Axiomatizing changing conceptions of the geometric continuuum I: Euclid-Hilbert

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

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Abstract

We begin with a general account of the goals of axiomatization, introducing several variants (e.g. modest) on Detlefsen's notion of 'complete descriptive axiomatization'. We examine the distinctions between the Greek and modern view of number, magnitude and proportion and consider how this impacts the interpretation of Hilbert's axiomatization of geometry. We argue, as indeed did Hilbert, that Euclid's propositions concerning polygons, area, and similar triangle are derivable (in their modern interpretation in terms of number) by from Hilbert's *first order* axioms plus 'circle-circle intersection'.

We argue that Hilbert's continuity properties show much more than the data set of Greek mathematics and thus are an immodest complete descriptive axiomatization of that subject.

By the *geometric continuum* we mean the line situated in the context of the plane. Consider the following two propositions[1].

(*) Euclid VI.1: Triangles and parallelograms which are under the same height are to one another as their bases.

Hilbert[2] gives the area of a triangle by the following formula.

(**) Hilbert: Consider a triangle ABC having a right angle at A. The measure of the area of this triangle is expressed by the formula

$$F(ABC) = \frac{1}{2}AB \cdot AC.$$

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[1]For diagrams illustrating the the Euclidean propositions about area, see Theorem 4.6.4 and Remark 4.6.6.

[2]Hilbert doesn't state this result as a theorem; and I have excerpted the statement below from an application on page 66 of [Hil62]. Hilbert defines proportionality in terms of segment multiplication on page 50. 'Negative' segments are introduced in Section 17 on page 53.

When formulating a new axiom set in the late 19th century Hilbert faced several challenges:

1. Identify and fill 'gaps' or remove 'extraneous hypotheses' in Euclid's reasoning.
2. Reformulate propositions such as VI.1 to reflect the 19th century understanding of real numbers as measuring both length and area.
3. Ground both the geometry of Descartes and 19th century analytic geometry[3].

Hilbert's third goal can be stated more emphatically: grounding calculus. We will argue that in meeting the third goal, Hilbert added axioms that were unnecessary for purely geometric considerations. We frame this discussion in terms of the notion of *descriptive axiomatization* from [Det14]. But the axiomatization of a theory of geometry that had been developing for over two millenia leads to several further considerations. First, while the sentences (*) and (**) clearly, in some sense, express the same proposition, the sentences are certainly different. How does one correlate such statements? Secondly, previous descriptions of complete descriptive axiomatization omit the possibility that the axioms might be too strong and obscure the 'cause' for a proposition to hold. We introduce the term 'modest' descriptive axiomatization to denote one which avoids this defect. We give several explicit lists of propositions from Euclid and draw from [Har00] for an explicit linking of subsets of Hilbert's axioms as justifications for these lists. In particular, we analyze the impact of the distinction between ratios in the language of Euclid and segment multiplication in [Hil62] or multiplication[4] of "numbers". Then, we examine in more detail, certain specific propositions that in the modern interpretation might appear to depend on Dedekind's postulate. We conclude that the first order axioms provide a modest complete descriptive axiomatization[5] for most of Euclid's geometry. In the sequel [Bal14a] we argue that the second order axioms aim at results that are beyond (and even in some cases antithetical to) the Greek and even the Cartesian view of geometry. So Hilbert's axioms are immodest as an axiomatization of traditional geometry.

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[3]Giovannini [Gio], translating from Hilbert's 1891 paper in [Hil04], writes that "[analytic geometry] correlates the points on a line with numbers from the very beginning, thereby reducing geometry to calculus." Following Giovannini, we then take this as one objective of the Grundlagen. We stress here the vast distinction between this goal and the others we have enumerated.

[4]That is, a multiplication on points rather than segments. See Heyting [Hey63]; the most thorough treatment is in [Art57].

[5]It is modest with respect to modern conceptions that straight line segments and arcs should be commensurable but not with earlier conceptions discussed in the text.

1 Introduction

Hilbert groups his axioms for geometry into 5 classes. The first four are first order. Group V, Continuity, contains Archimedes axiom which can be stated in the logic[6] $L_{\omega_1,\omega}$ and a second order completeness axiom equivalent (over the other axioms) to Dedekind completeness[7]of each line in the plane. Hilbert[8] closes the discussion of continuity with 'However, in what is to follow, no use will be made of the "axiom of completeness" '.

Why then did he include the axiom? Earlier in the same paragraph[9], he writes that 'it allows the introduction of limiting points' and enables one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. Hilbert's concern with the foundations of real analytic geometry is sometimes lost as the Grundlagen itself focuses on the reconstruction of Euclid and metamathematical considerations (e.g. independence of axioms) while the foundation of analytic geometry is restricted to the short section §17: equations of lines and planes.

In Section 2, we consider several accounts of the purpose of axiomatization. We adjust Detlefsen's definition to guarantee some 'minimality' of the axioms by fixing on a framework for discussing the various axiom systems: a *modest descriptively complete axiomatization*. One of our principal tools is the notion of 'data set', a collection of sentences to be accounted for by an axiomatization. 'The data set for area X' is time dependent; new sentences are added; old ones are reinterpreted. In Section 3.1, we will consider several concepts of the continuum so as to focus on the notions most relevant here: those which involve order and the embedding of the line in the plane. We review in Section 3.2 the Greek conceptions of proportion and ratio and the fundamental role they play in Euclid's geometry. With this background, Section 3.3 lists data sets (collections of mathematical 'facts') for which systems of axioms are discussed later. We focus on propositions that which might be thought to require the Dedekind axiom. Section 4.1 contrasts the arithmetization of geometry program of the 19th century with the grounding of algebra in geometry enunciated by Hilbert. We lay out in Section 4.2 various sets of axioms for geometry and correlate them with the data sets of Section 3.3 in Theorem 4.2.3. Section 4.3 sketches Hilbert's proof that the axiom set HP5 (see Notation 4.2.2) suffice to define a field. In Section 4.4 we note that several

[6] In the logic, $L_{\omega_1,\omega}$, quantification is still over individuals but now countable conjunctions are permitted so it is easy to formulate Archimedes axiom : $\forall x, y(\bigvee_{m\in\omega} mx > y$. By switching the roles of x and y we see each is reached by a finite multiple of the other.

[7] Dedekind defines the notion of a cut a linearly ordered set I (a partition of $\mathbb{Q}$ into two intervals (L, U)). He postulates that each cut has unique realization, a point above all members of L and below all members U -it may be in either L or U (page 20 of [Ded63]). If either the L contains a least upper bound or the upper interval U contains a greatest lower bound, the cut is called 'rational' and no new element is introduced. Each of the other (irrational) cuts introduces a new number. It is easy to see that the *unique* realization requirement implies the Archimedes axiom. By Dedekind completeness of a line, I mean the Dedekind postulate holds for the linear ordering of that line. See [Bal14a].

[8] Page 26 of [Hil71].

[9] For a thorough historical description, see the section The Vollständigkeit, on pages 426-435 of [Hil04]. We focus on the issues most relevant to this paper.

theorems, which at first glance (or first historical proof) used Dedekind's postulate, are consequences of HP5 plus the circle-circle intersection axiom. Section 4.5 has two purposes. On the one hand we distinguish the geometric conception of multiplication as similarity from repeated addition. On the other, we report Hilbert's first order proof that similar triangles have proportional sides. Again, in Section 4.6 we report Hilbert's argument for the computation of area of polygons (avoiding Euclid's implicit use of the Archimedes axiom).

