Fine Structure of strongly minimal sets with flat geometries
Conference in honor of Bektur Baizhanov, Almaty

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1. Strongly Minimal Theories
2. Quasi-groups and Steiner systems
3. Groups, definable closure, and elimination of imaginaries
4. The General Construction
5. The structure of $\mathrm{acl}(X)$

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Strongly Minimal Theories

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Definition

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e.g. acf, vector spaces, successor

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a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Theorem

If T is strongly minimal algebraic closure defines matroid/combinatorial geometry.

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. 'bi-interpretable' with an algebraically closed field (non-locally modular)

Zilber: geometries $\leftrightarrow$ canonical structures

Hrushovski gave a method of constructing strongly minimal sets that have flat geometries and admit no associative binary function.

There is no apparent canonical structure - only a (very flexible) method.
Zariski Geometries aim at canonical structures with more restrictions.

Baizhanov's Question

Question (1990's)

Does every strongly minimal set that admits elimination of imaginaries interpret an algebraically closed field?

Partial Answer

1. Infinite language: No! Verbovskiy
2. finite language:
    1. Yes! for constructions of [Hru93, BP20].
    2. A program for other flat geometries

The diversity of flat strongly minimal sets

The ‘Hrushovski construction’ actually has 5 parameters:

Describing Hrushovski constructions

1. σ: vocabulary
2. $\boldsymbol{L}_0$: A univerally axiomatized collection of finite σ-structures. (But generalizing to $\forall\exists$ is useful.)
3. ϵ: A submodular (hence flat) function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$.
4. $\boldsymbol{L}_0$: $\boldsymbol{L}_0^*$ defined using ϵ.
5. μ: a function bounding the number of 0-primitive extensions of an $A \in \boldsymbol{L}_0$ are in L_μ.

To organize the classification of the theories each choice of a class **U** of μ yields a collection of T_μ with similar properties.

Quasi-groups and Steiner systems

Definitions

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$ and allow infinite n.

Some History

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
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Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939) Skolem ($6n + 1$, 1958)

Linear Spaces

Definition: linear space

The vocabulary contains a single ternary predicate R, interpreted as collinearity. A linear space satisfies

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

α is the iso type of $(\{a, b\}, \{c\})$ where $R(a, b, c)$.

Groupoids and semigroups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

existentially closed Steiner Systems

Barbina-Casanovas

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are the graphs of a Steiner quasigroup.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T^*_{sq}.
2. T^*_{sq} has
    1. quantifier elimination
    2. $2^{\aleph_0}$ 3-types;
    3. the generic model is prime and locally finite;
    4. T^*_{sq} has TP_2 and $NSOP_1$.

Hrushovki's basic construction vs Steiner

Example

1. σ has a single ternary relation R;
2. $\boldsymbol{L}_0$: All finite σ-structures
   finite linear spaces
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
   $\delta(A) = |A| - \sum_{\ell \in L(A)}(|\ell| - 2)$.
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
   Replace ϵ by δ.
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.
   $\mu(\boldsymbol{\alpha}) = q - 2$ gives line length 2.

Strongly minimal linear spaces I

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k.

Specific Strongly minimal Steiner Systems

Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(2, k, \infty)$ Steiner-systems.

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Groups, definable closure, and elimination of imaginaries

This section is about arbitrary strongly minimal theories not just Hrushovski constructions.

Group Action and Definable Closure

Fix I, a finite set of independent points in the model $M \models T$.

2 groups

Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

Definition

1. $\mathrm{dcl}^*(I)$ consists of those elements that are fixed by G_I but not by G_X for any $X \subsetneq I$.
2. The *symmetric definable closure* of I, $\mathrm{sdcl}^*(\mathrm{I})$, consists of those elements that are fixed by $G_{\{I\}}$ but not by $G_{\{X\}}$ for any $X \subsetneq I$.

$\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ implies T does not admit elimination of imaginaries.

Finite Coding

Definition

A finite set $F = \{\bar{a}_1, \ldots, \bar{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples F is coded by some set S over $\emptyset$.

If there exists I with $\mathrm{dcl}^*(I) = \emptyset$, T does not have the finite set property.

dcl^* and elimination of imaginaries

Fact: Elimination of imaginaries

A theory T admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no

Fact

If T admits weak elimination of imaginaries then T satisfies the finite set property if and only T admits elimination of imaginaries.

Since every strongly minimal theory weak elimination of imaginaries.

If a strongly minimal T has only essentially unary definable binary functions it does not admit elimination of imaginaries.

