two pairs of homogenous magnitudes[35]. Some key points from Euclid's discussion of proportion in Book V are.

1. Definition V.4 of Euclid [Euclid 1956]: Magnitudes are said to *have a ratio* to one another, which are capable, when multiplied, of exceeding one another.
2. Definition V.5 'sameness of two ratios' (in modern terminology): The ratio of two magnitudes x and y are proportional to the ratio of two others z, w if for each m, n, $mx > ny$ implies $mz > nw$ (and also replacing $>$ by $=$ or $<$).
3. Definition V.6: Let magnitudes which have the same ratio be called proportional.
4. Proposition V.9: 'same ratio' is, in modern terminology, a transitive relation. Apparently Euclid took symmetry and reflexivity for granted and treats proportionality as an equivalence relation.

[33] More precisely, for the Greeks, natural numbers greater than 1.

[34] The Greeks accepted only potential infinity. From a modern perspective, the natural numbers are ordered in order type ω, and any collection of homogeneous magnitudes (e.g. areas) are in a dense linear order (which is necessarily infinite); however, this imposition of completed infinity is not the understanding of the Greeks.

[35] Homogeneous pairs means magnitudes of the same type. Ratios of numbers are described in Book VII while ratios of magnitudes are discussed in Book V.

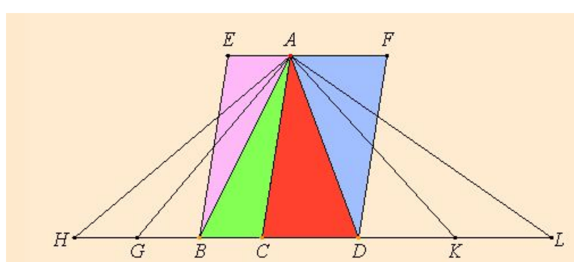
3.3 Bolzano's Challenge

In a discussion of the foundations of geometry Bolzano discusses the 'dissimilar objects' found in Euclid and finds Euclid's approach fundamentally flawed. His general position is that one must analyze conceptually prior notions (line) before more complex notions (plane). See ([Bolzano & Russ 2004], 33) and [Rusnock 2000] 53) for discussions of Bolzano's claim that the study of the line, as 'more fundamental' must precede that of the plane. More specifically, he objects to Euclid's proof in Book VI.2 that similar triangles have proportional sides:

> Firstly triangles, that are already accompanied by circles which intersect in certain points, then angles, adjacent and vertically opposite angles, then the equality of triangles, and only much later their similarity, which however, is derived by an atrocious detour [ungeheuern Umweg], from the consideration of parallel lines, and even of the area of triangles, etc[36]. ([Bolzano 1810])

What we call *Bolzano's challenge* has two aspects: a) the evil of using two dimensional concepts to understand the line and b) the 'atrocious detour' to similarity (VI.2). We consider the plane essential to understanding the *geometric continuum*; so, a) is irrevant to our project; sections 4.1-4.3 sketch how Hilbert meets the challenge by avoiding the detour.

The side-splitter theorem, VI.2. is taken to represent similarity. Euclid deduces VI.2 from VI.1. In his proof of VI.1 (*) Euclid applies Definition V.5 (above) to deduce that the area of two triangle with the same height is proportional to their bases. But this assumes that any two lengths (or any two areas) have a ratio in the sense of Definition V.4. This is an implicit assertion of Archimedes axiom for both area and length[37].



[36]This quotation is taken from [Franks 2014].

[37]Euclid's development of the theory of proportion and area requires the Archimedean axioms. Our assertion is one of many descriptions of the exact form and location of the dependence among such authors as [Euclid 1956, Mueller 2006, Stein 1990, Fowler 1979, Smorynski 2008]. Since our use of Euclid is as a source of sentences, not proofs, the details are not essential to our argument.