This paper expounds the consequences of Hilbert's first order axioms and argues they form a modest descriptive axiomatization of Euclidean geometry. The sequel [Bal14a] 1) discusses the role of the Archimedean axiom in Hilbert; 2) analyzes the distinctions between the completeness axioms of Dedekind and Hilbert, 3) argues that Hilbert's continuity axioms are overkill for strictly geometric propositions as opposed to Hilbert's intent of 'grounding real analytic geometry'; and 4) speculates about the use of 'definable analysis' to justify parts of analysis on first order grounds.

We will also discuss, Archimedes, Descartes and Tarski in [Bal14a] drawing on [Bal14b] where we provide a first order theory to justify the formulas for circumference and area of a circle. Hilbert implicity uses both continuity axioms to guarantee the existence of π.

2 The Goals of Axiomatization

In this section, we place our analysis in the context of recent philosophical work on the purposes of axiomatization. We investigate the connection between axiom sets and data sets of sentences for an area of mathematics. We introduce the notion of a modest descriptively complete axiomatization for a particular data set.

Hilbert begins the Grundlagen [Hil71] with:

> The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent axioms* and to deduce from them the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the groups of axioms and the scope of the conclusions to be derived from the individual axioms.

Hallett (page 434 of [Hil04]) presaged much of the intent of this article:

> Thus completeness appears to mean [for Hilbert] 'deductive completeness with respect to the geometrical facts'. ...In the case of Euclidean geometry there are various ways in which 'the facts before us' can be presented. If interpreted as 'the facts presented in school geometry' (or the

initial stages of Euclid's geometry), then arguably the system of the original Festschrift [i.e. 1899 French version] is adequate. If, however, the facts are those given by geometrical intuition, then matters are less clear.

We begin by considering several ways of construing the notion Hallett labels 'fact'. Hintikka doesn't use that word [10]; his 'descriptive use of logic' concerns a class of models:

> If we use logical notions (such as quantifiers, connectives, etc) for the purpose of capturing a class of structures studied in a particular mathematical theory, we are pursuing the descriptive use of logic. To be more precise, we exploit logic in the sense that we formulate an axiomatization of a mathematical theory in order to describe that class of structures and no other structures, as precisely as we can. Thus, a descriptive use of logic consists, for example, in formulating an axiom system in order to capture the class of structures which number-theory deals with, e.g. the series of natural numbers.
>
> If we want to systematize and formalize mathematicians reasoning about the mathematical structures they are interested in, we are interested in the deductive use of logic. ... This machinery appeals to the deductive consequence relation provided by the logic, which in turn is defined by a list of inference rules.

Note that Hintikka's 'descriptive use' is a semantic requirement - describe 'as precisely as we can' a *class of structures*. In general this is a hard problem; we don't really know *all* the models. Thus, it is natural that several authors (e.g. [Gio10] [Nov14]) have focused on the situation where, as in Hintikka's example, the class is categorical; we can have a strong conception of a particular structure: *the* natural numbers. This focus has historical roots in Peano, Dedekind and the early axiomatizations of geometry (e.g., Hilbert and Veblen). Blanchette [Bla14] provides an apt moniker for this view: model-centric.

Detlefson [Det14] provides a syntactic counterpart to Hinktikka's notion of descriptive use, which he calls *descriptive axiomatization*. He motivates the notion with this remark of Huntington (Huntington's emphasis) [Hun11]:

> [A] miscellaneous collection of facts ... does not constitute a *science*. In order to reduce it to a science the first step is to do what *Euclid* did in geometry, namely, to *select a small number of the given facts as axioms and then to show that all other facts can be deduced from these axioms by the methods of formal logic*.

[10] We quote in full Hintikka's discussion of the descriptive use of logic([Hin89] as quoted in [Gio10]) and truncate Hintikka's account of the deductive aspect.

Detlefsen describes a local descriptive axiomatization as an attempt to deductively organize a data set (a collection of commonly accepted sentences pertaining to a given subject area of mathematics[11]). The axioms are *descriptively complete* if all elements of the data set are deducible from them. This raises two questions. What is a sentence? Who commonly accepts?

From the standpoint of modern logic, a natural answer to the first would be to specify a logic and a vocabulary and consider all sentences in that language. Detlefsen argues (pages 5-7 of [Det14]) that this is the wrong answer. He thinks Gödel errs in seeing the problem as completeness in the now standard sense of a first order theory[12]. Rather, Detlefsen presents us with an empirical question. We (at some point in time) look at the received mathematical knowledge in some area and want to construct a set of axioms from which it can all be deduced. In general such a data set is more graspable than *all models*. Of course, the data set is inherently flexible; conjectures are proven from time to time. In a way this version reflects Hintikka's as the data set is indeed a description of the class of its models. But it is very far from being model-centric as there is no requirement of categoricity.

Geometry is an example of what Detlefsen calls a local as opposed to a foundational descriptive axiomatization. Beyond the obvious difference in scope, Detlefsen points out several other distinctions. In particular ([Det14] page 5), the axioms of local axiomatizations are generally among the given facts while those of a foundational axiomatization are found by (paraphrasing Detlefsen) tracing each truth in a data set back to the deepest level where it can be properly traced. Comparing geometry at various times opens a deep question we want to avoid. In what sense do (*) and (**) opening this paper express the same thought, concept etc. Rather than address the issue of what is expressed, we will simply show how to interpret (*) (and other propositions of Euclid) as propositions in Hilbert's system. See Section 3.2 for this issue and Section 3.3 for extensions to the data set over the centuries.

An aspect of choosing axioms seems to be missing from the account so far. Hilbert [Hil05] provides this insight into how axioms are chosen:

> If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework. ...
>
> These underlying propositions may from an initial point of view be regarded as the axioms of the respective field of knowledge ...

[11] There is an interesting subtlety here. Suppose our body of mathematics is group theory. One might think the data set was the sentences in the vocabulary of group theory true in all groups. (The axioms are evident). But these sentences are not in fact the data set of 'group theory'; that subject is concerned about the properties and relations between groups. So taking the commutative law as a sentence that might illegitimately be added as an axiom for groups is studying the wrong subject.

[12] We argue against this in [Bal14a].

We want to identify a 'few' distinguished propositions[13] from the data set that suffice for the deduction of the data set. By a *modest* axiomatization[14], we mean one that implies all the data and not too much more[15]. Of course, 'not too much more' is a rough term. One cannot expect a list of known mathematical propositions to be deductively complete. By more, we mean introducing essentially new concepts and concerns or by adding additional hypotheses proving a result that contradict the explicit understandings of the authors of the data set (See the end of Section 4.1). As we'll see below, Hilbert's first order axioms are a modest axiomatization of the data: the theorems in Euclid about polygons (not circles) in the plane. We give an example in Remark 4.4.6 of an immodest first order axiomatization.

The mathematical goal of this paper is to provide a modest descriptively complete axiomatization of plane geometry[16]. If the data set is required to be deductively closed, there would be an easy sufficient condition for a modest axiomatization: the axioms must come from the data set. There is a difficulty with this requirement. First, the data sets stem from eras before 'deductive closure' was clearly defined; so there is an issue of how to apply this requirement to such systems. Secondly, there are two ways in which data sets are destined to change. New theorems will be proved from the existing hypotheses; but, more subtly, new interpretations of the basic concepts may develop over time so that sentence attain essentially new meanings. As (**) illustrates, such is the case with Euclid's VI.1.

