Problems on 'Pathological' Structures

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July 29, 1997

The following three major results of Hrushovski create a new area for model theoretic studies.

0.1 Theorem. *There exist nontrivial strongly minimal sets which do not interpret a group.*

0.2 Theorem. *Any two reasonable strongly minimal sets can be amalgamated to form a new strongly minimal set.*

0.3 Theorem. *There exist countable stable and* $\aleph_0$*-categorical structures which are not* ω*-stable.*

The first and second examples strongly refuted Zil'ber's conjecture that the 'known' examples of strongly minimal sets exhausted the class. The third answered a similar question of Lachlan limiting the variety of stable theories. One direction of research, pursued with great success by Zil'ber and Hrushovski [14], [19], is to further refine the notion of strongly minimal set to see which strongly minimal sets are closely connected to such classical fields of study as algebraic geometry and complex manifolds. Another is to explore the realm of the more paradoxical examples. A general method of construction underlies all three of these results. We will describe this method

*Partially supported by NSF grant 90000139. This article represents an extension of talks by the author at MAMLS meeting at CMU on April 3, 1993 and at the 1993 Easter Model Theory Conference in Berlin.

below and indicate a number of test questions to determine the limits of the method. These questions are stimulated by a number of papers on the subject: [6], [1], [13], [12], [9], [15], [18] [17], [3].

1 Generic Structures

Let A be a structure for a finite language (for simplicity with one relation R). Then $e(A)$ is the number of tuples in A that satisfy R. Fix α, $0 < \alpha \leq 1$. Define

$$\delta_\alpha(A) = |A| - \alpha \cdot e(A)$$

For any L-structure N,

$$d_N(A) = \inf\{\delta(B) : A \subseteq B \subseteq N\}.$$

Define A is a strong substructure of N if $d_N(A) = \delta(A)$. The following axioms describe the properties of this notion of strong substructure.

Let $\mathbf{K_0}$ be a collection of finite relational structures. $\mathbf{K}$ is a class of structures whose finite members are in $\mathbf{K_0}$.

1.1 Axiom Group A.

A1. If $M \in K_0$ then $M \leq M$.

A2. If $M \leq N$ then $M \subseteq N$.

A3. If $A, B, C \in K_0$, then $A \leq B \leq C \Longrightarrow A \leq C$.

A4. If $A, B, C \in K_0$, $A \leq C$, $B \subseteq C$ and $A \subseteq B$ then $A \leq B$.

A5. $\emptyset \in \mathbf{K_0}$ and $\emptyset \leq A$ for all $A \in \mathbf{K_0}$.

1.2 Axiom A6. Let $A, B, C \in K_0$. If $A \leq B$, $C \subseteq B$, and $A \bigcap C \in \mathbf{K_0}$ then $A \bigcap C \leq C$.

Previous discussion of this formulation (e.g. in [6]) assumed that in Axiom A6, $A \bigcap C \in K_0$. This requires that $\mathbf{K_0}$ is a universal class. This condition is both unnecessary [15] and overly narrows the situation considered [7].

1.3 Definition. The countable model M is $(\mathbf{K_0}, \leq)$-*generic* if

i) If $A \leq M, A \leq B \in \mathbf{K_0}$, then there exists $B' \leq M$ such that $B \cong_A B'$,

ii) M has finite closures.

Obviously, if a class satisfies A5 and the amalgamation property then $(\mathbf{K_0}, \leq)$ has JEP.

It is easy to see (assuming A1-A5) that if the model M is $(\mathbf{K_0}, \leq)$-*generic* then M is ω-$\mathbf{K_0}$ universal and ω-$\mathbf{K_0}$ homogeneous. We have the following slight generalization of the Jónsson-Fraissé construction.