If, for example, BC, GB and HG are congruent segments then the area of ACH is triple that of ABC. But without assuming BC and BD are commensurable, Euclid calls on Definition V.5 to assert that $ABD : ABC :: BD : BC$. In VI.2, he uses these results to show that similar triangles have proportional sides. From VI.2, Euclid constructs in VI.12 the fourth proportional to three lines but does not regard it as a definition of multiplication of segments. As we will see in more detail, Hilbert's treatment of area and similarity has no such dependence on Archimedes axiom. By interpreting the field with segment arithmetic, he defines proportionality directly.

In contrast, Descartes defines the multiplication of line segments to give another segment[38], but he *still relies* on Euclid's theory of proportion to justify the multiplication. Hilbert's innovation is use to segment multiplication to gain the notion of proportionality, which is defined in Subsection 4.2.

3.4 From Arithmetic to geometry or from geometry to algebra?

On the first page of *Continuity and the Irrational Numbers*, Dedekind wrote:

> Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful from the didactic standpoint ... But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. [Dedekind 1963]

I do not contest Dedekind's claim. I quote this passage to indicate that Dedekind's motivation was to provide a basis for analysis, not geometry. But I will argue that the second-order Dedekind completeness axiom is not needed for the geometry of Euclid and indeed for the grounding of the algebraic numbers, although it is in Dedekind's approach.

Dedekind provides a theory of the continuum (the continuous) line by building up in stages from the structure that is fundamental to him: the natural numbers under successor. This development draws on second-order logic in several places. The well-ordering of the natural numbers is required to define addition and multiplication by recursion. Dedekind completeness is a second appeal to a second-order principle. Perhaps in response to Bolzano's insistence, Dedekind constructs the line without recourse to two dimensional objects and from arithmetic. Thus, he succeeds in the 'arithmetization of analysis'.

We proceed in the opposite direction for several reasons. Most important is that we are seeking to ground geometry, not analysis. Further, we adopt as a principle that the concept of line arises only in the perception of at least two dimensional space.

[38]He refers to the construction of the fourth proportional ('ce qui est meme que la multiplication' [Descartes 1637]). See also Section 21 page 296 of [Bos 2001].

Dedekind's continuum knows nothing of being straight or breadthless. Hilbert's proof of the existence of the field is the essence of the *geometric continuum*. By virtue of its lying in a plane, the line acquires algebraic properties.

Moreover, the distinction between the arithmetic and geometric intuitions of multiplication is fundamental. The basis of the first is iterated addition; the basis of the second is scaling or proportionality. The late 19th century developments provide a formal reduction of the second to the first but the reduction is only formal; the 'scaling' intuition is lost. In this paper we view both intuitions as fundamental and develop the second (Section 4.1), with the understanding that development of the first through the Dedekind-Peano treatment of arithmetic is in the background.

4 Axiomatizing the geometry of polygons and circles

Section 4.1 sketches the key step in making an independent foundation of geometry: Hilbert's definition of a field in a geometry. Section 4.2 describes the transition from segments to points as the domain of that field and gives Hilbert's definition of proportional. Section 4.3 distinguishes the role of the CCP and analyzes several problems that can be approached by limits but have uniform solutions in any ordered field; order completeness of the field is irrelevant. We return to Bolzano's challenge and derive Theorem 4.3.4, the properties of similar triangles and, in Section 4.4, the area of polygons.

4.1 From geometry to segment arithmetic to numbers

One of Hilbert's key innovations is his *segment arithmetic* and his definition of the semi-field[39] of segments with partial subtraction and multiplication. We assume the axiom system we called HP5 in Notation 2.2. The details can be found in e.g. [Hilbert 1971, Hartshorne 2000, Baldwin 2017b, Giovannini 2016].