We return to our question, "what is a sentence?". The first four groups of Hilbert's axioms are sentences of first order logic: quantification is over individuals and only finite conjunctions are allowed. As noted in Footnote 6, Archimedes axiom can be formulated in $L_{\omega_1,\omega}$. But the Dedekind postulate in any of its variants is a sentence of a kind of second order logic[17]. All three logics are deductive systems so that the set of provable sentences is recursively enumerable. Second order logic (in the standard semantics) fails the completeness theorem but, by the Gödel and Keisler [Kei71] completeness theorems, every valid sentence of $L_{\omega,\omega}$ or $L_{\omega_1,\omega}$ is provable. In the next few paragraphs we focus on the second order axiom. We consider the role of Archmedean axiom and $L_{\omega_1,\omega}$ in [Bal14a].

Adopting this syntactic view, there is a striking contrast between the data set in earlier generations of such subjects as number theory and geometry and the axiom systems advanced at the turn of the twentieth century; except for the Archimedean axiom,

[13]Often, few is interpreted as finite. Whatever Hilbert meant, we should now be satisfied with a small finite number of axioms and axiom schemes. At the beginning of the Grundlagen, Hilbert adds 'simple, independent, and complete'. Such a list including schemes is simple.

[14]We considered replacing 'modest' by 'precise or'safe' or 'adequate'. We chose 'modest' rather than one of the other words to stress that we want a sufficient set and one that is as necessary as possible. As the examples show, 'necessary' is too strong. Later work finds consequences of the original data set undreamed by the earlier mathematicians. Thus just as, 'descriptively complete', 'modest' is a description, not a formal definition.

[15]This concept describes normal work for a mathematician. "I have a proof; what are the actual hypotheses so I can convert it to a theorem."

[16]We extend this to include formulas for the circumference and area of a circle in [Bal14a, Bal14b].

[17]See the caveats on 'second order' in [Bal14a].

the data sets are expressed in first order logic. But through the analysis of the concepts involved, Dedekind arrived at second order axioms that formed the capstone of each axiomatization: induction and Dedekind completeness. These axioms answered real problems (especially in analysis). But a primary goal was to describe a particular structure, to attain categoricity.

In the quotation above, Hilbert takes the axioms to come from the data set. But this raises a subtle issue about what comprises the data set. For examples such as geometry and number theory, it was taken for granted that there was a unique model. In one sense this reflects a model-centric view. But even Hilbert (Blanchette's representative of the deductivist view) adds his completeness axiom to guarantee categoricity and to connect with the real numbers. So one could certainly argue that the early 20th century axiomatizers took categoricity as part of the data[18]. But is it essential? Can one obtain the first order data set without making second order assumptions?

Hallett (page 429 of [Hil04]) formulates this issue in words that fit strikingly well in the 'descriptive axiomatization' framework, "Hilbert's system with the Vollständigkeit is complete with respect to 'Cartesian' geometry'." But by no means is Cartesian geometry[19] a part of Euclid's data set.

3 Descriptions of the Geometric Continuum

In the first subsection, we distinguish the 'geometric continuum' from the set theoretic continuum. In Section 3.2 we sketch the background shift from the study of various types of magnitudes by the Greeks, to the modern notion of a collection of real numbers which are available to measure any sort of magnitude. In the third subsection we set out various data sets for 'plane geometry' and discuss the distinctions among them. In the remainder of the paper we will analyze axiomatizations for each data set.

3.1 Conceptions of the continuum

In this section, we motivate our restriction to the *geometric* continuum, we defined it as a linearly ordered structure that is situated in a plane. Sylvester[20] describes the three divisions of mathematics:

> There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits

[18]In fact Huntington invokes Dedekind's postulate in his axiomatization of the complex field in the article quoted above [Hun11].

[19]See [Bal14a].

[20]As quoted in [Mat92].

> of being referred; these are the three cardinal notions, of Number, Space and Order.

This is a slightly unfamiliar trio. We are all accustomed to the opposition between arithmetic and geometry. While Newton famously founded the calculus on geometry (see e.g. [DA11]) the 'arithmetization of analysis' in the late 19th century reversed the priority. From the natural numbers the rational numbers are built by taking quotients and the reals by some notion of completion. And this remains the normal approach today. We want here to consider reversing the direction again: building a firm grounding for geometry and then finding first the field and then some completion and considering incidentially the role of the natural numbers. In this process, Sylvester's third cardinal notion, order, will play a crucial role. In the first section, the notion that one point lies between two others will be fundamental and an order relation will naturally follow; the properties of space will generate an ordered field and the elements of that field will be numbers (but definitely not the set of natural numbers).

We here argue briefly that there is a problem: there are different conceptions of the continuum (the line); hence different axiomatizations may be necessary to reflect these different conceptions. These different conceptions are witnessed by such collections as [Ehr94, SE92] and further publications concerned with the constructive continuum and various non-Archimdean notions of the continuum.

Feferman [Fef08] lists six[21] different conceptions of the continuum: (i) the Euclidean continuum, (ii) Cantor's continuum, (iii) Dedekind's continuum, (iv) the Hilbertian continuum, (v) the set of all paths in the full binary tree, and (vi) the set of all subsets of the natural numbers. For our purposes, we will identify ii), v), and vi) as essentially cardinality based as they have lost the order type imposed by the geometry; so, they are not in our purview. We want to contrast two essentially geometrically based notions of the continuum: those of Euclid and Hilbert. And we identify Dedekind's and Hilbert's conceptions for reasons described in [Bal14a].

We began by stipulating that by 'geometric continuum', we meant the line situated in the plane. One of the fundamental results of 20th century geometry is that *any* (projective[22] for convenience) plane can be coordinatized by a 'ternary field'. A ternary plane is a structure with one ternary function $f(x, y, z)$ such that f has the properties that $f(x, y, z) = xy + z$ would have if the right hand side were interpreted in a field. In accord with our concerns with Euclidean geometry here, we assume the axioms of congruence and the parallel postulate; this implies the ternary field is actually a field. But these geometric hypotheses are necessary. In [Bal94], I constructed an $\aleph_1$-categorical projective plane where the ternary field is a wild as possible (in the precise sense of the Lenz-Barlotti classification [Yaq67]).

[21] Smorynski [Smo08] notes that Bradwardine already reported five in the 14th century.

[22] That is, any system of points and lines such that two points determine a line, any two lines intersect in a point, and there are 4 non-collinear points.

3.2 Ratio, magnitude, and number

In this section we give a short review of Greek attitudes toward magnitude and ratio as described for example in [Mue06, Euc56, Ste90]. We by no means follow the 'geometric algebra' interpretation decried in [GG09]. We attempt to contrast the Greek meanings of propositions with Hilbert's understanding. When we rephrase a sentence into algebraic notation we try to make clear this is a modern formulation, not the intent of Euclid.

Euclid develops arithmetic in chapters VII-IX. What we think of as the 'number' one, was the unit: a number (Definition VII.2) is a multitude of units. These are counting numbers. So from our standpoint (considering the unit as the number 1) Euclid's numbers (in the arithmetic) can be thought of as the 'natural numbers'. The numbers[23] are a discretely ordered collection of objects.

Following Mueller[24] we work from the interpretation of magnitudes in the Elements as "abstractions from geometric objects which leave out of account all properties of those objects except quantity" : length of line segments, area of plane figures, volume of solid figures etc. Mueller emphasizes the distinction between the properties of proportions of magnitudes developed in Chapter V and those of number in Chapter VII. The most easily stated is implicit in Euclid's proof of Theorem V.5; for every m, every magnitude can be divided in m equal parts[25]. This is of course, false for the (natural) numbers.

