Axiomatizing changing conceptions of the geometric continuuum II: Archimedes – Descartes –Tarski – Hilbert

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Abstract

In Part I of this paper we argued that the first-order systems HP5 and EG (defined below) are modest complete descriptive axiomatization of most (described more precisely below) of Euclidean geometry. In this paper we discuss two further modest complete descriptive axiomatizations: Tarksi's for Cartesian and geometry and new systems for adding π. In contrast we find Hilbert's full second order system immodest for geometrical purposes but appropriate as a foundation for mathematical analysis.

Part I [Baldwin 2017] dealt primarily with Hilbert's first order axioms for geometry; Part II deals with his 'continuity axioms' – the Archimedean and completeness axioms. Part I argued that the first-order systems HP5 and EG (defined below) are 'modest' complete descriptive axiomatization of most (described more precisely below) of Euclidean geometry. In this paper we consider some extensions of Tarski's axioms for elementary geometry to deal with circles. In this paper we argue: 1) that Tarski's first-order axiom set $\mathcal{E}^2$ is a modest complete descriptive axiomatization of Cartesian geometry; 2) that the theories $EG_{\pi,C,A}$ and $\mathcal{E}^2_{\pi,C,A}$ are modest complete descriptive axiomatizations of the extensions of these geometries obtained to describe area and circumference of the circle; and 3) that, in contrast, Hilbert's full second-order system in the *Grundlagen* is an immodest axiomatization of any of these geometries but a modest descriptive axiomatization the late 19th century conception of the real plane.

We recall some of the key material and notation from Part I. The paper involved two key elements. The first was the following quasi-historical description. Eu-

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clid founded his theory of area (of circles and polygons) on Eudoxus' theory of proportion and thus (implicitly) on the axiom of Archimedes. Hilbert showed any plane satisfying his axioms HP5 (below) interprets a field and recovered Euclid's theory for polygons in a first-order theory. The Greeks and Descartes dealt only with geometric objects. The Greeks regarded multiplication as an operation from line segments to plane figures. Descartes interpreted it as an operation from line segments to line segments. In the late 19th century, multiplication became an operation on points (that is 'numbers' in the coordinatizing field).

Secondly, we built on Detlefsen's notion of complete descriptive axiomatization and defined a *modest complete descriptive axiomatization of a data set* Σ (essentially, of facts in the sense of Hilbert) to be a collection of sentences that imply all the sentences in Σ and 'not too many more'. Of course, this set of facts will be open-ended, since over time more results will be proved. But if this set of axioms introduces essentially new concepts to the area and certainly if it contradicts the understanding of the original era, we deem the axiomatization immodest. We clarify these definitions in terms of specific axiomatizations of various areas of geometry that we now describe.

1 Terminology and Notations

We begin by distinguishing several topics in plane geometry[1] that represent distinct data sets in Detlefsen's sense. In cases where *certain axioms are explicit, they are included in the data set*. Each set includes its predecessors.

Euclid I, polygonal geometry: Book I (except I.22), Book II.1-II.13, Book III (except III.1 and III.17), Book VI.)

Euclid II, circle geometry: I.22, II.14, III.1, III.17 and Book IV.

Archimedes, arc length and π: XII.2, Book IV (area of circle proportional to square of the diameter), approximation of π, circumference of circle proportional to radius, Archimedes' axiom.

Descartes, higher degree polynomials: nth roots; coordinate geometry

Hilbert: The Dedekind plane

Our division of the data sets is somewhat arbitrary and is made with the subsequent axiomatizations in mind. We explain placing the Axiom of Archimedes in the Archimedes data set in discussing Hilbert's analysis of the relation between axiom

[1]In the first instance we draw from Euclid: Books I-IV, VI and XII.1, 2 clearly concern plane geometry; XI, the rest of XII and XIII deal with solid geometry; V and X deal with a general notion proportion and with incommensurability. Thus, below we put each proposition Books I-IV, VI, XII.1,2 in a group and consider certain geometrical aspects of Books V and X.

groups in Sections 3 and 4. Further, we distinguish the Cartesian data set, in Descartes's historical sense, from Hilbert's identification of Cartesian geometry with the Dedekind line and explain the reason for that distinction in Section 3.

In Part I, we formulated our formal system in a two-sorted vocabulary τ chosen to make the Euclidean axioms (either as in Euclid or Hilbert) easily translatable into first-order logic. This vocabulary includes unary predicates for points and lines, a binary incidence relation, a ternary collinearity relation, a quaternary relation for line congruence and a 6-ary relation for angle congruence. The *circle-circle intersection postulate* asserts if the interiors of two circles (neither contained in the other) have a common point, the circles intersect in two points.

The following axiom sets[2] are defined to organize these data sets.

1. first-order axioms

   HP, HP5: We write HP for Hilbert's incidence, betweenness[3], and congruence axioms. We write HP5 for HP plus the parallel postulate. A *Pythagorean field* is any field associated[4] with a model of HP5; such fields are characterized by closure under $\sqrt{(1+a^2)}$.

   EG: The *axioms for Euclidean geometry*, denoted EG[5], consist of HP5 and in addition the circle-circle intersection postulate. A *Euclidean plane* is a model of EG; the associated *Euclidean field* is closed under $\sqrt{a}$ $(a > 0)$.

   $\mathcal{E}^2$**:** Tarski's axiom system [Tarski 1959] for a plane over a real closed field (RCF[6]).

   EG_π **and** $\mathcal{E}_\pi$**:** Two new systems extend EG and $\mathcal{E}^2$.

2. Hilbert's continuity axioms, infinitary and second-order

   AA: The sentence in $L_{\omega_1,\omega}$ expressing the Archimedean axiom.

   Dedekind: Dedekind's second-order axiom that there is a point in each irrational cut in the line.

Notation 1.1. Closing a plane under ruler and compass constructions corresponds to closing the coordinatizing ordered field under square roots of positive numbers to give a Euclidean field[7]. As in Example 4.2.2.2 of Part I , F_s (surd field) denotes the minimal

[2]The names HP, HP5, and EG come from [Hartshorne 2000] and $\mathcal{E}^2$ from [Tarski 1959].

[3]These include Pasch's axiom (B4 of [Hartshorne 2000]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[4]The field F is *associated* with a plane Π if Π is the Cartesian plane on F^2.

[5]In the vocabulary here, there is a natural translation of Euclid's axioms into first-order statements. The construction axioms have to be viewed as 'for all– there exist sentences. The axiom of Archimedes is of course not first-order. We write Euclid's axioms for those in the original [Euclid 1956] vrs (first-order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [Avigad et al. 2009], namely, planes over fields where every positive element as a square root). The latter system builds the use of diagrams into the proof rules.

[6]RCF abbreviates 'real closed field'; these are the ordered fields such that every positive element has a square root and every odd degree polynomial has at least one root.

[7]We call this process 'taking the Euclidean closure' or adding *constructible* numbers.

field whose geometry is closed under ruler and compass construction. Having named $0, 1$, each member in F_s is definable over the emptyset[8]

We referred to [Hartshorne 2000] to assert in Part I the sentences of Euclid I are provable in HP5 and the additional sentences of Euclid II are provable in EG. Here we consider the data sets of Archimedes, Descartes, and Dedekind and argue for the following claim.

1. Tarski's axioms $\mathcal{E}^2$ are a modest descriptive axiomatization of the Cartesian data set.
2. EG^2_π ($\mathcal{E}_\pi$) are a modest descriptive axiomatization of Euclidean Geometry (Cartesian geometry) extended by the Archimedean data set.
3. Hilbert's axioms groups I-V give a modest descriptive axiomatization of the second-order geometrical statements concerning the plane $\Re^2$ but the system is immodest for even the Cartesian data set.