Note that congruence forms an equivalence relation on line segments. Fix a ray ℓ with one end point 0 on ℓ. For each equivalence class of segments, we consider the unique segment $0A$ on ℓ in that class as the representative of that class. We will often denote the segment $0A$ (ambiguously its congruence class) by a. We say a segment CD (on any line) has length a if $CD \cong 0A$. Following Hartshorne [Hartshorne 2000], here is our *official definition* of segment multiplication[40].

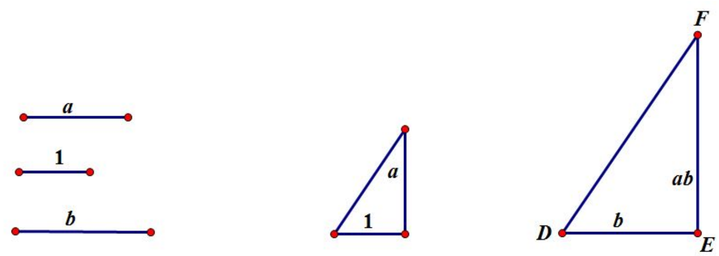
Fix a unit segment class, 1. Consider two segment classes a and b. To determine their product, define a right triangle with legs of length 1 and a. Denote the angle between

[39] In a semi-field there is no requirement of an additive inverse.

[40] Hilbert's definition goes directly via similar triangles. The clear association of a particular angle with right multiplication by a recommends Hartshorne's version.

the hypoteneuse and the side of length 1 by α.

Now construct another right triangle with base of length b with the angle between the hypoteneuse and the side of length 1 congruent to α. The product ab is defined to be the length of the vertical leg of the triangle.



Note that we must appeal to the parallel postulate to guarantee the existence of the point F. It is clear from the definition that there are multiplicative inverses; use the triangle with base a and height 1. Hartshorne has a roughly three page proof[41] that shows multiplication is commutative, associative, distributes over addition, and respects the order. It uses only the cyclic quadrilateral theorem and connections between central and inscribed angles in a circle.

To summarize the effect of the axiom sets, we introduce two definitions.

1. An ordered field F is *Pythagorean* if it is closed under addition, subtraction, multiplication, division and for every $a \in F$, $\sqrt{(1 + a^2)} \in F$.
2. An ordered field F is *Euclidean* if it is closed under addition, subtraction, multiplication, division and for every positive $a \in F$, $\sqrt{a} \in F$.

When the model is taken as geometry over the reals, it is easy[42] to check that the multiplication defined on the positive reals by this procedure is exactly the usual multiplication on the positive reals because they agree on the positive rational numbers.

As in section 21 of [Hartshorne 2000], we have:

Theorem 4.1.1. *1. HP5 is bi-interpretable with the theory of ordered pythagorean planes.*

[41] See [Baldwin & Mueller 2012] or ([Hartshorne 2000],170), which explain the background cyclic quadrilateral theorem. That theorem asserts: Let $ACED$ be a quadrilateral. The vertices of $ACED$ lie on a circle (the ordering of the name of the quadrilateral implies A and E are on opposite sides of CD) if and only if $\angle EAC \cong \angle CDE$. Hilbert uses Pascal's theorem, a relative of Desargues' theorem.

[42] One has to verify that segment multiplication is continuous but this follows from the density of the order since the addition respects order.

2. *Similarly EG is bi-interpretable with the theory of ordered Euclidean planes.*

Formally bi-interpretability means there are formulas in the field language defining the geometric notions (point, line, congruence, etc) and formulas in the geometric language (plus constants) defining the field operations $(0, 1, +, \times)$ such that interpreting the geometric formulas in a Pythagorean field gives a model of HP5 and conversely. See chapter 5 of [Hodges 1993] for general background on interpretability.

With this information we can explain why Proposition I.1 (equilateral triangle) is in Euclid I rather than II. Both it and I.22 (construct a triangle given three lines with the sum of the length of two greater than the length of the third) use circle-circle intersection in Euclid. However, Hilbert proves the first in HP5, but the second requires the field to be Euclidean and so uses CCP.