There is a second use of 'number' in Euclid. It is possible to count unit magnitudes, to speak of, e.g. four copies of a unit magnitude. So (in modern language) Euclid speaks of multiples of magnitudes by positive integers. See Remark 4.3.3 where we give a modern mathematical interpretation of this usage.

Magnitudes of the same type are also linearly ordered and between any two there is a third.[26] Multiplication of line segments yielded rectangles. Ratios are not objects; equality of ratios is a 4-ary relation between two pairs of homogenous magnitudes[27]. Here are four important definitions or propositions from Chapter V of Euclid.

Definition 3.2.1. *1. Definition V.4 of Euclid [Euc56] asserts: Magnitudes are said to* have a ratio *to one another, which are capable, when multiplied, of exceeding one another.*

2. Definition V.5 defines 'sameness of two ratios' (in modern terminology): The

[23] More precisely, natural numbers greater than 1.

[24] page 121 of [Mue06].

[25] page 122 of [Mue06].

[26] The Greeks accepted only potential infinity. So, while from a modern perspective, the natural numbers are ordered in order type ω, and any collection of homogeneous magnitudes (e.g. areas) are in a dense linear order (which is necessarily infinite); this completed infinity is not the understanding of the Greeks.

[27] Homogeneous pairs means magnitudes of the same type. Ratios of numbers are described in Chapter VII while ratios of magnitudes are discussed in Chapter V.

ratio of two magnitudes x and y are proportional to the ratio of two others z, w if for each m, n, $mx > ny$ implies $mz > nw$ (and also if $>$ is replaced by $=$ or $<$).

3. *Definition V.6 says, Let magnitudes which have the same ratio be called proportional.*

4. *Proposition V.9 asserts that 'same ratio' is, in modern terminology, a transitive relation. Apparently Euclid took symmetry and reflexivity for granted and treats proportional as an equivalence relation.*

We now contrast Euclid's notion of proportionality with the (distinct) segment multiplications of Descartes and Hilbert. Bolzano[28] objects to 'dissimilar objects' found in Euclid and finds Euclid's approach fundamentally flawed.

Remark 3.2.2 (Bolzano's Challenge)**.**

> *Firstly triangles, that are already accompanied by circles which intersect in certain points, then angles, adjacent and vertically opposite angles, then the equality of triangles, and only much later their similarity, which however, is derived by an atrocious detour [ungeheuern Umweg], from the consideration of parallel lines, and even of the area of triangles, etc.! (1810, Preface)*

Bolzano's 'atrocious detour' has two aspects: a) the evil of using two dimensional concepts to understand the line[29] and b) the long path to similarity[30]. In VI.1, using the technology of proportions from chapter V, Euclid determines the area of a triangle or parallelogram; in VI.2, he uses these results to show that similar triangles have proportional sides. The role of the theory of proportion is to show that the area of two parallelograms whose respective base and top are on the same parallel lines (and so the parallelograms have the same height) have proportionate areas even if the bases are incommensurable[31]. From VI.2, he constructs in VI.12 the fourth proportional to three lines.

Descartes defines the multiplication of line segments to give another segment[32], but he is still relying on Euclid's theory of proportion to justify the multiplication. Hilbert's innovation is use to segment multiplication to gain the notion of proportionality.

[28]This quotation is taken from [Fra13].
[29]In contrast, we take such concepts as fundamental in understanding the *geometric* continuum.
[30]Section 4 reports how Hilbert avoids this detour.
[31]See the diagram in Remark 4.6.6.
[32]He refers to the construction of the fourth proportional ('ce qui est meme que la multiplication' [Des54]). See also Section 21 page 296 of [Bos01].

Definition 3.2.3 (Proportionality)**.** *We write the ratio of CD to CA is proportional to that of CE to CB,*

$$CD : CA :: CE : CB$$

which is defined as

$$CD \times CB = CE \times CA.$$

where $\times$ is taken in the sense of segment multiplication as in Definition 4.3.2.

Now (*) (from the first page) (VI.1) is interpreted as a variant of (**):

$$F(ABC) = \frac{1}{2}\alpha \cdot AB \cdot AC.$$

Here $F(ABC)$ is an area function satisfying the properties discussed in Definition 4.6.5. But the cost is that while Euclid does not specify what we now call the proportionality constant while Hilbert must. As we'll see in Definition 4.6.5, Hilbert assigns a proportionality constant (in this case the constant α is one).

In his proof of VI.1 (our *) Euclid applies Definition V.5 (Definition 3.2.1) to deduce the proportionality of the area of the triangle to its base. But this assumes that any two lengths (or any two areas) have a ratio in the sense of Definition V.4. This is an implicit assertion of Archimedes axiom for both area and length[33]. As expounded in Section 4, Hilbert's treatment of area and similarity has no such dependence. It is widely understood[34] that Dedekind's analysis is radically different from that of Eudoxus. A principle reason for this, discussed in [Bal14a], is that Eudoxus applies his method to specific situations; Dedekind demands that every cut be filled. Secondly, Dedekind develops addition and multiplication on the cuts. Thus, *Dedekinds's postulate should not be regarded as part of the Euclidean data set*.

Remark 3.2.4 (Naming $0, 1$)**.** Hilbert shows the multiplication on segments of a line through points $0, 1$ satisfies the semi-field axioms[35]. Hilbert has defined segment multiplication on the ray from 0 through 1. But to get negative real numbers he must reflect through 0. Then addition and multiplication can be defined on directed segments of the line through $0, 1$[36] and thus all axioms for a field are obtained. The next step is to identity the points on the line and the domain of an ordered field by mapping A to

[33] Euclid's development of the theory of proportion and area requires the Archimedean axioms. Our assertion is one way of many descriptions of the exact form and location of the dependence among such authors as [Euc56, Mue06, Ste90, Fow79, Smo08]. Since our use of Euclid is as a source of sentences, not proofs, this reliance is not essential to our argument.

[34] Stekeler-Weithofer [SW92] writes, "It is just a big mistake to claim that Eudoxus's proportions were equivalent to Dedekind cuts." Feferman [Fef08] avers, "The main thing to be emphasized about the conception of the continuum as it appears in Euclidean geometry is that the general concept of set is not part of the basic picture, and that Dedekind style continuity considerations of the sort discussed below are at odds with that picture." Stein [Ste90] gives a long argument for at least the compatibility of Dedekind's postulate with Greek thought "reasons ... plausible, even if not conclusive- for believing the Greek geometers would have accepted Dedekinds's postulate, just as they did that of Archimedes, once it had been stated."

[35] In a semi-field there is no requirement of an additive inverse. See Definition 4.3.2.

[36] Hilbert had done this in lecture notes in 1894 [Hil04].

OA. This naturally leads to thinking of a segment as a set of points, which is foreign to both Euclid and Descartes. Although in the context of Grundlagen, Hilbert is aiming for coordinatizing planes by the real numbers, his methods open the path to thinking of the members of any field as 'numbers' that coordinatize the associated geometries. We will discuss this shift in [Bal14a].

In [Bal14a] we go further and study (as Dedekind's or Birkhoff's postulates demand) the identification of a straight line segment of the same length as the circumference of a circle. But this contradicts the 4th century view of Eutocius[37], 'Even if it seemed not yet possible to produce a straight line equal to the circumference of the circle, nevertheless, the fact that there exists some straight line by nature equal to it is deemed by no one to be a matter of investigation.'