2 From Descartes to Tarski

Descartes and Archimedes represent distinct and indeed orthogonal directions for making the geometric continuum precise. These directions can be distinguished as follows. Archimedes goes directly to transcendental numbers while Descartes investigates curves defined by polynomials. Of course, neither thought in these terms, although Descartes' resistance to squaring the circle shows his implicit awareness of the problem. We deviate from chronological order and discuss Descartes before Archimedes; as, in Section 3 we will extend both Euclidean and Cartesian geometry by adding π.

As was highlighted above in describing the data sets, the most important aspects of the Cartesian data set are: 1) the explicit definition ([Descartes 1954], 1) of the multiplication of line segments to give a line segment, which breaks with Greek tradition[9]; and 2) on the same page to announce constructions for the extraction[10] of nth roots for all n. Marco Panza formulates a key observation about the ontological importance of these innovations

> The first point concerns what I mean by 'Euclid's geometry'. This is the theory expounded in the first six books of the *Elements* and in the *Data*. To be more precise, I call it 'Euclid's plane geometry', or EPG, for short. It is

[8]That is for each point a constructible by ruler and compass there is a formula $\phi_a(x)$ such that $EG \vdash (\exists!1x)\phi(x)$. in EG. That is, there is a unique solution to ϕ.

[9]His proof is still based on Eudoxus.

[10]This extraction cannot be done in EG, since EG is satisfied in the field which has solutions for all quadratic equations but not those of odd degree. See section 12 of [Hartshorne 2000].

> not a formal theory in the modern sense, and, a fortiori, it is not, then, a deductive closure of a set of axioms. Hence, it is not a closed system, in the modern logical sense of this term. Still, it is no[11] more a simple collection of results, nor a mere general insight. It is rather a well-framed system, endowed with a codified language, some basic assumptions, and relatively precise deductive rules. And this system is also closed, in another sense ([Julien 1964] 311-312), since it has sharp-cut limits fixed by its language, its basic assumptions, and its deductive rules. In what follows, especially in section 1, I shall better account for some of these limits, namely for those relative to its ontology. More specifically, I shall describe this ontology as being composed of objects available within this system, rather than objects which are required or purported to exist by force of the assumptions that this system is based on and of the results proved within it. This makes EPG radically different from modern mathematical theories (both formal and informal). One of my claims is that Descartes geometry partially reflects this feature of EPG. ([Panza 2011], 43)

In our context we interpret Panza's 'composed of objects available within this system' model theoretically as the existence of certain starting points and the closure of each model of the system under admitted constructions.

We take Panza's 'open' system to include Descartes' 'linked constructions'[12] which greatly extend the ruler and compass constructions that are licensed by EG. Descartes endorses such 'mechanical' constructions as those used in the duplication of the cubic as geometric. According to Molland ([Molland 1976], 38) "Descartes held the possibility of representing a curve by an equation (specification by property)" to be equivalent to its "being constructible in terms of the determinate motion criterion (specification by genesis)". But as Crippa points out ([Crippa 2014a], 153), Descartes did not prove this equivalence and there is some controversy as to whether the 1876 work of Kempe solves the precise problem. Descartes rejects as non-geometric any method for quadrature of the circle.

Descartes' proposal to organize geometry via the degree of polynomials ([Descartes 1954], 48) is reflected in the modern field of 'real' algebraic geometry, i.e., the study of polynomial equalities and inequalities in the theory of real closed ordered fields. To justify this geometry we adapt Tarski's 'elementary geometry'. This move makes a significant conceptual step away from Descartes whose constructions were on segments and who did not regard a line as a set of points, while Tarski's axiom are given entirely formally in a one-sorted language of relations on points. In our modern understanding of an axiom set the translation is routine, but anachronistic; Tarski [Tarski 1959] gives a fully-formalize theory for elementary geometry and proves it is complete. We will describe the theory using the following equivalent and much more understandable set of axioms.

[11] There appears to be an error here. Probably 'more a' should be deleted. jb

[12] The types of constructions allowed are analyzed in detail in Section 1.2 of [Panza 2011] and the distinctions with the Cartesian view in Section 3. See also [Bos 2001].

Tarski's elementary geometry The theory $\mathcal{E}^2$ is axiomatized by the following set of axioms in our vocabulary.

1. Euclidean plane geometry[13] (HP5);
2. Either of the following two sets of axioms which are equivalent over HP5:
   (a) An infinite set of axioms declaring the field is formally real and that every polynomial of odd-degree has a root.
   (b) The axiom schema of continuity described just below.

Tarski's system differs from Descartes in several ways. First, Tarski prescribes a ternary relation on points, thus making explicit that a line is viewed as a set of points[14]. Secondly, with Tarski's model theory we can specify a minimal model, the plane over the *real algebraic numbers*[15] of the theory, one that contains exactly (as we now understand) the objects Descartes viewed as solutions of those problems that it was 'possible to solve' (Chapter 6 of [Crippa 2014b]).

Tarski observed that Dedekind's axiom has an alternative formulation. Require that for any two sets A and B, if beyond some point a all elements of A are below all elements of B, there there is a point b which is (beyond a) above all of A and below all of B. Tarski [Givant & Tarski 1999] imposes a first-order version of this by an *Axiom Schema of Continuity*:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(axy)] \rightarrow (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$$

where α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x. Recalling that $B(x, z, y)$ represents 'z is between[16] x and y', the hypothesis asserts the solutions of the formulas α and β behave as the A, B above. This schema allows the solution of odd degree polynomials. By the completeness of real closed fields, this theory is also complete[17].

In Detlefsen's terminology Tarski has laid out a *Gödel complete* axiomatization, that is, the consequences of his axioms are a complete first-order theory of (in our terminology) Cartesian plane geometry. This completeness guarantees that if we keep the vocabulary and continue to accept the same data set no axiomatization[18] can account for more of the data. There are certainly open problems in plane geometry

[13]Note that circle-circle intersection is implied by the axioms in 2).

[14]Writing in 1832, Bolyai ([Gray 2004], appendix) wrote in his 'explanation of signs', 'The straight AB means the aggregate of all points situated in the same straight line with A and B.' This is the earliest indication I know of the transition to an extensional version of incidence. William Howard showed me this passage.

[15]That is, a real number that satisfies a polynomial with rational coefficients. A real number that satisfies no such polynomial is called *transcendental*.

[16]More precisely in terms of the linear order $B(xyz)$ means $x \leq y \leq z$.

[17]Tarski [Tarski 1959] proves that planes over real closed fields are exactly the models of his elementary geometry, $\mathcal{E}^2$.

[18]Of course, more perspicuous axiomatizations may be found. Or one may discover the entire subject is better viewed as an example in a more general context.

[Klee & Wagon 1991]. But however they are solved, the proof will be formalizable in $\mathcal{E}^2$. Thus, in our view, the axioms are descriptively complete.

The axioms $\mathcal{E}^2$ assert, consistently with Descartes' conceptions and theorems, the solutions of certain equations. So they provide a *modest* complete descriptive axiomatization of the Cartesian data set. In the case at hand, however, there are more specific reasons for accepting the geometry over real closed fields as 'the best' descriptive axiomatization. It is the only one which is decidable and 'constructively justifiable'.

Remark 2.1 (Undecidability and Consistency)**.** Ziegler [Ziegler 1982] has shown that every nontrivial finitely axiomatized subtheory[19] of RCF is *not decidable*. Thus both to approximate more closely the Dedekind continuum and to obtain decidability we restrict to the theory of planes over RCF and thus to Tarski's $\mathcal{E}^2$ [Givant & Tarski 1999]. The biinterpretability between RCF and the theory of all planes over real closed fields yields the decidability of $\mathcal{E}^2$ and a *finitary proof of its consistency*[20]. The crucial fact that makes decidability possible is that the natural numbers are *not first-order definable* in the real field.