1.4 Theorem. *If $(\mathbf{K_0}, \leq)$ is a collection of finite relational structures that satisfy A1-A5 and has the amalgamation property then there is a countable $\mathbf{K_0}$-generic model M.*

1.5 Assumption. Henceforth $\mathbf{K_0}$ is a collection of finite structures satisfying A1 through A6 that has the amalgamation property. $\mathbf{K}$ is the class of substructures of models of the theory of the generic model.

1 Question. *What is the most suitable framework for considering this construction. The stability properties of the theory cannot be expressed solely in terms of $(\mathbf{K}, \leq)$. Either the dimension function must be mentioned explicitly or the properties of a stable amalgamation should be part of the framework as in [5]*

2 Question. *Find a rubric which includes both algebraic examples (straightforward) and the Hrushovski method of amalgamating strongly minimal sets (not straightforward).*

3 Question. *What is a generic structure? Are there properties of the structure M which guarantee that it is constructed in the way discussed in this section?*

The work of [15] is a start on this question.

2 Variations: ω-stable examples

There are many examples depending on the variation of two parameters: α and $\mathbf{K_0}$. In this section we consider various rational choices for α and some possible ways to vary to $\mathbf{K_0}$ for given α.

Let $\mathbf{K'}$ be the collection of all finite structures for the finite relational language L.

I. $\delta(A) = |A| - e(A)$. $\mathbf{K_0} = \{A \in \mathbf{K}' : \delta(A') \geq 0 \text{ if } A' \subseteq A\}$. The theory of the generic is ω-stable of infinite Morley rank.

The following definition is needed to control the rank of the generic structure. We consider it here for the special case of graphs.

2.1 Definition. Let AB be a finite graph.

i) A is 0-*simply algebraic* over B if $A \cap B = \emptyset$, $y(A/B) = 0$ and for every C properly contained in A, $y(C/B) > 0$.

ii) A is *minimally* 0-*simply algebraic* over B if A is 0-simply algebraic over B and A is not 0-simply algebraic over any proper subset of B.

iii) Fix a function μ from pairs A/B where A is minimally 0-simply algebraic over$B$ into $\omega \cup \{\infty\}$.

Here is the original Hrushovski example.

II. $\delta(A) = |A| - e(A)$. $\mathbf{K_0}$ is further restricted by requiring $\mu(A/B) < \infty$ for each A/B. The theory of the generic is strongly minimal. It is 'new' if L has a ternary relation.

III. $\delta(A) = |A| - 1/2 \cdot e(A)$. $\mathbf{K_0} = \{A \in \mathbf{K}' : \delta(A') \geq 0 \text{ if } A' \subseteq A\}$. The theory of the generic is an ω-stable projective plane of infinite Morley rank.

IV. $\delta(A) = |A| - 1/2 \cdot e(A)$. $\mathbf{K_0}$ is further restricted by requiring $\mu(A/B) < \infty$ for each A/B. The theory of the generic is an almost strongly minimal non-Desarguesian projective plane ([1]). This construction is studied further in [2].

V. Further restricting $\mathbf{K_0}$ in IV by omitting all graphs A with $|A| > 3$ and $\delta(A) \leq 3$ produces an almost strongly minimal non-Desarguesian projective plane that is in the definable closure of any line (without parameters!) [2]

4 Question. *Find an explanation in terms of the projective geometry for this phenomena.*

The last few examples were projective planes. Each of them is constructed from a graph in the following way. Each plane M is derived from a graph $M^* = (M^*, R)$. Formally this transformation sets $P = M^* \times \{0\}$, $L = M^* \times \{1\}$ and $\langle m, 0\rangle$ is on $\langle n, 1\rangle$ if and only if $R(m, n)$. Thus, if G^* denotes the automorphism group of M^*, there is a natural injection of G^* into the automorphism (collineation) group G of M. There is a natural polarity ρ of M ($\rho(\langle m, i\rangle) = \langle m, 1-i\rangle$ for $i = 0, 1$). Then G^* is naturally regarded as the subgroup of G consisting of those collineations that are automorphisms of the structure (M, P, L, I, ρ).