3.3 Some geometric Data sets

We begin by distinguishing a number of topics in geometry; these represent distinct data sets in Detlefsen's sense. We distinguish two data sets from the elements: polygonal geometry [38] and circle geometry[39] The relevant propositions are specified in Notation 4.2.2 and Theorem 4.2.3 spells out an appropriate axiom system for each topic. Now we outline some specific theorems, will be addressed below, that might be thought to depend on the continuity axioms.

Remark 3.3.1. *1. Euclid I*

(a) Similar triangles have proportional sides (Theorem 4.5.2)

(b) Area of polygons Section 4.6

(c) Pythagorean theorem Theorem 4.4.2

(d) laws of sines and cosine Theorem 4.4.2

2. Euclid II

(a) Euclid 3: circle intersection (Postulate4.2.1

(b) Construction of an equilateral triangle. Euclid I.1

We deal in detail below with Euclid I; the crucial point is that the arguments in Euclid go through the theory of area which depends on Eudoxus and so have an implicit dependence on the Archimedean axiom; Hilbert eliminates this dependence.

[37] Archimedis Opera Omnia cum commentariis Eutociis, vol. 3, p. 266. Quoted in: Davide Crippa (Sphere, UMR 7219, Universit Paris Diderot) Reflexive knowledge in mathematics: the case of impossibility. See his thesis [Cri14].

[38] Roughly speaking,The Elements: Chapter I (except I.1 and (I.22), chapter II, III except for III.1 and III.17, chapter VI. See a more careful description in Theorem 4.2.3.

[39] The Elements: Chapter IV.

The role of Euclid II appears already in Proposition I of Euclid[40] where Euclid makes the standard construction of an equilateral triangle on a given base. Why do the two circles intersect? While some[41] regard the absence of this axiom as a gap in Euclid, Manders (page 66 of [Man08]) asserts: 'Already the simplest observation on what the texts do infer from diagrams and do not suffices to show the intersection of two circles is completely safe[42].' For our purposes, here we are content to accept that adopting the circle-circle intersection axiom resolves those continuity issues around circles and lines[43]. We separate this case because Hilbert's first order axioms do not resolve this issue[44]; he chooses to resolve it (implicitly) by an appeal to Dedekind[45].

Showing a particular set of axioms is descriptively complete is inherently empirical. One must check whether each of a certain set of results is derivable from a given set of axioms. Hartshorne [Har00] carried out this project without using this language and we organize his results in Theorem 4.2.3.

4 Axiomatizing the geometry of polygons and circles

In the first section we contrast the goal here of an independent basis for geometry with the 19th century arithmetization project. The second section lays out the first order axioms that will be employed. Section 4.3 sketches Hilbert's definition of a field in a geometry. Section 4.4 distinguishes the role of the circle-circle intersection axiom and notices that a number of problems that can be approached by limits have uniform solutions in any ordered field; completeness of the field is irrelevant. We then return to Bolzano's challenge and derive first, Section 4.5, the properties of similar triangles and then, Section 4.6, the area of polygons.

[40] `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI1.html`

[41] Veblen [Veb14], page 4

[42] Manders develops the use of diagrams as a coherent mathematical practice; Avigad and others [ADM09] have developed the idea of formalizing a deductive system which incorporates diagrams. Here is a rough idea of this program. Properties that are *not* changed by minor variations in the diagram such as subsegment, inclusion of one figure in another, two lines intersect, betweenness are termed *inexact*. Properties that *can be* changed by minor variations in the diagram, such as whether a curve is a straight line, congruence, a point is on a line, are termed *exact*. We can rely on reading inexact properties from the diagram. We must write exact properties in the text. The difficulty in turning this insight into a formal deductive system is that, depending on the particular diagram drawn, after a construction, the diagram may have different inexact properties. The solution is case analysis but bounding the number of cases has proven difficult.

Although I agree with the approach of Manders, Avigad et al, or Miller [Mil07], the goal of this paper entails comparison with the axiom systems of Hilbert and Tarski. Reformulating those systems via proof systems formally incorporating diagrams would not affect the specific axioms addressed in this paper.

[43] Circle-circle intersection implies line-circle intersection. Hilbert already in [Hil71] shows (page 204-206 of [Hil04]) that circle-circle intersection holds in what we call a Euclidean plane. See Section 4.4.

[44] Hilbert is aware of this and of the alternative discussed here.

[45] Moore suggests in [Moo88] that Hilbert may have added the completeness axiom to the second edition specifically because Sommer in his review of the first edition pointed out it did not prove the line-circle intersection principle.

4.1 From Arithmetic to geometry or from geometry to algebra?

On the first page of *Continuity and the Irrational Numbers*, Dedekind writes:

> Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful from the didactic standpoint ... But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny.

I have no intention of denying that claim. I quote this passage to indicate that Dedekind's motivation was to provide a basis for calculus not geometry. But I will argue that the second order Dedekind completeness axiom is not needed for the geometry of Euclid and indeed for the grounding of the algebraic numbers, although it is for Dedekind's approach.

Dedekind provides a theory of the continuum (the continuous) line by building up in stages from the structure that is fundamental to him: the natural numbers under successor. This development draws on second order logic in several places. The well-ordering of the natural numbers is required to define addition and multiplication by recursion. Dedekind completeness is a second appeal to a second order principle. Perhaps in response to Bolzano's insistence, Dedekind constructs the line without recourse to two dimensional objects and from arithmetic. Thus, he succeeds in the 'arithmetization of analysis'.

We proceed in the opposite direction for several reasons. Most important is that we are seeking to ground geometry, not analysis. Further, we would assert that the concept of line arises only in the perception of at least two dimensional space. Dedekind's continuum knows nothing of being straight or breadthless. Hilbert's proof of the existence of the field is the essence of the *geometric continuum*. By virtue of its lying in a plane, the line acquires algebraic properties.

The distinction between the arithmetic and geometric intuitions of multiplication is fundamental. The first is as iterated addition; the second is as scaling or proportionality. The late 19th century developments provide a formal reduction of the second to the first but the reduction is only formal; the intuition is lost. In this paper we view both intuitions as fundamental and develop the second (Section 4.3), with the understanding that development of the first through the Dedekind-Peano treatment of arithmetic is in the background. See Remark 4.3.3 for the connection between the two.

4.2 The geometry of Euclid/Hilbert

We identify two levels of formalization in mathematics. By the Euclid-Hilbert style we mean the axiomatic approach of Euclid along with the Hilbert insight that postulates are implicit definitions of classes of models[46]. By the Hilbert-Gödel-Tarski-Vaught style, we mean that that syntax and semantics have been identified as mathematical objects; Gödel's completeness theorem is a standard tool, so methods of modern model theory can be applied[47]. We will give our arguments in English; but we will be careful to specify the vocabulary and the postulates in a way that the translation to a first order theory is transparent.

Postulate 4.2.1. Circle Intersection Postulate *If from points A and B, circles with radius AC and BD are drawn such that one circle contains points both in the interior of one and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

Notation 4.2.2. We follow Hartshorne[Har00] in the following nomenclature. A *Hilbert plane* is any model of Hilbert's incidence, betweenness[48], and congruence axioms. We abbreviate these axioms by HP. We will write HP5 for these axioms plus the parallel postulate. By the *axioms for Euclidean geometry* we mean HP5 and in addition the circle-circle intersection postulate 4.2.1. We will abbreviate this as EG[49]. By definition, a *Euclidean plane* is a model of EG: Euclidean geometry. We write $\mathcal{E}^2$ for a geometrical axiomatization of the plane over a real closed field (RCF) developed in [Bal14a].