As we know, the crucial contribution of Descartes to geometry, is coordinate geometry. Tarski provides a converse; his interpretation of the plane into the coordinatizing line [Tarski 1951] unifies the study of the 'geometry continuum' with axiomatizations of 'geometry'. We have used Tarski's axioms for plane geometry from [Tarski 1959]. However, they extend by a family of axioms for higher dimensions [Givant & Tarski 1999] to ground modern real algebraic geometry. This natural extension demonstrates the fecundity of Cartesian geometry. Descartes used polynomials in at most two variables. But once the field is defined, the semantic extension to spaces of arbitrary finite dimension, i.e. polynomials in any finite number of variables, is immediate. Thus, every n-space is controlled by the field so the plane geometry determines the geometry of any finite dimension. Although the Cartesian data set concerns polynomials of very few variables and arbitrary degree, all of real algebraic geometry is latent.

There are three post-Descartes innovations that we have largely neglected in these papers: a) higher dimensional geometry, b) projective geometry c) definability by analytic functions. The first is a largely nineteenth century innovation which significantly impact Descartes's analytic geometry by introducing equations in more than three variables. The second is essentially bi-interpretable. So both of these threads are more or less orthogonal to our development here which concerns the actual structure of the line (and moves more or less automatically to higher dimensional or projective geometry). We addressed c) briefly in Part I while discussing Dieudonné's definition of *analytic geometry*. In a sense, the question is anachronistic; Hilbert writes almost thirty years before Artin-Schreier isolate the notion of real closed field, thirty-five before Tarski proves the theory is complete and ninety-five before o-minimality provides

[19] A nontrivial subtheory is one satisfied by one of $\Re$, or a p-adic field Q_p.

[20] The geometric version of this result was conjectured by Tarski in [Tarski 1959]: The theory RCF is complete and recursively axiomatized so decidable. For the context of Ziegler result and Tarski's quantifier elimination in computer science see [Makowsky 2013].

a unifying scheme capturing real algebraic and some of Dieudonné's analytic geometry (e^x and the restriction of any analytic function to a compact domain).

3 Archimedes: π and the circumference and area of circles

We begin with our rationale for placing various facts in the Archimedean data set[21]. Three propositions encapsulate the issue: Euclid VI.1 (area of a triangle), Euclid XII.2 (area of a circle), and Archimedes proof that the circumference of a circle is proportional to the diameter. Hilbert showed that VI.1 is provable already in HP5 (Part I). While Euclid implicitly relies on the Archimedean axiom, Archimedes makes it explicit in a recognizably modern form. Euclid does *not* discuss the circumference of a circle. To deal with that issue, Archimedes develops his notion of arc length. By beginning to calculate approximations of π, Archimedes is moving towards the treatment of π as a number. Consequently, we distinguish VI.1 from the Archimedean axiom and the theorems on measurement of a circle, and place the latter instead in the Archimedean data set. The validation in the theories EG_π and $\mathcal{E}^2_\pi$ set out below of the formulas $A = \pi r^2$ and $C = \pi d$ answer questions of Hilbert and Dedekind not questions of Euclid though possibly Archimedes. But, we think the theory EG_π is closer to the Greek origins than to Hilbert's second-order axioms.

The geometry over a Euclidean field (every positive number has a square root) may have no straight line segment of length π, since the model over the surd field (Notation 1.1) does not contain π. We want to find a theory which proves the circumference and area formulas for circles. Our approach is to extend the theory EG so as to guarantee that there is a point in every model which behaves as π does. While for Archimedes and Euclid, sequences constructed in the study of magnitudes in the *Elements* are of geometric objects, not numbers, in a modern account, as we saw already while discussing areas of polygons in Part I, we must identify the proportionality constant and verify that it represents a point in any model of the theory[22]. Thus this goal diverges from a 'Greek' data set and indeed is orthogonal to the axiomatization of Cartesian geometry by Tarski's $\mathcal{E}^2$.

This shift in interpretation drives the rest of this section. We search first for the solution of a specific problem, finding π in the underlying field. We established in Part I that for each model of EG and any line of the model, the surd field F_s is embeddable in the field definable on that line. On this basis we can interpret the Greek theory of limits by way of cuts in the ordered surd field F_s.

[21]It is not in any sense chronological, as Archimedes attributes the method of exhaustion to Eudoxus who precedes Euclid. Post-Heath scholarship by Becker, Knorr, and Menn [Menn 2017] have identified four theories of proportion in the generations just before Euclid. [Menn 2017] led us to the three propositions.

[22]For this reason, Archimedes needs only his postulate while Hilbert would also need Dedekind's postulate to prove the circumference formula.

Euclid's third postulate, 'describe a circle with given center and radius', entails that a circle is uniquely determined by its radius and center. In contrast, Hilbert simply defines the notion of circle and proves the uniqueness. (See Lemma 11.1 of [Hartshorne 2000].) In either case we have: two segments of a circle are congruent if they cut the same central angle. As the example of geometry over the real algebraic numbers shows, there is no guarantee that there is a straight line segment whose 'length' is π. We remedy this with the following extensions, EG_π and $\mathcal{E}^2(\pi)$, of the systems EG and $\mathcal{E}^2$.

Axioms for π: Add to the vocabulary a new constant symbol π. Let i_n (c_n) be the perimeter of a regular $3 * 2^n$-gon inscribed[23] (circumscribed) in a circle of radius 1. Let $\Sigma(\pi)$ be the collection of sentences (i.e. a type[24])

$$i_n < 2\pi < c_n$$

for $n < \omega$. Now, we can define the new theories.

1. EG_π denotes the deductive closure of the following set of axioms in the vocabulary τ augmented by constant symbols $0, 1, \pi$.

   (a) the axioms EG of a Euclidean plane;

   (b) $\Sigma(\pi)$.

2. $\mathcal{E}^2(\pi)$ is formed by adding $\Sigma(\pi)$ to $\mathcal{E}^2$ and taking the deductive closure.

Second dicta on constants: Here we named a further single constant π. But the effect is very different than naming 0 and 1 (Compare the Dicta on constants just after Theorem 4.2.1 of Part I.) The new axioms specify the place of π in the ordering of the definable points of the model. So the data set is seriously extended.

Theorem 3.1. *EG_π is a consistent but not finitely axiomatizable[25] incomplete theory.*

Proof. A model of EG_π is given by closing $F_s \cup \{\pi\} \subseteq \Re$ to a Euclidean field. To see the theory is not finitely axiomatizable, for any finite subset $\Sigma_0(\pi)$ of $\Sigma(\pi)$ choose a real algebraic number p satisfying Σ_0 when p is substituted for π; close $F_s \cup \{p\} \subseteq \Re$ to a Euclidean field to get a model of $EG \cup \Sigma_0$ which is not a model of EG_π. $\square_{3.1}$

[23] I thank Craig Smorynski for pointing out that is not so obvious that that the perimeter of an inscribed n-gon is monotonic in n and reminding me that Archimedes avoided the problem by starting with a hexagon and doubling the number of sides at each step.

[24] Let $A \subset M \models T$. A type over A is a set of formulas $\phi(\mathbf{x}, \boldsymbol{a})$ where $\mathbf{x}$, ($\boldsymbol{a}$) is a finite sequence of variables (constants from A) that is consistent with T. Taking T as EG, a type over all F_s is a type over $\emptyset$ since each element is definable without parameters in EG.