5 Question. *Is ρ ever (always?) definable in M?*

VI. $\delta(A) = |A| - 1/2 \cdot e(A)$. If any value of μ is allowed to be infinite the plane has infinite rank.

6 Question. *How many countable models do the examples in VI have? How do these theories fit into the Shelah classification?*

Each of these examples will produce at least one 'flat' strongly minimal set.

7 Question. *Develop a general theory for the classification of strongly minimal sets. In particular, find model theoretic criteria for distinguishing among these examples (or give a sense in which they are all the same).*

The only serious work in this direction that I know of is due to Kitty Holland [10]; here are a couple of her results. Let T be the theory of a strongly minimal set and $\mathcal{M}$ a large saturated model of T. For any model N of T, $L(N)$ denotes the lattice of elementary submodels of $\mathcal{M}$ that contain N. Let $\mathcal{P}$ be the prime model of T.

2.2 Theorem. *$L(\mathcal{M})$ is modular if and only if for every N, $L(N) \approx L(\mathcal{P})$.*

2.3 Theorem. *$L(\mathcal{M})$ is not modular if and only if for every N, N' which have infinite dimension over* $\mathrm{acl}(\emptyset)$, *$L(N) \approx L(N')$ if and only if $N \approx N'$.*

8 Question. *Is the requirement that N and N' have infinite dimension over* $\mathrm{acl}(\emptyset)$ *essential for the last result?*

Holland [11] has shown this assumption is not necessary for the theory of algebraically closed fields. But that proof uses a lot of algebraic information.

Baudisch [7] has found a more complicated algebraic application of this construction.

VII. There is an $\aleph_1$-categorical two-step nilpotent groups that is not Abelian by finite and does not interpret a field. (Here K_0 is a class of finite groups (with a bilinear map) and δ is defined in terms of the dimension of certain related groups.)

3 Variations: strictly stable examples

In this section we will discuss several different strictly stable theories constructed by this method. They are all variants on the original example of Hrushovski [12].

VIII. $\delta(A) = |A| - \alpha \cdot e(A)$ where α is a suitably chosen irrational number between $1/2$ and $2/3$. $\mathbf{K_0}$ is carefully chosen on the basis of rational approximations to α. The theory of the generic is $\aleph_0$-categorical and strictly stable. Thus, Lachlan's conjecture is refuted.

IX. (Herwig) There is a small stable theory which has a type with preweight ω with respect to itself. This example is a limit over a sequence of finite languages of Example VIII.

9 Question. *The proof of the amalgamation in these examples depends upon a approximation sequence for the irrational α. Is there any model theoretic significance to varying the approximation?*

It is relevant to note that a further slight variant on this example in [12] provides an unstable example. A third variation is expounded in [6]. The proof of its stability is complicated by the lack of an ω-saturated model.

X. $\delta(A) = |A| - \alpha \cdot e(A)$ where α is any irrational number between $1/2$ and 1. $\mathbf{K_0} = \{A \in \mathbf{K}' : \delta(A') \geq 0 \text{ if } A' \subseteq A\}$. The generic is strictly stable but not ω-saturated. In fact, the theory is not small.

There are two major variations on the Hrushovski construction: one to obtain strongly minimal sets; one for $\aleph_0$-categorical theories. The obvious distinction between these is the use of rational versus irrational α. However, we can make the distinction in a framework without the dimension function.

3.1 Definition. The structure M *has finite closures* if for every finite $A \subseteq M$ there is a finite $B \leq M$ with $A \subseteq B$. The class $\mathbf{K}$ has finite closures if each member of $\mathbf{K}$ does.