With these definitions we align various subsystems of Hilbert's geometry with certain collections of propositions in Euclidean geometry as spelled out in Section 12 and Sections 20-23 of [Har00]. In addition to these much of theory of proportion in Chapters V and X follows from existence of the field. Chapters VII-IX are number theory and chapters X-XII are solid geometry.

Theorem 4.2.3.

1. *The sentences of Euclid I, polygonal geometry: Chapter I (except I.1 and (I.22), chapter II, III except for III.1 and III.17, chapter VI.) are provable in HP5.*
2. *The additional sentences of Euclid II, circle geometry: Chapter IV, I.1 and I.22, III.1 and III.17 are provable in EG.*

In [BM12] and [Bal13] we give an equivalent set of postulates to EG, which return to Euclid's construction postulates and stress the role of Euclid's axioms (common notions) [50] in interpreting the geometric postulates. For aesthetic reasons we used

[46] The priority for this insight is assigned to such slightly earlier authors as Pasch, Peano, Fano, in works such as [Fre57] as commented on in [Bos93] and chapter 24 of [Gra03].

[47] See [Bal14a, Bal14c] for further explication of this method and [Bal14b] for an application.

[48] These include Pasch's axiom (B4 of [Har00]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[49] In the vocabulary here, there is a natural translation of Euclid's axioms into first order statements. The construction axioms have to be viewed as 'for all– there exist sentences. The axiom of Archimedes is of course not first order. We write Euclid's axioms for those in the original [Euc56] vrs (first order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [ADM09], namely, planes over fields where every positive element as a square root). The latter system builds the use of diagrams into the proof rules.

[50] See `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#cns`

SSS rather than SAS as the basic congruence postulate in those notes. Below we explicitly state the postulates only if it seems essential for the development.

We will frequently switch from syntactic to semantic discussions so we stipulate precisely the vocabulary in which we take the axioms above to be formalized. I freely used defined terms such as collinear, segment, and angle in giving the reading of these relation symbols.

Notation 4.2.4. *The fundamental relations of plane geometry make up the following vocabulary τ.*

1. *two-sorted universe: points (P) and lines (L).*
2. *Binary relation $I(A, \ell)$: Read: a point is incident on a line;*
3. *Ternary relation $B(A, B, C)$: Read: B is between A and C (and A, B, C are collinear).*
4. *quaternary relation, $C(A, B, C, D)$: Read: two segments are congruent, in symbols $\overline{AB} \cong \overline{CD}$.*
5. *6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\angle ABC \cong \angle A'B'C'$.*

4.3 From geometry to segment arithmetic to numbers

We introduce in this section *segment arithmetic* and sketch Hilbert's definition of the (semi)-field of segments with partial subtraction and multiplication. We assume what we called HP5 in Notation 4.2.2. The details can be found in e.g. [Hil71, Har00, Bal13]

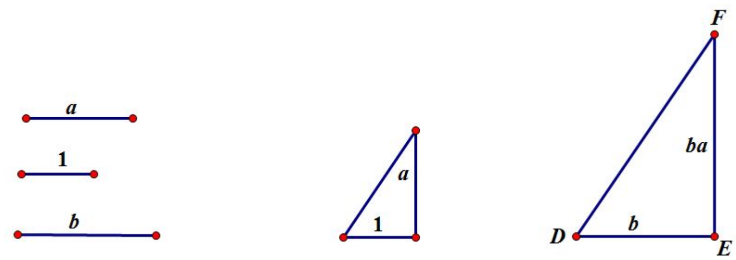
Notation 4.3.1. *Note that congruence forms an equivalence relation on line segments. We fix a ray ℓ with one end point 0 on ℓ. For each equivalence class of segments, we consider the unique segment $0A$ on ℓ in that class as the representative of that class. We will often denote the segment $0A$ (ambiguously its congruence class) by a . We say a segment CD (on any line) has length a if $CD \cong 0A$.*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. This procedure can be extended to a field structure on segments on a line not a ray (so with negatives), either directly as sketched in [BM12] or by passing through the theory of ordered fields as in Section 19 of [Har00]. Following Hartshorne [Har00], here is our official definition of segment multiplication[51].

[51] Hilbert's definition goes directly via similar triangles. The clear association of a particular angle with right muliplication by a recommends Hartshorne's version.

Definition 4.3.2. *[Multiplication] Fix a unit segment class,* 1*. Consider two segment classes* a *and* b*. To define their product, define a right triangle with legs of length* 1 *and* a*. Denote the angle between the hypoteneuse and the side of length* a *by* α*.*

Now construct another right triangle with base of length b *with the angle between the hypoteneuse and the side of length* b *congruent to* α*. The product* ab *is defined to be the length of the vertical leg of the triangle.*



Note that we must appeal to the parallel postulate to guarantee the existence of the point F. It is clear from the definition that there are multiplicative inverses; use the triangle with base a and height 1. The roughly 3 page proof that multiplication is commutative, associative, distributes over addition, and respects the order uses only the cyclic quadrilateral theorem and connections between central and inscribed angles in a circle.

It is easy[52] to check that the multiplication defined on the positive reals by this procedure is exactly the usual multiplication on the positive reals because they agree on the positive rational numbers.

Remark 4.3.3. We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a. It is an easy exercise to show these are the same. But these distinct constructions make an important point. The (inductive) definition of multiplication by a natural number is indeed 'multiplication as repeated addition'. But the multiplication by another field element is based on similarity, implies the existence of multiplicative inverses, and so is a very different operation.

The first notion of multiplication in the last paragraph, where the multiplier is a natural number, is a kind of 'scalar multiplication' by positive integers that can be viewed mathematically as a rarely studied object: a semiring (the natural numbers) acting on a semigroup (positive reals under addition). There is no uniform definition[53] of this *scalar* multiplication within the semiring; multiplication by 17 is defined in the

[52]One has to verify that segment multiplication is continuous but this follows from the density of the order since the addition respects order.

[53]Instead, there are infinitely many formulas $\phi_n(x, y)$ defining $nx = y$ for each $n > 0$.

geometry but not multiplication by -17. A mathematical structure more familiar to modern eyes is obtained by extending to the negative numbers and has a well-defined notion of subtraction, both of the scalars are now in the ring $(Z, +, \cdot)$ act on the module $(\Re, +)$. Now we can multiply by $-\frac{17}{27}$ but the definition is still non-uniform.

4.4 Initial consequences for field arithmetic

In this section we investigate statements from: 1) Euclid's geometry that depended in his development on the Archimedean axiom and 2) Dedekind's development of the properties of real numbers that he deduces from his postulate. In each case, they are true in any field associated with a geometry modeling HP5.

We established in Section 4.3 that one could define an ordered field in any plane satisfying HP5. The converse is routine, the ordinary notions of lines and incidence in F^2 creates a geometry over any ordered field, which is easily seen to satisfy HP5. We now exploit this equivalence.

We will prove some algebraic facts using our defined operations, thus basing them on geometry. We begin with justifying that taking square root commutes with multiplication for algebraic numbers. Dedekind (page 22 of [Ded63]) wrote '... in this way we arrive at real proofs of theorems (as, e.g. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before.'