[25] Ziegler ([Ziegler 1982], Remark 2.1) shows that EG is undecidable. Almost surely his proof can be modified to show the undecidability of EG_π, but I haven't done so.

Dicta on Definitions or Postulates: We now extend the ordering on segments by adding the lengths of 'bent lines' and arcs of circles to the domain. Two approaches[26] to this step are:

a) our approach to introduce an explicit but inductive definition;

b) or add a new predicate to the vocabulary and new axioms specifying its behavior. This alternative reflects in a way the trope that Hilbert's axioms are *implicit definitions*.

We can make choice a) in Definitions 3.2, 3.3 etc. is available only because we have already established a certain amount of geometric vocabulary. Crucially the definition of bent lines (and thus the perimeter of certain polygons) is not a single definition but a schema of formulas ϕ_n defining the property for each n.

Definition 3.2. *Let $n \geq 2$. By a* bent line[27] $b = X_1 \dots X_n$ *we mean a sequence of straight line segments X_iX_{i+1} such that each end point of one is the initial point of the next.*

1. *Each bent line $b = X_1 \dots X_n$ has a length $[b]$ given by the straight line segment composed of the sum of the segments of b.*
2. *An* approximant *to the arc $X_1 \dots X_n$ of a circle with center P, is a bent line satisfying:*
   (a) *$X_1, \dots X_n, Y_1, \dots Y_n$ are points such that all PX_i are congruent and each Y_i is in the exterior of the circle.*
   (b) *Each of X_1Y_1, Y_iY_{i+1}, Y_nX_n is a straight line segment.*
   (c) *X_1Y_1 is tangent to the circle at X_1; $Y_{n-1}X_n$ is tangent to the circle at X_n.*
   (d) *For $1 \leq i < n$, Y_iY_{i+1} is tangent to the circle at X_i.*

Definition 3.3. *Let $\mathcal{S}$ be the set (of equivalence classes of) straight line segments. Let $\mathcal{C}_r$ be the set (of equivalence classes under congruence) of arcs on circles of a given radius r. Now we extend the linear order on $\mathcal{S}$ to a linear order $<_r$ on $\mathcal{S} \cup \mathcal{C}_r$ as follows. For $s \in \mathcal{S}$ and $c \in \mathcal{C}_r$*

1. *The segment $s <_r c$ if and only if there is a chord XY of a circular segment $AB \in c$ such that $XY \in s$.*
2. *The segment $s >_r c$ if and only if there is an approximant $b = X_1 \dots X_n$ to c with length $[b] = s$ and with $[X_1 \dots X_n] >_r c$.*

[26]We could define $<$ on the extended domain or, in style b), we could add an $<^*$ to the vocabulary and postulate that $<^*$ extends $<$ and satisfies the properties of the definition.
[27]This is less general than Archimedes (page 2 of [Archimedes 1897]) who allows segments of arbitrary curves 'that are concave in the same direction'.

It is easy to see that this order is well-defined since each chord of an arc is shorter than the arc and the arc is shorter than any approximant to it.

Lemma 3.4 (Encoding a second approximation of π)**.** *Let I_n and C_n denote the area of the regular 3×2^n-gon inscribed or circumscribing the unit circle.*

$$I_n < \pi < C_n$$

for $n < \omega$. Then EG_π proves[28] *each of these sentences is satisfied by π.*

Proof. The (I_n, C_n) define the cut for π in the surd field F_s reals and the (i_n, c_n) define the cut for 2π and it is a fact (i.e. for every natural number t, there exists an N_t such that if $k, \ell, m, n \geq N$ the distances between any pair of i_k, c_ℓ, I_m, I_n is less than $1/t$.) about the surd field that these are the same cut. $\square_{3.4}$

To argue that π, as implicitly defined by the theory EG_π, serves its geometric purpose, we add new unary function symbols C and A mapping our fixed line to itself and satisfying a scheme asserting that these functions do in fact produce the required limits. The definitions are identical except for the switch from the area to the perimeter of the approximating polygons. This strategy is analogous to that in an introductory calculus course of describing the properties of area and proving that the integral satisfies them.

Definition 3.5. *A unary function $C(r)$ (($A(r)$) mapping $\mathcal{S}$, the set of equivalence classes (under congruence) of straight line segments, into itself that satisfies the conditions below is called* a circumference function *(*area function).

1. $C(r)$ ($A(r)$) is less than the perimeter (area) of a regular 3×2^n-gon circumscribing circle of radius r.
2. $C(r)$ ($A(r)$) is greater than the perimeter (area) of a regular 3×2^n-gon inscribed in a circle of radius r.

We extend EG_π to include definitions of $C(r)$ and $A(r)$.

Definition 3.6. *1. The theory $EG_{\pi,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory EG_π obtained by the explicit definition $A(r) = \pi r^2$.*

2. The theory $EG_{\pi,A,C}$ is the extension of the $\tau \cup \{0, 1, \pi, A\}$-theory $EG_{\pi,A}$, obtained by the explicit definition $C(r) = 2\pi r$.

In any model of $EG_{\pi,A,C}$ for each r there is an $s \in \mathcal{S}$ whose length[29] $C(r) = 2\pi r$ is less than the perimeters of all circumscribed polygons and greater than those of

[28] Note that we have not attempted to justify the convergence of the i_n, c_n, I_n, C_n in the formal system EG_π. We are relying on mathematical proof, not logical deduction; see item 5 in Section 4.3 for elaboration.

[29] A similar argument works for area and $A(r)$.

the inscribed polygons. We can verify that by choosing n large enough we can make i_n and c_n as close together as we like (more precisely, for given m differ by $< 1/m$). In phrasing this sentence I follow Heath's description[30] of Archimedes statements, "But he follows the cautious method to which the Greeks always adhered; he never says that a given curve or surface is the *limiting form* of the inscribed or circumscribed figure; all that he asserts is that we can approach the curve or surface as nearly as we please."

Our definition of EG_π then makes the following metatheorem immediate. In the vocabulary with these functions named we have, since the $I_n(C_n)$ converge to one half of the limit of the $i_n(C_n)$ and we describe the same cut:

Theorem 3.7. *In $EG^2_{\pi,A,C}$, $C(r) = 2\pi r$ is a circumference function and $A(r) = \pi r^2$ is an area function.*

In an Archimedean field there is a unique interpretation of π and thus a unique choice for a circumference function with respect to the vocabulary without the constant π. By adding the constant π to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad of $2\pi r$ would equally well fit our condition for being the circumference.

We omit the technical details of 1) modifying the development of the area function of polygons described in Section 4.5 of Part I, by extending the notion of figure to include sectors of circles and 2) formalizing a notion of equal area, including a schema for approximation by finite polygons. These details complete the argument that the formal area function $A(r)$ does indeed compute the area. We did the harder case of circumference to emphasize the innovation of Archimedes in defining arc length. Unlike area it is not true that the perimeter of a polygon containing a second is larger that the perimeter of the enclosed field. By dealing with a special case, we suppressed Archimedes anticipation of the notion of bounded variation.

We have extended our descriptively complete axiomatization from the polygonal geometry of Hilbert's first-order axioms (HP5) to Euclid's results on circles and beyond. Euclid doesn't deal with arc length at all and we have assigned straight line segments to both the circumference and area of a circle. So this would not qualify as a modest axiomatization of Greek geometry but only of the modern understanding of these formulas. This distinction is not a problem for the notion of descriptive axiomatization. The facts are sentences. The formulas for circumference and area not the same sentences as the Euclid/Archimedes statement in terms of proportions, but they are implied by the modern equational formulations.