Dicta on Constants: Note that to fix the field we had to add constants 0, 1. These constants can name any pair of points in the plane[43]. But this naming induces an extension of the data set. We have in fact specified the unit. This specification has little effect on the data set but a major change in view from either the Greeks or Descartes.

4.2 Points and Numbers, Multiplication and Proportionality

Hilbert shows the multiplication on segments of a line through points $0, 1$ satisfies the semi-field axioms. He defines segment multiplication on the ray from 0 through 1 as in Section 4.1. But to get negative numbers one must reflect through 0. Then addition and multiplication can be defined on directed segments of the line through $0, 1$[44] and thus all axioms for a field are obtained. The next step is to identity the points on the line and the domain of an ordered field by mapping A to OA. This naturally leads to thinking of a segment as a set of points, which is foreign to both Euclid and Descartes. Although in the context of the *Grundlagen*, Hilbert's goal is to coordinatize the plane by the real numbers; his methods open the path to thinking of the members of any field as 'numbers' that coordinatize the associated geometries. Boyer traces the origins of numerical coordinates to 1827-1829 and writes,

> It is sometimes said that Descartes arithmetized geometry but this is not strictly correct. For almost two hundred years after his time coordinates were essentially geometric. Cartesian coordinates were line segments . . . The arithmetization of coordinates took place not in 1637 but in the crucial years 1827-1829. ([Boyer 1956], 242)

[43] The automorphism group of the plane acts 2-transitively on the plane (any pair of distinct points can be mapped by an automorphism to any other such pair); this can be proven in HP5. This transitivity implies that a sentence $\phi(0, 1)$ holds just if either or both of $\forall x \forall y \phi(x, y)$ and $\exists x \exists y \phi(x, y)$ hold.

[44] Hilbert had done this in lecture notes in 1894 [Hallett & Majer 2004]. Hartshorne constructs the field algebraically from the semifield rather than in the geometry.

Boyer points to Bobillier, Möbius, Feurbach and most critically Plücker as introducing several variants of what constitute numerical (signed distance) barycentric coordinates of a point.

Multiplication is not repeated addition: We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a. It is an easy exercise to show these give the same answer. But these distinct constructions make an important point. The (inductive) definition of multiplication by a natural number is indeed 'multiplication as repeated addition'. But the multiplication by another field element is based on similarity and has multiplicative inverses; so from a modern standpoint they yield very different structures: no extension of natural number arithmetic is decidable but important theories of fields are.

The first notion of multiplication in the last paragraph, where the multiplier is a natural number, is a kind of *scalar multiplication* by positive integers that can be viewed mathematically as a rarely studied object: a semiring (the natural numbers) acting on a semigroup (positive reals under addition). There is no uniform definition[45] of this binary operation of *scalar* multiplication within the semiring action.

A mathematical structure more familiar to modern eyes is obtained by adding the negative numbers to get the ring $\mathbb{Z}$, which has a well-defined notion of subtraction. The scalars are now in the ring $(Z, +, \cdot)$ and act on the module $(\Re, +)$. Now we can multiply by $-\frac{17}{27}$ but the operations is still not uniform but given by a family of unary functions.

Looking more closely, we have three different referents for *segment multiplication*: Euclid's computation of the area of rectangle, his construction of the fourth proportional, and Hilbert's definition. These three computations give the 'same answer'. We now consider the fourth proportional in $1\colon a :: b\colon x$ to measure the area of the rectangle with sides of length a and b. In view of this we do not view Hilbert as introducing a new concept of multiplication - but as reinterpreting the notion and indeed the same geometric construction as applying to line segments, *which we now read as numbers*, rather than mapping from linear magnitudes to planar magnitudes. With this in hand, Hilbert redefines proportionality:

Proportionality We write the ratio of CD to CA is proportional to that of CE to CB,

$$CD : CA :: CE : CB$$

which is defined as

$$CD \times CB = CE \times CA.$$

where $\times$ is taken in the sense of segment multiplication as defined in Section 4.1.