No definable binary function/elimination of imaginaries: Sufficient

Lemma

Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and M is a generic model of a strongly minimal theory.

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then I is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter free definable binary function.*

‘Non-trivial definable functions’

Definition

Let T be a strongly minimal theory. function $f(x_0 \ldots x_{n-1})$ is called *essentially unary* if there is an $\emptyset$-definable function $g(u)$ such that for some i, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.

Lemma

For a strongly minimal T the following conditions are equivalent:

1. for any $n > 1$ and any independent set $I = \{a_1, a_2, \ldots a_n\}$, $\mathrm{dcl}^*(I) = \emptyset$;
2. every $\emptyset$-definable n-ary function ($n > 0$) is essentially unary;
3. for each $n > 1$ there is no $\emptyset$-definable truly n-ary function in any $M \models T$.

The main result: Classifying dcl [BV21]

Theorem

Let T_μ be a strongly minimal theory as in Hrushovski's original paper. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B)\}$). Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

G_I If T_μ triples

$$\mathcal{U} \supseteq \mathcal{T} = \{\mu : \mu(A/B) \geq 3\}$$

then $\mathrm{dcl}^*(I) = \emptyset$
$\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$
and every definable function is essentially unary (Definition 18).

$G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$
$\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$
and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly v-ary function.

In both cases T_μ does not admit elimination of imaginaries and the algebraic closure geometry is not disintegrated.

The General Construction

Amalgamation and Generic model

We study classes $\boldsymbol{K}_0$ of finite structures A
with $\delta(A') \geq 0$, for every $A' \subset A$.
$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

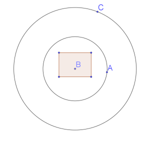
$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

α is the isomorphism type of $(\{a, b\}, \{c\})$,

Overview of construction

Realization of good pairs

1. A good pair C/B *well-placed* by A in a model M, if $B \subseteq A \leq M$ and C is 0-primitive over X.
2. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.
3. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).

If C/B is well-placed by $A \leq M$, $\chi_M(B, C) = \mu(B/C)$

The structure of $\mathrm{acl}(X)$

G-decomposable sets

Definition

$\mathcal{A} \subseteq M$ is G-decomposable if

1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is G-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

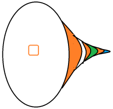
Fact

There are G-decomposable sets.
Namely for any finite U with $d(U/I) = 0$,

$$\mathcal{A} = \mathrm{icl}(\mathrm{I} \cup \mathrm{G}(\mathrm{U}))$$

Constructing a *G*-decomposition

Linear Decomposition



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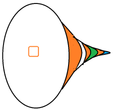
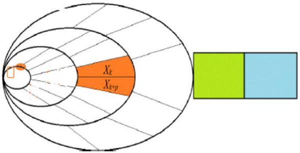


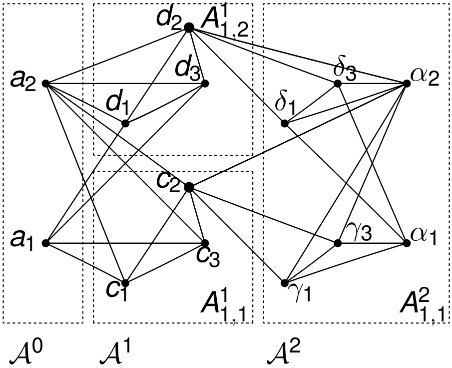
Tree Decomposition



Prove by induction on levels that $\mathrm{dcl}^*(I) = \emptyset$. $(\mathrm{sdcl}^*(\mathrm{I}) = \emptyset)$

A non-trivial definable binary function

In the diagrams, we represent a triple satisfying R by a triangle.



Conclusion

Strongly minimal theories with non-locally modular algebraic closure

1. Diversity
    1. $2^{\aleph_0}$ theories of strongly minimal Steiner systems (M, R) with no $\emptyset$-definable binary function
    2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + an example of Hrushovski
    3. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
    4. 2-ample but not 3-ample sm sets (not flat) [MT19]
    5. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)

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2. Classifying
   1. discrete
   2. non-trivial but no binary function
   3. non-trivial but no commutative binary function
   4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]

Combinatorial connections

Unlike many construction in infinite combinatorics these methods give a family of infinite structures with similar properties [Bal21a, Bal21b]. Among the properties investigated are:

1. cycle graphs in 3-Steiner systems [CW12] generalized to paths in Steiner k-system;
2. preventing or demanding 2-transitivity
3. sparse Steiner systems: forbidding specific configurations [CGGW10]

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