We now want to make a similar extension of $\mathcal{E}^2$. Dedekind (page 37-38 of [Dedekind 1963]) observes that the field of real algebraic numbers is 'discontinuous everywhere' but 'all constructions that occur in Euclid's elements can . . . be just as accurately effected as in a perfectly continuous space'. Strictly speaking, for *constructions* this is correct. But the proportionality constant between a circle and its circumference π is absent, so, even more, not both a straight line segment of the same length

[30] Archimedes, Men of Science [Heath 2011], Chapter 4.

as the circumference and the diameter are in the model[31]. We want to find a theory which proves the circumference and area formulas for circles and countable models of the geometry over RCF, where ‘arc length behaves properly’.

In contrast to Dedekind and Hilbert, Descartes eschews the idea that there can be a ratio between a straight line segment and a curve. As [Crippa 2014b] writes, “Descartes excludes the exact knowability of the ratio between straight and curvilinear segments”:

> ... la proportion, qui est entre les droites et les courbes, n’est pas connue, et mesme ie croy ne le pouvant pas estre par les hommes, on ne pourroit rien conclure de là qui fust exact et assuré[32].

Hilbert[33] asserts that there are many geometries satisfying his axioms I-IV and V1 but only one, ‘namely the Cartesian geometry’ that also satisfies V2. Thus the conception of ‘Cartesian geometry[34]’ changed radically from Descartes to Hilbert; even the symbol π was not introduced until 1706 (by Jones). Nevertheless, we now define a theory $\mathcal{E}^2_\pi$ analogous to EG_π which does not depend on the Dedekind axiom but can be obtained in a first-order way.

Given Descartes’ proscription of π, the new system will be immodest with respect to the Cartesian data set. But we will argue at the end of this section that both of our additions of π are closer to Greek conceptions than the Dedekind axiom. At this point we need some modern model theory to guarantee the *completeness* of the theory we are defining. A first-order theory T for a vocabulary including a binary relation $<$ is *o-minimal* if every model of T is linearly ordered by $<$ and every 1-ary formula is equivalent in T to a Boolean combination of equalities and inequalities [Van den Dries 1999]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [Tarski 1931].

Theorem 3.8. *Form $\mathcal{E}^2_\pi$ by adjoining $\Sigma(\pi)$ to $\mathcal{E}^2$. $\mathcal{E}^2_\pi$ is first-order complete for the vocabulary τ augmented by constant symbols $0, 1, \pi$.*

Proof. We have established that there is definable ordered field with domain the line through 01. By Tarski, the theory of this real closed field is complete. The field is bi-interpretable with the plane [Tarski 1951] so the theory of the geometry T is complete as well. Further by Tarski, the field is o-minimal. Therefore, the type over the empty set of any point on the line is determined by its position in the linear ordering of the subfield F_s (Notation 1.1). Each i_n, c_n is an element of the field F_s. This position

[31] Thus, Birkhoff’s protractor postulate is violated.

[32] Descartes, Oeuvres, vol. 6, p. 412. Crippa also quotes Averros as emphatically denying the possibility of such a ratio and notes Vieta held similar views.

[33] See pages 429-430 of [Hallett & Majer 2004].

[34] One wonders whether it had changed when Hilbert wrote. That is, had readers at the turn of the 20th century already internalized a notion of Cartesian geometry which entailed Dedekind completeness and so was at best formulated in the 19th century (Bolzano-Cantor-Weierstrass-Dedekind).

in the linear order of 2π in the linear order on the line through 01 is given by Σ. Thus $T \cup \Sigma(\pi)$ is a complete theory. $\square_{3.1}$

Building on Definition 3.2 we extend the theory $\mathcal{E}^2_\pi$.

Definition 3.9. *We define two new theories expanding $\mathcal{E}^2_\pi$.*

1. *The theory $\mathcal{E}^2_{\pi,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory $\mathcal{E}^2_\pi$ obtained by the explicit definition $A(r) = \pi r^2$*
2. *The theory $\mathcal{E}^2_{\pi,A,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory $\mathcal{E}^2_{\pi,A}$ obtained by adding the explicit definition $C(r) = 2\pi r$.*

Theorem 3.10. *The theory $\mathcal{E}^2_{\pi,A,C}$ is a complete, decidable extension of $EG_{\pi,A}$ $\mathcal{E}^2_\pi$ that is coordinatized by an o-minimal field. Moreover, in $\mathcal{E}^2_{\pi,A,C}$, $C(r) = 2\pi r$ is a circumference function (i.e. satisfies all the ι_n and γ_n) and $A(r) = \pi r^2$ is an area function.*

Proof. We are adding definable functions to $\mathcal{E}^2_\pi$ so o-minimality and completeness are preserved. The theory is recursively axiomatized and complete so decidable. The formulas continue to compute area and circumference correctly (as in Theorem 3.7) since they extend $EG_{\pi,A,C}$. $\square_{3.10}$

This theory is sufficient to prove π is transcendental. Lindemann proved that π does not satisfy a polynomial of degree n for any n. Thus for any polynomial over the rationals $p(\pi) \neq 0$ is a consequence of the complete type generated by $\Sigma(\pi)$ and so a theorem of $\mathcal{E}^2_{0,1,\pi}$. We explore this type of argument in point 5 of Section 4.3.

We now extend the known fact that the theory of real closed fields is 'finitistically justified' (in the list of such results on page 378 of [Simpson 2009]) to $\mathcal{E}^2_{\pi,A,C}$. For convenience, we lay out the proof with reference to results[35] recorded in [Simpson 2009].

Fact 3.11. The theory $\mathcal{E}^2$ is bi-interpretable with the theory of real closed fields. And thus it (as well as $\mathcal{E}^2_{\pi,A,C}$) is finitistically consistent, in fact, provably consistent in primitive recursive arithmetic (PRA).

Proof. By Theorem II.4.2 of [Simpson 2009], RCA_0 proves the system $(Q, +, \times, <)$ is an ordered field and by II.9.7 of [Simpson 2009], it has a unique real closure. Thus the existence of a real closed ordered field and so $Con(RCOF)$ is provable in RCA_0. (Note that the construction will imbed the surd field F_s.)

Lemma IV.3.3 [Friedman et al. 1983] asserts the provability of the completeness theorem (and hence compactness) for countable first-order theories from WKL_0.

[35]We use RCOF here for what we have called RCF before. Model theoretically adding the definable ordering of a formally real field is a convenience. Here we want to be consistent with the terminology in [Simpson 2009]. Note that Friedman[Friedman 1999] strengthens the results for PRA to exponential function arithmetic (EFA). Friedman reports Tarski had observed the constructive consistency proof much earlier. The theories discussed here, in increasing proof strength are EFA, PRA, RCA_0 and WKL_0.

Since every finite subset of $\Sigma(\pi)$ is easily seen to be satisfiable in any RCOF, it follows that the existence of a model of $\mathcal{E}^2_\pi$ is provable in WKL_0. Since WKL_0 is π^0_2-conservative over PRA, we conclude PRA proves the consistency $\mathcal{E}^2_\pi$. As $\mathcal{E}^2_{\pi,C,A}$ is an extension by explicit definitions its consistency is also provable in PRA. $\square_{3.11}$

It might be objected that such minor changes as adding to $\mathcal{E}$ the name of the constant π, or adding the definable functions C and A undermines the earlier claim that $\mathcal{E}^2$ is descriptively complete for Cartesian geometry. But π is added because the modern view of 'number' requires it and increases the data set to include propositions about π which are inaccessible to $\mathcal{E}^2$.

We have so far tried to find the proportionality constant only for a specific situation. In the remainder of the section, we consider several ways of systematizing the solution of families of such problems. First, still in a specific case we look for models where every angle determines an arc that corresponds to the length of a straight line segment. We consider several model theoretic schemes to organize such problems.