Note that this is a problem for Dedekind but not for Descartes. Already Euclid, in constructing the fourth proportional, constructs from segments of length 1, a and b, one of length ab; but he doesn't regard this operation as multiplication. When Descartes interprets this procedure as multiplication of segments, he has no problem. But Dedekind has presented the problem as multiplication in his continuum and so he must prove a theorem to find the product as a real number; that is, he must show the limit operation commutes with product. We report Hilbert's equally rigorous but much more simple proof that any field arising from geometry (e.g. the reals) is closed under multiplication (of any segments).

Theorem 4.4.1. *In an ordered field, for any positive a, if there is an element $b > 0$ with $b^2 = a$, then b is unique (and denoted $\sqrt{a}$). Moreover, for any positive a, c with square roots, $\sqrt{a} \cdot \sqrt{c} = \sqrt{ac}$.*

This fact holds for any field coordinatizing a plane satisfying HP5.

Thus, the algebra of square roots in the real field is established without any appeal to limits. The usual (e.g. [Spi80, Apo67]) developments of the theory of complete ordered fields invoke the least upper bound principle to obtain the existence of the roots although the multiplication rule is obtained by the same algebraic argument as here. Our approach (like Hilbert's) contrasts with Dedekind[54]; our treatment is

[54]Dedekind objects to the introduction of irrational numbers by measuring an extensive magnitude in

based on the geometric concepts and in particular regards 'congruence' as an equally fundamental intuition as 'number'. The justification here for either the existence or operations on roots does not invoke limits.

The shift here is from 'proportional segments' to 'product of numbers'. Euclid had a rigorous proof of the existence of a line segment which is the fourth proportional of $1 : a = b : x$. Dedekind demands a product of numbers; Hilbert provides this by a combination of his interpretation of the field in the geometry and geometrical definition of multiplication.

We now consider .1c, and .1d of Remark 3.3.1. Euclid's proof of Pythagoras I.47 uses an area function as we will justify in Section 4.6. His second proof uses the theory of similar triangles that we will develop Section 4.5. In both cases Euclid depends on the theory of proportionality (and thus implicitly on Archimedes axiom) to prove the Pythagorean theorem; Hilbert avoids this appeal[55]. Similarly, since the right angle trigonometry in Euclid concerns the ratios of sides of triangles, the field multiplication justifies basic right angle trigonometry.

Theorem 4.4.2. *The Pythagorean theorem as well as the law of cosines, Euclid II.11 and the law of sines, Euclid II.13 hold in HP5.*

To summarize, we introduce two definitions (details in section 21 of [Har00]):

Definition 4.4.3.

1. *A field F is* Pythagorean *if it is closed under addition, subtraction, multiplication, division and for every $a \in F$, $\sqrt{(1 + a^2)} \in F$.*
2. *A field F is* Euclidean *if it is closed under addition, subtraction, multiplication, division and for every $a \in F$, $\sqrt{a} \in F$.*

Hartshorne [Har00] describes two instructive examples, connecting the notions of Pythagorean and Euclidean planes.

Example 4.4.4.

1. The Cartesian plane over a Pythagorean field may fail to be closed under square root (Exercise 39.30 of [Har00]).
2. On page 146, Hartshorne[56] observes that the smallest ordered field closed under addition, subtraction, multiplication, division and square roots of positive numbers satisfies the circle-circle intersection postulate is a Euclidean field. We denote this field by F_s for surd field.

terms of another of the same kind (page 9 of [Ded63]).

[55]But Hilbert does not avoid the parallel postulate since he uses it establishing multiplication and thus similarity. Note also that Euclid's theory of area depends heavily on the parallel postulate. It is a theorem in 'neutral geometry' in the metric tradition that the Pythagorean Theorem is equivalent to the parallel postulate (See Theorem 9.2.8 of [MP81].). But this approach basically assumes the issues dealt with in this paper as the 'ruler postulate' ([Bir32]) also provides a multiplication on the 'lengths' (since they are real numbers). Thanks to Julien Narboux for pointing out issues in stating the Pythagorean theorem in the absence of the parallel postulate.

[56] Hartshorne and Greenberg [Gre10] calls this the constructible field, but given the many meanings of constructible, we use Moise's term: surd field.

Recall that we distinguished a Hilbert plane from a Euclidean plane in Notation 4.2.2. As in [Har00], we have:

Theorem 4.4.5. *1. (HP5) is biinterpretable[57] with theory of ordered pythagorean planes.*

2. A Hilbert plane satisfies the circle-circle intersection postulate, 4.2.1, (i.e. is a Euclidean plane) if and only the coordinatizing field is Euclidean.

Remark 4.4.6. *Note that if HP5 + CCI were proposed as an axiom set for polygonal geometry it would be a complete descriptive but not modest axiomatization.*

Heron's formula shows a hazard of the kind of organization of data sets attempted here. In every Euclidean plane such that every positive element of the coordinatizing plane has a square root, Heron's formula computes the area of a triangle from the lengths of its sides. But, the geometric proof of Heron doesn't involve the square roots of the modern formula [Hea21]. Since in EG we have the field with square roots, we can prove the modern form of Heron's formula from EG. Thus, as in from (*) to (**), the different means of expressing the geometrical property requires different proofs.

Dicta 4.4.7 (Constants 1)**.** To fix the field we have to add constants 0, 1. These constants can name any pair of points in the plane (since the automorphism group acts two transitively (any pair of distinct points can be mapped by an automorphism to any other such pair) as can be proven in EG). But this naming induces an extension of the data set. We have in fact specified the unit. This reflects a major change in view from either the Greeks or Descartes. In this situation, the change is small. A sentence $\phi(0, 1)$ holds just if either or both of $\forall x \forall y \phi(x, y)$ and $\exists x \exists y \phi(x, y)$ hold since the automorphism group of a Hilbert plane is two transitive.

In each case we have considered in this section, Greeks give geometric constructions for what in modern days becomes a calculation involving the field operations and square roots. We still need to complete the argument that HP5 is descriptively complete for polygonal Euclidean geometry. In particular, is the notion of proportional included in our analysis? The test question is the similar triangle theorem. We turn to that issue now.

4.5 Multiplication is not repeated addition

In the natural numbers, addition can be defined as iterated succession and multiplication as iterated addition. But the resulting structure is essentially undecidable. Moreover, this structure does not illuminate the essential aspect of multiplication as similarity; many elements have no multiplicative inverse.

[57] Formally this means there are formulas in the field language defining the geometric notions (point, line, congruence, etc) and formulas in the geometric language (plus constants) defining the field operations $(0, 1, +, \times)$ such that interpreting the geometric formulas in a Pythagorean field gives a model of HP5 and conversely. See chapter 5 of [Hod93] for the general background on interpretability.

Definition 4.5.1. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g. $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Various texts define 'similar' as we did, or as corresponding sides are proportional or require both (Euclid). We now meet Bolzano's challenge by showing that in Euclidean Geometry (without the continuity axioms) the choice doesn't matter. Recall that we defined proportional in terms of segment multiplication in Definition 3.2.3. Hartshorne proves this fundamental result from the development laid out here on page 175-177 of [Har00].

Theorem 4.5.2. *Two triangles are similar if and only if corresponding sides are proportional.*

There is no assumption that the field is Archimedean or satisfies any sort of completeness axiom. There is no appeal to approximation or limits.