While in Book V Euclid provides a *general* account of proportionality,

[45] Instead, there are infinitely many formulas $\phi_n(x, y)$ defining unary operations $nx = y$ for each $n > 0$.

Hilbert's ability to avoid the Archimedean axiom depends both on the geometrical construction of the field and the reinterpretation of 'number'.

4.3 Field arithmetic and basic geometry

In this section we investigate some statements from: 1) Euclid's geometry that depended in his development on the Archimedean Axiom and some from 2) Dedekind's development of the properties of real numbers that he deduces from his postulate. In each case, they are true in any field associated with a geometry modeling HP5.

We established in Section 4.1 that one could define an ordered field F in any plane satisfying HP5. The converse is routine, the ordinary notions of line and incidence in F^2 creates a geometry over any Pythagorean ordered field, which is easily seen to satisfy HP5. We now exploit this equivalence to show some important algebraic facts using our defined operations, thus basing them on geometry. First, note that taking square root commutes with multiplication for algebraic numbers. Dedekind ([Dedekind 1963], 22) wrote ' ...in this way we arrive at real proofs of theorems (as, e.g. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before.'

Note that this commutativity is a problem for Dedekind but not for Descartes. Euclid had already, in constructing the fourth proportional, constructed from segments of length 1, a and b, one of length ab; but he doesn't regard this operation as multiplication. When Descartes interprets this procedure as multiplication of segments, the reasoning above shows multiplying square roots is not an issue. But Dedekind has presented the problem as multiplication in his continuum and so he must prove a theorem which allows us to find the product as a real number; that is, he must show the limit operation commutes with product.

But, in an ordered field, for any positive a, if there is an element $b > 0$ with $b^2 = a$, then b is unique (and denoted $\sqrt{a}$). Moreover, for any positive a, c with square roots, $\sqrt{a} \cdot \sqrt{c} = \sqrt{ac}$, since each side of the equality squares to ac. This fact holds for any field coordinatizing a plane satisfying HP5.

Thus, the algebra of square roots in the real field is established without any appeal to limits. The usual (e.g. [Spivak 1980, Apostol 1967]) developments of the theory of complete ordered fields follow Dedekind and invoke the least upper bound principle to obtain the existence of the roots although the multiplication rule is obtained by the same algebraic argument as here. Hilbert's approach contrasts with Dedekind's[46]. The justification here for either the existence of, or operations on, roots does not invoke limits. Hilbert's treatment is based on the geometric concepts and in particular regards 'congruence' as an equally fundamental notion as 'number'.

[46]Dedekind objects to the introduction of irrational numbers by measuring an extensive magnitude in terms of another of the same kind (page 9 of [Dedekind 1963]).

In short, the shift here is from 'proportional segments' to 'product of numbers'. Euclid had a rigorous proof of the existence of a line segment which is the fourth proportional of $1 : a = b : x$. Dedekind demands a product of numbers; Hilbert provides this by a combination of his interpretation of the field in the geometry and the geometrical definition of multiplication.

Euclid's proof of Pythagoras' theorem I.47 uses the properties of area that we will justify in Section 4.4. His second proof (Lemma for X.33) uses the property of similar triangles that we prove in Theorem 4.3.4. In both cases Euclid depends on the theory of proportionality (and thus implicitly on Archimedes' axiom) to prove the Pythagorean theorem; Hilbert avoids this appeal[47]. Similarly, since the right-angle trigonometry in Euclid concerns the ratios of sides of triangles, the field multiplication justifies basic right-angle trigonometry. We have:

Theorem 4.3.1. *The Pythagorean theorem as well as the law of cosines (Euclid II.12, II.13) and the law of sines ([Maor 1993], 216-17) hold in HP5.*

Hartshorne [Hartshorne 2000] describes two instructive examples, connecting the notions of Pythagorean and Euclidean planes.