Birkhoff [Birkhoff, George 1932] posited the following *protractor postulate* in his system[36].

> POSTULATE III. The half-lines ℓ, m, through any point O can be put into $(1,1)$ correspondence with the real numbers $a(\mathrm{mod} 2\pi)$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m respectively, the difference $a_m - a_\ell(\mathrm{mod} 2\pi)$ is $\angle AOB$. Furthermore, if the point B varies continuously in a line r not containing the vertex 0, the number a_m varies continuously also[37].

This is a parallel to his 'ruler postulate' which assigns each segment a real number length. Thus, Birkhoff takes the real numbers as an unexamined background object. At one swoop he has introduced addition and multiplication, and assumed the Archimedean and completeness axioms. So even 'neutral' geometries studied on this basis are actually greatly restricted. He argues that his axioms define a categorial system isomorphic to $\Re^2$. So it is equivalent to Hilbert's.

This particular postulate conflates three distinct ideas: i) the rectifiability of arcs – each arc of a circle has the same length as a straight line segment, ii) rectification of arcs, an algorithm for attaining i) and iii) the measurement of angles.

The next task is to find a more modest version of Birkhoff's postulate: a first-order theory with countable models which assign to each angle a measure between 0 and 2π. Recall that we have a field structure on the line through 01 and the number π on that line.

Definition 3.12. *A* measurement of angles *function is a map μ from congruence classes*

[36]This is the axiom system used in virtually all U.S. high schools since the 1960's.
[37]I slight modified the last sentence, in lieu of reproducing the diagram.

of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side BC, $\mu(\angle ABD) = \mu(\angle ABC) + \mu(\angle CBD)$

If we omitted the additivity property this would be trivial: Given an angle $\angle ABC$ less than a straight angle, let C' be the intersection of a perpendicular to AC through B with AC and let $\mu(\angle ABC) = \frac{BC'}{AB}$. (It is easy to extend to the rest of the angles.)

Here we use approach b) of the Dicta on definitions rather than the explicit definition approach a) used for $C(r)$ and $A(r)$. We define a new theory with a function symbol μ which is 'implicitly defined' by the axioms.

Definition 3.13. *The theory $\mathcal{E}^2_{\pi,A,C,\mu}$ is obtained by adding to $\mathcal{E}^2_{\pi,A,C,\mu}$, the assertion μ is a continuous*[38] *additive map from congruence classes of angles to* $(0, 2\pi]$.

Now we have to address the consistency and completeness of $\mathcal{E}^2_{0,1,\pi,A,C,\mu}$. Consistency is easy, we can easily define (in the mathematical sense, not as a formally definable function in $\mathcal{E}^2_{0,1,\pi,A,C}$) such a function μ^* on the real plane. So the axioms are consistent. And by taking the theory of this structure we would get a complete first-order theory. But, *a priori*, we don't have an axiomatization[39].

Crippa describes Leibniz's distinguishing two types of quadrature,

> 'universal quadrature of the circle, namely the problem of finding a general formula, or a rule in order to determine an arbitrary sector of the circle or an arbitrary arc; and on the other he defines the problem of the particular quadrature, . . . , namely the problem of finding the length of a given arc or the area of a sector, or the whole circle . . . (page 424 of [Crippa 2014a])

Thus, while we have solved i) the rectifiability problem, merely assuming the existence of a μ does not solve ii) as we have no idea how to compute μ. However the addition of the restricted cosine, as in footnote 39 does so by calculating arc length as in calculus. But a nice axiom system remains a dream.

Blanchette [Blanchette 2014] distinguishes two approaches to logic: deductivist and model-centric. Hilbert represents the deductivist school and Dedekind the model-centric. Essentially, the second comes to theories trying to describe an intuition of a particular structure. We briefly consider the opposite procedure; are there 'canonical' models of the various theories we have been considering.

[38] With a little effort we can express continuity of μ in $\mathcal{E}^2_{\pi,A,C,\mu}$ and it could fail in a non-Archimedean model so we have to require it to have chance at a complete theory.

[39] In fact, by coding a point on the unit circle by its x-coordinate and setting $\mu((x_1, y_1), (x_2, y_2)) = \cos^{-1}(x_1 - x_2)$ one gets such a function which definable in the theory of the real field expanded by the sin function restricted to $(0, 2\pi]$. This theory is known [Van den Dries 1999] to be o-minimal. But there is no known axiomatization and Marker tells me it is unlikely to be decidable without assuming the Schanuel conjecture.

By modern tradition, the continuum is the real numbers and geometry is the plane over it. Is there a smaller model which reflects the geometric intuitions discussed here? For Euclid II, there is a natural candidate, the Euclidean plane over the surd field F_s. Remarkably, this does not conflict with Euclid XII.2 (the area of a circle is proportional to the square of the diameter). The model is Archimedean and π is not in the model. But Euclid only requires a proportionality which defines a type, not a realization of the type. Plane geometry over the real algebraic numbers plays the same role for $\mathcal{E}^2_{0,1}$. Both are categorical in $L_{\omega_1,\omega}$. In the second case, the axiomatization is particularly nice. Add the Archimedean axiom[40].

Now we argue that the methods of this section better reflect the Greek view that does Dedekind. Mueller ([Mueller 2006], 236) makes an important point distinguishing the Euclid/Eudoxus use from Dedekind's use of cuts.

> One might say that in applications of the method of exhaustion the limit is given and the problem is to determine a certain kind of sequence converging to it, ... Since, in the *Elements* the limit always has a simple description, the construction of the sequence can be done within the bounds of elementary geometry; and the question of constructing a sequence for any given arbitrary limit never arises.

But what if we want to demand the realization of various transcendentals? Mueller's description suggests the principle that we should only realize cuts in the field order that are recursive over a finite subset. So a candidate would be a recursively saturated model[41] of $\mathcal{E}^2$. Remarkably, almost magically[42] this model would also satisfy $\mathcal{E}^2_{\pi,A,C,\mu}$. A recursively saturated model is necessarily non-Archimedean. There are however many different countable recursively saturated models depending on which transcentals are realized

Here is a more canonical candidate for a natural model which admits the 'Eudoxian transcendentals'; take the smallest elementary submodel of $\Re$ closed[43] under A, C, μ containing the real algebraic numbers and all realizations of recursive cuts in F_s. The Scott sentence[44] of this sentence is a categorical sentence in $L_{\omega_1,\omega}$.

The models in the last paragraph were all countable; we cannot do this with the Hilbert model; it has no countable $L_{\omega_1,\omega}$-elementary submodel.

We turn to the question of modesty. Mueller's distinction can be expressed in another way. Eudoxus provides a technique to solve certain problems, which are

[40] It is easy to see that any transcendental adds an infinitesimal to the field.

[41] A model is recursively saturated if every recursive type over a finite set is realized. [Barwise 1975]

[42] The magic is called resplendency. Every recursively saturated model is resplendent [Barwise 1975]. M is resplendent if any formula $\exists A\phi(A, \boldsymbol{c})$ that is satisfied in an elementary extension of M is satisfied by some A' on M. Examples are the formulas defining C, A, μ.

[43] Interpret A, C, μ on $\Re$ in the standard way.

[44] For any countable structure M there is a 'Scott' sentence ϕ_M such that all countable models of ϕ_M are isomorphic to M; see chapter 1 of [Keisler 1971].

specified in each application. In contrast, Dedekind's postulate provides solves $2^{\aleph_0}$ problems at one swoop. Each of the theories $\mathcal{E}_{0,1,\pi}$, $\mathcal{E}_{\pi,A,C}$, $\mathcal{E}_{\pi,A,C,\mu}$ and the later search for their canonical models reflect this concern. Each solves at most a countable number of recursively stated problems. In summary, we regard the replacement of 'congruence class of segment', by 'length represented by an element of the field' as a modest reinterpretation of Greek geometry. But it becomes immodest relative to even Descartes when this length is a transcendental. And most immodest is to demand arbitrary transcendentals.