4.6 Area of polygonal figures

In Section 4.5 we saw Bolzano's challenge 3.2.2 is answered by a proof that similar triangles have proportional sides without resorting to the concept of area. But area is itself a vital geometric notion. We show now that using segment multiplication Hilbert grounds the now familiar methods of calculating the area of polygons by multiplication. As Hilbert writes[58], "We ... establish Euclid's theory of area *for the plane geometry and that independently of the axiom of Archimedes.*"

In this section, we sketch Hartshorne's [Har00] exposition of this topic. We stress the connections with Euclid's common notions and are careful to see how the notions defined here are expressible in first order logic; supporting our 5th objection to second order axiomatization in [Bal14a]; although these arguments are not carried out as direct deductions from the first order axioms, the results are derivable by a direct deduction. Here is an informal definition of those configurations whose areas are considered in this section.

Definition 4.6.1. *A* figure *is a subset of the plane that can be represented as a finite union of disjoint triangles.*

There are serious issues concerning the formalization in first order logic of the notions in this section. Notions such as polygon involve quantification over integers; this is strictly forbidden within the first order system. We can approach these notions with axiom schemes[59]. We sketch a uniform metatheoretic definition of the relevant

[58]Emphasis in the original: (page 57 of [Hil71]).

[59]In order to justify the application of the completeness theorem we have to introduce inductively a scheme giving the definition of an n-decomposable figure as the disjoint union of an $(n-1)$-decomposable figure A with a triangle such that a portion of one side of the triangle lies on a portion of one side of the figure A. In [Bal14b], we give such formal definitions to find area and circumference of a circle.

concepts to prove that the theorems hold in all models of the axioms. Hilbert raised a 'pseudogap' in Euclid[60] by distinguishing area and content. In Hilbert two figures have

1. *equal area* if they can be decomposed into a finite number of triangles that are pairwise congruent
2. *equal content* if we can transform one into the other by adding and subtracting congruent triangles.

Euclid treats the equality of areas as a special case of his common notions. The properties of equal content, described next, are consequences for Euclid of the common notions and need no justification.

Fact 4.6.2 (Properties of Equal Content)**.** *The following properties of area are used in Euclid I.35 through I.38 and beyond. They hold for equal content in HP5.*

1. *Congruent figures have the same content.*
2. *The content of two 'disjoint' figures (i.e. meet only in a point or along an edge) is the sum of the two content of the polygons. The analogous statements hold for difference and half.*
3. *If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.*

Observe that while these properties concern 'figure', a notion that is not definable by a single formula in first order geometry, we can replace 'figures' by n-gons for each n. For the crucial proof that the area of a triangle or parallelogram is proportional to the base and the height, we need only 'triangles or quadrilaterals'. In general we could formalize formalize these notions with either equi-area predicate symbols[61] or by a schema and a function mapping into the line as in Definition 4.6.5. Here is the basic step.

Definition 4.6.3. *Two figures α and β (e.g. two triangles or two parallelograms) have equal content in one step there exist figures α' and β' such that the disjoint union of α and α' is congruent to the disjoint union of β and β' and $\beta \cong \beta'$.*

Reading equal content for Euclid's 'equal', Euclid's I.35 (for parallelogram) and the derived I.37 (triangles) become the following. With this formulation Hilbert accepts Euclid's proof.

[60] Any model with infinitessimals shows the notions are distinct and Euclid I.35 and I.36 fail for what Hilbert calls area. Hilbert shows they are equivalent under the axiom of Archimedes. Since Euclid includes preservation under both addition and subtraction in his common notions, his term 'area' clearly refers to what Hilbert calls 'equal content', I call this a pseudogap.

[61] For example, we could have 8-ary relation for quadrilaterals have the same area, 6-ary relation for triangles have the same area and 7-ary for a quadrilateral and a triangle have the same area.

Theorem 4.6.4. *[Euclid/Hilbert] If two parallelograms (triangles) are on the same base and between parallels they have equal content in 1 step.*

Euclid shows the result by adding and subtracting figures, but with a heavy dependence on the parallel postulate to derive properties of parallegrams. See the diagram and proof of Proposition 35 of Euclid (available on line at the url in reference [Euc56]). Varying Hilbert, Hartshorne (Sections 19-23 of [Har00]) shows that these properties of equal content for a notion of figure (essentially a finite nonoverlapping union of triangles) are satisfied in the system EG (Notation 4.2.2). The key tool is:

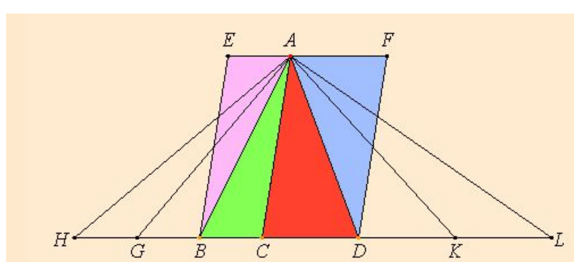
Definition 4.6.5. *An* area function *is a map α from the set of figures, $\mathcal{P}$, into an ordered additive abelian group with* 0 *such that*

1. *For any nontrivial triangle T, $\alpha(T) > 0$.*
2. *Congruent triangles have the same content.*
3. *If P and Q are disjoint figures $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.*

This formulation hides the quantification over arbitrary n-gons. We clarify the method of translating to first order in [Bal14b] by a more detailed discussion in a slightly different context.

It is evident that if a plane admits an area function then the conclusions of Lemma 4.6.2 hold. This obviates the need for positing separately De Zolt's axiom that one figure properly included in another has smaller area[62]. In particular this implies Common Notion 4 for 'area'. Using the segment multiplication, Hilbert (compare the exposition in Hartshorne) establishes the existence of an area function for any plane satisfying HP5. The key point is to show that formula $A = \frac{bh}{2}$ does not depend on the choice of the base and height. Thus, Hilbert proves (**) without recourse to the axiom of Archimedes.

Remark 4.6.6. In contrast, recall the diagram for Euclid's, VI.1.



[62]Hartshorne notes that (page 210 of [Har00]) that he knows no 'purely geometric' (without segment arithmetic and similar triangles) proof for justifying the omission of De Zolt's axiom.

If, for example, BC, GB and HG are congruent segments then the area of ACH is triple that of ABC. But without assuming BC and BD are commensurable, Euclid calls on Definition V.5 of the proportionality chapter to assert that $ABD : ABC :: BD : BD$.

We have now shown that the axioms for Euclidean planes (HP5 + circle-circle intersection) suffice to prove Properties 3.3.1 a to d.

In this paper we defined the notion of a *modest* descriptive axiomatization to emphasize that the primary goal of an axiomatization is to distill what is 'really going on'. One can axiomatize any structure by taking as axioms all the first order sentences true in it; such a choice makes a farce of axiomatizing. Historically, we stress one of Hilbert's key points. As we discussed here, Hilbert eliminater. the use of the Axiom of Archimedes in Euclid's polygonal and circle geometry. He [Hil71] was finding the 'distinguished propositions of the field of knowledge that underlie the construction of the framework of concepts' and showing what we now call first order axioms sufficed. As we elaborate in [Bal14a], Hilbert showed the Archimedean and Dedekind axioms were not needed for 'geometry' but only to base a modern theory of the real numbers on a geometric footing.

In that sequel to this paper, we extend the historical analysis from Euclid and Hilbert to Descartes and Tarski. We explore several variants on Dedekind's axiom and the role of first order, infinitary, and second order logic. Then based on our [Bal14b], we expound a first order basis for the formulas for area and circumference of a circle.

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