3.2 Definition. Let $Q \in \mathbf{K}$. The theory T=Th(Q) is said to have *amalgamation over closed sets* if for any infinite models N_0, N_1 of T, any set A in $\mathbf{K}$, if there are strong embeddings $f : A \to N_0$ and $g : A \to N_1$, then there are $S \models T$ and elementary embeddings $f' : N_0 \to S$, $g' : N_1 \to S$ such that $f'f = g'g$. If this condition holds for finite A then T is said to have *amalgamation over finite closed sets.*

3.3 Definition. We say A is n-strong in B, written $A \leq_n B$, if for any B' with $A \subseteq B' \subseteq B$ and $|B' - A| \leq n$, $A \leq B'$.

3.4 Definition. $\mathbf{K}$ has the *uniform amalgamation property* if the following condition holds for every $A \leq B$. For every $m \in \omega$ there is an $n = f_B(m)$ such that if $A \leq_n C$ then there is a D, a strong embedding of C into D and an m-strong embedding of B into D such that these embeddings commute.

The following result summarises work by [8], [18], [9], [6].

3.5 Lemma. *Let M be $\mathbf{K}_0$-generic and $T = \mathrm{Th(M)}$. Assume $\mathbf{K} = \mathrm{mod}(T)$ has finite closures. The following are equivalent.*

i) M is ω-saturated

ii) T has amalgamation over finite closed sets.

iii) M is ω_1-universal.

iv) $\mathbf{K}_0$ has the uniform amalgamation property.

10 Question. *Can either the construction in IV or VIII be modified to get an $\aleph_0$-categorical projective plane or an $\aleph_0$-categorical stable nilpotent group that is not Abelian by finite?*

11 Question. *Is it possible to modify these constructions to find stable theories with finitely many countable models? Alternatively, does the existence of a nice dimension function (which determines a real valued rank as in these examples) force a theory which is not $\aleph_0$-categorical to have infinitely many countable models.*

12 Question. *Is there a 'generic' structure which is superstable but not ω-stable?*

This problem can be refined in several ways. The following result is proved in [6]

3.6 Theorem. *Let M be $\mathbf{K_0}, \leq$-generic and $T = \mathrm{Th(M)}$ where $\leq$ is derived from $\delta(A) = |A| - \alpha \cdot e(A)$. Suppose T has amalgamation over closed sets.*

i) T is stable.

ii) If, in addition, α is rational then T is ω-stable.

3.7 Definition. T is finitely (countably) based if for any a and any M a model of T there exists a $B \subseteq M$ with $|B|$ finite (countable) and $d(a/B) = d(a/M)$.

The hypothesis that distinguishes the two cases of Theorem 3.6 is that in the first case T is countably based and in the second finitely based. If α is rational $\mathbf{K}$ has finite closures and is finitely based. If, in addition, $\mathbf{K_0}$ has the uniform amgamation property, M is ω-saturated. But we are left with the question.

13 Question. *Is there a class $\mathbf{K_0}$ that has amalgamation and a rational dimension function but does not have the uniform amalgamation property? Can such an example be stable?*

14 Question. *Is there a class $(\mathbf{K}, \leq)$ which is finitely based but does not have finite closures.*

[6] show that the generic model associated with all finite graphs such that $|A| - \alpha e(A)$ is hereditarily positive is stable even though the generic model is not ω-saturated. Further work on this subject ([4]), using methods from [16] on $0 - 1$ laws for random graphs show that $\mathrm{Th}(M)$ is 1-model complete.

15 Question. *Can the strong submodel notion be used to explicate which classes of graphs satisfy the $0 - 1$ law?*

These examples are all distinguished from classical examples of non-locally modular structures because they do not interpret groups. Since there are continuum many examples, they clearly cannot all be interpreted in algebraically closed fields (since they admit only countably many interpreted theories). There are two reasonable variants to this question.

16 Question. *Show none of the strongly minimal sets defined in the method outlined here can be interpreted in an algebraically closed field.*

17 Question. *Can these examples be interpreted in the Zariski structures described by Zil'ber [19] which are compact complex manifolds.*

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