4 And back to Hilbert

The non-first-order postulates of Hilbert play complementary roles. The Archimedean axiom is minimizing; each cut is realized by at most one point so each model has cardinality at most $2^{\aleph_0}$. The Veronese postulate (See Footnote 47.) or Hilbert's Vollständigkeitaxiom is maximizing; in the absence of the Archimedean axiom each cut is realized, the set of realizations could have arbitrary cardinality.

4.1 The role of the Axiom of Archimedes in the *Grundlagen*

Recall the following from Hilbert's introduction, 'bring out as clearly as possible the significance of the groups of axioms.' Much of his book is devoted to this metamathematical investigation. In particular this includes Sections 9-12 (from [Hilbert 1971]) concerning the consistency and independence of the axioms. Further examples[45], in Sections 31-34 shows that without the congruence axioms, the axiom of Archimedes is necessary to prove what Hilbert labels as Pascal's (Pappus) theorem. Moreover, in the Conclusion he explores the connection between the angle sum theorem (sum of the angles of a triangle is 180^o) and the fifth postulate and reports on Dehn's result that Archimedes axiom behaves very differently in relating the sum of the angle to the hypotheses of no or more than one parallel to a given line through a fixed point. These sorts of results demonstrate the breadth of Hilbert's program. However, with respect the problem studied here, they do not affect the conclusion that Hilbert's full axiom set is an immodest axiomatization[46] of Euclid I or Euclid II or of the Cartesian data set since those data sets contain and are implied by the appropriate first-order axioms.

This conclusion is not affected by a further use of the Archimedean axiom by Hilbert. In Sections 19 and 21, it is shown that the Archimedean axiom is necessary to show equicomplementable (equal content) is the same as equidecomposable (in 2 dimensions). These are all metatheoretical results. The use of the Archimedean axiom

[45] I thank the referee for pointing to the next two examples and emphasizing Hilbert's more general goals of understanding the connections among organizing principles.

[46] I could add the data set Archimedes, but that would be a cheat. I restricted to Archimedes on the circle; Archimedes proposed a general notion of arc length and studied many other transcendental curves.

to prove equidecomposable is the same as equicomplementable is certainly a proof in the system. But an unnecessary one. As we argued in Section 4.4 of Part I, Hilbert could just have easily defined 'same area' as 'equicomplementable' (as is a natural reading of Euclid).

Thus, we find no *geometrical* theorems in the Grundlagen that essentially depend on the Axiom of Archimedes. Rather Hilbert's use of the axiom of Archimedes is i) to investigate the interaction of the various principles and ii) in conjunction with the Dedekind axiom, identify the field defined in the geometry with the independent existence of the real numbers as conceived by Dedekind. Hilbert wrote that V.1 and V.2 allow one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. Archimedes makes the axiom one-one and the Vollständigkeitsaxiom makes it onto. We have noted here that the grounding of real algebraic geometry (the study of systems of polynomial equations in a real closed field) is fully accomplished by Tarski's axiomatization. And we have provided a first-order extension to deal with the basic properties of the circle. Since Dedekind, Weierstrass, and others pursued the 'arithmetization of analysis' precisely to ground the theory of limits, identifying the geometrical line as the Dedekind line reaches beyond the needs of geometry.

4.2 Hilbert and Dedekind on Continuity

Hilbert's formulation of the completeness axiom reads [Hilbert 1971]:

> Axiom of Completeness (Vollständigkeitsaxiom): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

We have used in this article the following adaptation of Dedekind's postulate for geometry (DG):

> DG: Any cut in the linear ordering imposed on any line by the betweenness relation is realized.

While this formulation is convenient for our purposes, it misses an essential aspect of Hilbert's version. DG implies the Archimedean axiom and Hilbert was aiming for an independent set of axioms. Hilbert's axiom does not imply Archimedes. A variant VER[47] on Dedekind's postulate that does not imply the Archimedean axiom

[47] The axiom VER (see [Cantú 1999]) asserts that for a partition of a linearly ordered field into two intervals L, U (with no maximum in the lower L or minimum in the upper U) and third set in between at

was proposed by Veronese in [Veronese 1889]. If we substituted VER for DG, our axioms would also satisfy the independence criterion.

Hilbert's completeness axiom in [Hilbert 1971] asserting any model of the rest of the theory is maximal, is inherently model-theoretic. The later line-completeness [Hilbert 1962] is a technical variant[48]. Giovannini's account [Giovannini 2013] includes a number of points already made here; but he makes three more. First, Hilbert's completeness axiom is not about deductive completeness (despite having such consequences), but about maximality of every *model* (page 145). Secondly (last line of 153) Hilbert expressly rejects Cantor's intersection of closed intervals axiom because it relies on a sequence of intervals and 'sequence is not a geometrical notion'. A third intriguing note is an argument due to Baldus in 1928 that the parallel axiom is an essential ingredient in the categoricity of Hilbert's axioms [49].

Here are two reasons for choosing Dedekind's (or Veronese's) version. The most basic is that one cannot formulate Hilbert's version as sentence Φ in second-order logic[50] with the intended interpretation $(\Re^2, \mathbf{G}) \models \Phi$. The axiom requires quantification over subsets of an extension of the model which putatively satisfies it. Here is a second-order statement[51] of the axiom, where ψ denotes the conjunction of Hilbert's first four axiom groups and the axiom of Archimedes.

$$(\forall X)(\forall Y)\forall \mathbf{R})[[X \subseteq Y \wedge (X, \mathbf{R}{\restriction}X) \models \psi \wedge (Y, \mathbf{R}) \models \psi] \rightarrow X = Y]$$

This anomaly has been investigated by Väänänen who makes the distinction (on page 94 of [Väänänen 2012]) between $(\Re^2, \mathbf{G}) \models \Phi$ and the displayed formula and expounds in [Väänänen 2014] a new notion, 'Sort Logic', which provides a logic with a sentence Φ which by allowing a sort for an extension axiomatizes geometry formalizes Hilbert's V.2. The second reason is that Dedekind's formulation, since it is about the geometry, not about its axiomatization, directly gives the kind of information about the existence of transcendental numbers that we observe in the paper.

most one point, there is a point between L and U just if for every $e > 0$, there are $a \in A, b \in B$ such that $b - a < e$. Veronese derives Dedekind's postulate from his plus Archimedes in [Veronese 1889] and the independence in [Veronese 1891]. In [Levi-Civita 2 93] Levi-Civita shows there is a non-Archimean ordered field that is Cauchy complete. I thank Philip Ehrlich for the references and recommend section 12 of the comprehensive [Ehrlich 2006]. See also the insightful reviews [Pambuccian 2014a] and [Pambuccian 2014b] where it is observed that Vahlen [Vahlen 1907] also proved this axiom does not imply Archimedes.

[48] Since any point is in the definable closure of any line and any one point not one the line, one can't extend any line without extending the model. Since adding either the Dedekind postulate and or Hilbert completeness gives a categorical theory satisfied by a geometry whose line is order isomorphic to $\Re$ the two axioms are equivalent (over HP5 + Arch).

[49] Hartshorne (sections 40-43 of [Hartshorne 2000] gives a modern account of Hilbert's argument that replacing the parallel postulate by the axiom of limiting parallels gives a geometry that is determined by the underlying (definable) field. With V.2 this gives a categoricical axiomatization for hyperbolic geometry.