Example 4.3.2. 1. The Cartesian plane over a Pythagorean field may fail to be closed under square root[48].

2. On page 146, Hartshorne[49] observes that the smallest ordered field closed under addition, subtraction, multiplication, division and square roots of positive numbers and satisfying the CCP is a Euclidean field, denoted by F_s for *surd field*.

Note that if HP5 + CCP were proposed as an axiom set for polygonal geometry it would be a complete descriptive but not modest axiomatization since it would prove CCP which is not in the polygonal geometry data set.

In a Euclidean plane every positive element of the coordinatizing plane has a square root, so Heron's formula ($A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is $1/2$ the perimeter and a, b, c are the side lengths) computes the area of a triangle from the lengths of its sides. This fact demonstrates the hazards of the kind of organization of data sets attempted here. The geometric proof of Heron doesn't involve the square

[47] However, Hilbert does not avoid the parallel postulate since he uses it to establish multiplication and thus similarity. Note also that Euclid's theory of area depends heavily on the parallel postulate. It is a theorem in 'neutral geometry' in the metric tradition that the Pythagorean Theorem is equivalent to the parallel postulate (See Theorem 9.2.8 of [Millman & Parker 1981].) But, this approach basically assumes the issues dealt with in these papers; as, the 'ruler postulate' (Remark 3.12 of the sequel) also provides a multiplication on the 'lengths' (since they are real numbers). Julien Narboux pointed out the issues in stating the Pythagorean theorem in the absence of the parallel postulate.

[48] See Exercises 39.30, 30.31 of [Hartshorne 2000]. This was known to Hilbert ([Hallett & Majer 2004], 201-202).

[49] Hartshorne and Greenberg [Greenberg 2010] calls this the constructible field, but given the many meanings of constructible, we use Moise's term: surd field.

roots of the modern formula[50] ([Heath 1921], 321). But since in EG we have the field and we have square roots, the modern form of Heron's formula can be proved from EG. Thus, as in the shift from (*) to (**) at the beginning of the paper, the different means of expressing the geometrical property requires different proofs.

In each case considered in this section, Greeks give geometric constructions for what in modern days becomes a calculation involving the field operations and square roots. However, we still need to complete the argument that HP5 is descriptively complete for polygonal Euclidean geometry. In particular, is our notion of proportional correct? The test question is the similar triangle theorem. We turn to this issue now.

Definition 4.3.3. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Various texts define 'similar' as we did, or focus on corresponding sides are proportional or require both (Euclid). We now meet *Bolzano's challenge* by showing that in Euclidean Geometry (without the continuity axioms) the choice doesn't matter. We defined 'proportional' in terms of segment multiplication in Section 4.2. Hartshorne proves the fundamental result (Euclid VI.2) ([Hartshorne 2000], 176-77).

Theorem 4.3.4 (EG)**.** *Two triangles are similar if and only if corresponding sides are proportional. Euclid VI.2 follows.*

There is no assumption that the field is Archimedean or satisfies any sort of completeness axiom. There is thus no appeal to approximation or limits. We have avoided Bolzano's 'atrocious detour' through area. But area is itself a vital geometric notion and that is the topic of the next section.

4.4 Area of polygonal figures

Hilbert wrote[51], "We ... establish Euclid's theory of area *for the plane geometry and that independently of the axiom of Archimedes*." We now sketch Hartshorne's [Hartshorne 2000] exposition of this topic, stressing the connections with Euclid's Common Notions. We show the notions defined here are expressible in first-order logic, which supports our fifth objection in Section 4.3 of the sequel: Although these arguments are not carried out as direct deductions from the first-order axioms, the results are derivable by a direct deduction. That is, we develop area in first order logic and even though some of the arguments are semantical the conclusions are theorems of first order logic. This is further evidence of the immodesty of the second order axiomatization.