[50] Of course, this analysis is anachronistic; the clear distinction between first and second-order logic did not exist in 1900. By $\mathbf{G}$, we mean the natural interpretation in $\Re^2$ of the predicates of geometry introduced in Section 1.

[51] I am leaving out many details, $\mathbf{R}$ is a sequence of relations giving the vocabulary of geometry and the sentence 'says' they are relations on Y; the coding of the satisfaction predicate is suppressed.

In [Väänänen 2012], Väänänen discusses the categoricity of natural structures such as real geometry when axiomatized in second-order logic (e.g. DG). He has discovered the striking phenomena of 'internal categoricity'. Suppose the second-order categoricity of a structure A is formalized by the existence of sentence Ψ_A such that $A \models \Psi_A$ and any two models of Ψ are isomorphic. If this second clause in provable in a standard deductive system for second-order logic, then it is valid in the Henkin semantics, not just the full semantics.

Philip Ehrlich has made several important discoveries concerning the connections between the two 'continuity axioms' in Hilbert and develops the role of maximality. First, he observes (page 172) of [Ehrlich 1995] that Hilbert had already pointed out that his completeness axiom would be inconsistent if the maximality were only with respect to the first-order axioms. Secondly, he [Ehrlich 1995, Ehrlich 1997] systematizes and investigates the philosophical significance of Hahn's notion of Archimedean completeness. Here the structure (ordered group or field) is not required to be Archimedean; the maximality condition requires that there is extension which fails to extend an Archimedean equivalence class[52]. This notion provides a tool (not yet explored) for investigating the non-Archimedean models studied in Section 3.

In a sense, our development is the opposite of Ehrlich's in [Ehrlich 2012], *The absolute arithmetic continuum and the unification of all numbers great and small*. Rather than trying to unify all numbers great and small, we are interested in the minimal collection of numbers that allow the development of a geometry according with our fundamental intuitions.

4.3 Against the Dedekind Posulate for Geometry

Our fundamental claim is that (slight variants on) Hilbert's first-order axiom provide a modest descriptively complete axiomatization of most of Greek geometry.

As we pointed out in Section 3 of [Baldwin 2014] various authors have proved under $V = L$, any countable or Borel structure can be given a categorical axiomatization. We argued there that this fact undermines the notion of categoricty as an independent desiderata for an axiom system. There, we gave a special role to attempting to axiomatize canonical systems. Here we go further, and suggest that even for a canonical structure there are advantages to a first-order axiomatization that trump the loss of categoricity.

We argue then that the Dedekind postulate is inappropriate (in particular immodest) in any attempt to axiomatize the Euclidean or Cartesian or Archimedean data sets for several reasons:

1. The requirement that there be a straight-line segment measuring any circular arc

[52]In an ordered group, a and b are *Archimedes-equivalent* if there are natural numbers m, n such that $m|a| > |b|$ and $n|b| > |a|$.

clearly contradicts the intentions of Euclid and Descartes.

2. Since it yields categoricity, it is not part of the data set but rather an external limitative principle. The notion that there was 'one' geometry (i.e. categoricity) was implicit in Euclid. But it is not a geometrical statement. Indeed, Hilbert described his metamathematical formulation of the completeness axiom (page 23 of [Hilbert 1962]), 'not of a purely geometrical nature'.

3. As we have pointed out repeatedly, it is not needed to establish the properly geometrical propositions in the data set.

4. Proofs from Dedekind's postulate obscure the true geometric reason for certain theorems. Hartshorne writes[53]:

   > '... there are two reasons to avoid using Dedekind's axiom. First, it belongs to the modern development of the real number systems and notions of continuity, which is not in the spirit of Euclid's geometry. Second, it is too strong. By essentially introducing the real numbers into our geometry, it masks many of the more subtle distinctions and obscures questions such as constructibility that we will discuss in Chapter 6. So we include the axiom only to acknowledge that it is there, but with no intention of using it.

5. The use of second-order logic undermines a key proof method – informal (semantic) proof. A crucial advantage of a first-order axiomatization is that it licenses the kind of argument[54] described in Hilbert and Ackerman[55]:

   > Derivation of Consequences from Given Premises; Relation to Universally Valid Formulas
   >
   > So far we have used the predicate calculus only for deducing valid formulas. The premises of our deductions, viz Axioms a) through f), were themselves of a purely logical nature. Now we shall illustrate by a few examples the general methods of formal derivation in the predicate calculus . . . It is now a question of deriving the consequences from any premises whatsoever, no longer of a purely logical nature.
   >
   > The method explained in this section of formal derivation from premises which are not universally valid logical formulas has its main application in the setting up of the primitive sentences or axioms for any particular field of knowledge and the derivation of the remaining theorems from them as consequences . . . We will examine, at the end of this section, the question of whether every statement which would intuitively be regarded as a consequence of the axioms can be obtained from them by means of the formal method of derivation.

[53] page 177 of [Hartshorne 2000]

[54] We noted that Hilbert proved that a Desarguesian plane embeds in 3 space by this sort of argument in Section 2.4 of [Baldwin 2013].

[55] Chapter 3, §11 Translation taken from [Blanchette 2014].

We exploited this technique in Section 3 to provide axioms for the calculation of the circumference and area of a circle. Väänänen[56] makes a variant of this apply to those sentences in second-order logic that are internally categorical. He shows certain second-order propositions can be derived from the formal system of second-order logic by employing 3rd (and higher) order arguments to provide semantic proofs.

Venturi[57] formulates a distinction, which nicely summarises our argument: 'So we can distinguish two different kinds of axioms: the ones that are *necessary* for the development of a theory and the *sufficient* one used to match intuition and formalization.' In our terminology only the necessary axioms make up a '*modest* descriptive axiomatization'. For the geometry Euclid I (basic polygonal geometry), Hilbert's first-order axioms meet this goal. With $\mathcal{E}^2_{\pi,A,C}$, a less immodest complete descriptive axiomatization is provided even including the basic properties of π. The Archimedes and Dedekind postulates have a different goal; they secure the 19th century conception of $\Re^2$ to be the unique model and thus ground elementary analysis.

4.4 But what about analysis?

We expounded a procedure [Hartshorne 2000] to define the field operations in an arbitrary model of HP5. We argued that the first-order axioms of EG suffice for the geometrical data sets Euclid I and II, not only in their original formulation but by finding proportionality constants for the area formulas of polygon geometry. By adding axioms to require the field is real closed we obtain a complete first-order theory that encompasses many of Descartes innovations. The plane over the real algebraic numbers satisfies this theory; thus, there is no guarantee that there is a line segment of length π. Using the o-minimality of real closed fields, we can guarantee there is such a segment by adding a constant for π and requiring it to realize the proper cut in the rationals. However, guaranteeing the uniqueness of such a realization requires the $L_{\omega_1,\omega}$ Archimedean axiom.

Hilbert and the other axiomatizers of 100 years ago wanted more; they wanted to secure the foundations of calculus. In full generality, this surely depends on second-order properties. But there are a number of directions of work on 'definable analysis'. One of the directions of research in o-minimality has been to prove the expansion of the real numbers by a particular functions (e.g. the Γ-function on the positive reals [Speissinger & van den Dries 2000]).

Peterzil and Starchenko study the foundations of calculus in [Peterzil & Starchenko 2000]. They approach complex analysis through o-minimality of the real part in [Peterzil & Starchenko 2010]. The impact of o-minimality on number theory was recognized by the Karp prize of 2014. And a non-logician, suggests using methods of Descartes to teach Calculus [Range 2014]. For an interesting

[56] Discussion with Väänänen.
[57] page 96 of [Venturi 2011]

perspective on the historical background of the banishment of infinitesimals in analysis see [Borovik & Katz 2012].

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