[50] Heron's proof invokes a peculiar squaring of areas. But [Taisbak 2014] argues that this operation can be replaced by inferences from VI.1 or X.53.

[51] Emphasis in the original: (page 57 of [Hilbert 1971]).

Informally, those configurations whose areas are considered in this section are figures, where a *figure* is a subset of the plane that can be represented as a finite union of disjoint triangles. There are serious issues concerning the formalization in first order logic of such notions as figure or polygon that involve quantification over integers; such quantification is strictly forbidden within a first order system. We can approach these notions with axiom schemes[52] and sketch a uniform metatheoretic definition of the relevant concepts to prove that the theorems hold in all models of the axioms.

Hilbert raised a *pseudogap* in Euclid[53] by distinguishing area and content. In Hilbert, two figures have

1. *equal area* (*are equidecomposable*) if they can each be decomposed into a finite number of triangles that are pairwise congruent
2. *equal content* (*are equicomplementable*) if we can transform one into the other by adding and subtracting congruent triangles.

Hilbert showed: under the Archimedean Axiom the two notions are equivalent; and, without it they are not. Euclid treats the equality of areas as a special case of his Common Notions. The properties of equal content, described next, are consequences for Euclid of the Common Notions and need no justification. We introduce the notion of area function to show they hold in all models of HP5.

Fact 4.4.1 (Properties of Equal Content)**.** *The following properties of area are used in Euclid I.35 through I.38 and beyond.*

1. *Congruent figures have the same content.*
2. *The content of two 'disjoint' figures (i.e. meet only in a point or along an edge) is the sum of the contents of the two polygons. The analogous statements hold for difference and half.*
3. *If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.*

More precisely,

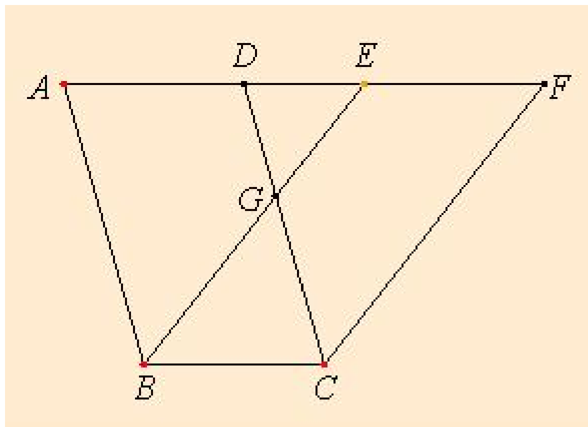
Definition 4.4.2 (Equal content)**.** *Two figures P, Q have* equal content *(are equicomplementable) in n steps if there are figures $P'_1 \ldots P'_n$, $Q'_1 \ldots Q'_n$ such that none of the figures overlap, each pair P'_i and Q'_i are equi-decomposable and $P \cup P'_1 \ldots \cup P'_n$ is equi-decomposable with $Q \cup Q'_1 \ldots \cup Q'_n$.*

[52] In order to justify the application of the completeness theorem we have to produce a scheme giving the definition of an n-decomposable figure as the disjoint union of an $(n-1)$-decomposable figure A with an appropriately placed triangle. The axioms for π in the sequel illustrate such a scheme.

[53] Any model with infinitessimals shows the notions are distinct and Euclid I.35 and I.36 (triangles on the same (congruent) base(s) and same height have the same area) fail for what Hilbert calls area. Since Euclid includes preservation under both addition and subtraction in his Common Notions, his term 'area' clearly refers to what Hilbert calls 'equal content', I call this a pseudogap.

Reading equal content for Euclid's 'equal', Euclid's I.35 (for parallelograms) and the derived I.38 (triangles) become Theorem 4.4.3 and in this formulation Hilbert accepts Euclid's proof.

Theorem 4.4.3. *[Euclid/Hilbert] If two parallelograms (triangles) are on the same base and between parallels they have equal content in 1 step.*



Proof. $P = ADCB$ has the same content as $P' = EFCB$ in one step as letting $Q = Q' = DGE$, $P + Q = P' + Q'$ which are decomposable as $CBG + ABE$ and $CBG + CDF$ and $ABE \approx DCF$. $\quad \square_{4.4.3}$

To show area is well-defined for figures, varying Hilbert, Hartshorne (Sections 19-23 of [Hartshorne 2000]) shows Fact 4.4.1 in the first order axiom system EG (Notation 2.2). The key tool is:

Definition 4.4.4. *An* area function *is a map α from the set of figures, $\mathcal{P}$, into an ordered additive abelian group with* 0 *such that*

1. *For any nontrivial triangle T, $\alpha(T) > 0$.*
2. *Congruent triangles have the same content.*
3. *If P and Q are disjoint figures $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.*

Semiformally, the idea is straightforward. Argue a) that every n-gon is triangulated into a collection of disjoint triangles and b) establish an area function on such collections. The complication for formalizing is describing the area function by first order formulas. For this[54], add to the vocabulary for geometry a 3-ary function α_1 from points into line segments that satisfies 1) and 2). For $n > 1$, add a $3n$-ary function α_n

[54]See Definition 3.2 of the sequel for a more detailed discussion in a slightly different context.

and assert that α_n maps n-disjoint triangles into the field of line segments by summing α_1 on the individual triangles; this will satisfy 3). This handles disjoint unions of triangles. Hartshorne (23.2-23.5) gives an inductive proof showing every polygon can be decomposed into a finite number of disjoint triangles and the area does not depend on the decomposition.

It is evident that if a plane admits an area function then the conclusions of Lemma 4.4.1 hold. This obviates the need for positing separately De Zolt's axiom, (Fact 4.4.1.3). In particular this implies Common Notion 4 for 'area'. As we just saw Hilbert established the existence of an area function and thus a theory of area for any plane satisfying HP5.

Now, letting $F(ABC)$ be an area function as in Definition 4.4.4 (* from the first page), VI.1 is interpreted as a variant of (**):

$$F(ABC) = \frac{1}{2}\alpha \cdot AB \cdot AC.$$

But the cost is that while Euclid does not specify what we now call the proportionality constant, Hilbert must. In (**) Hilbert assigns α to be one. A short argument ([Hartshorne 2000], 23.3) shows that the formula $A = \frac{bh}{2}$ does not depend on the choice of the base and height. Thus, Hilbert proves (**) without recourse to the axiom of Archimedes.

5 Conclusion

In this paper we defined the notion of a *modest* descriptive axiomatization to emphasize that the primary goal of an axiomatization is to distill what is 'really going on'. One can axiomatize the first-order theory of any structure by taking as axioms all the first-order sentences true in it; such a choice makes a farce of axiomatizing. Historically, we stress one of Hilbert's key points. Hilbert eliminated the use of the Axiom of Archimedes in Euclid's polygonal and circle geometry (except for area of circle), thus exhibiting a first order *modest descriptive axiomatization* of that geometry. His proof is impure [Baldwin 2013a] as he introduces the concept of field; but it remains modest; the axioms are in the data set. As we expand on in [Baldwin 2017b], extensions by *explicit first order definition* are (often unconsciously) standard mathematical tools that may impair purity but not provability. He [Hilbert 1971] was finding the 'distinguished propositions of the field of knowledge that underlie the construction of the framework of concepts' and showing what we now call first-order axioms sufficed. The sequel 1) expounds Tarski's system $\mathcal{E}^2$ as a modest descriptive axiomatization of Cartesian geometry, 2) analyzes the distinctions between the completeness axioms of Dedekind and Hilbert and argues that Hilbert's continuity axioms are overkill for strictly geometric propositions and 3) supports conclusion 2) by providing a first-order theory to justify the formulas for circumference and area of a circle.

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