José Iovino, Editor

Beyond First Order Model Theory

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Part I

Model Theory of Strong Logics

Chapter 1

Expressive power of infinitary $[0,1]$-logics

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Abstract

We consider model-theoretic properties related to the expressive power of three analogues of $\mathcal{L}_{\omega_1,\omega}$ for metric structures. We give an example showing that one of these infinitary logics is strictly more expressive than the other two, but also show that all three have the same elementary equivalence relation for complete separable metric structures. We then prove that a continuous function on a complete separable metric structure is automorphism invariant if and only if it is definable in the more expressive logic. Several of our results are related to the existence of Scott sentences for complete separable metric structures.

1.1 Introduction

In the last several years there has been considerable interest in the *continuous first-order logic for metric structures* introduced by Ben Yaacov and Usvyatsov in the mid-2000's and published in [BYU10]. This logic is suitable for studying structures based on metric spaces, including a wide variety of structures encountered in analysis. Continuous first-order logic is a generalization of first-order logic, and shares many of its desirable model-theoretic properties, including the compactness theorem. While earlier logics for considering metric structures, such as Henson's logic of positive bounded formulas (see [HI03]), were equivalent to continuous first-order logic, the latter has emerged as the current standard first-order logic for developing the model theory of metric structures. The reader interested in a detailed history of the interactions between model theory and analysis can consult [Iov14].

In classical discrete logic there are many examples of logics that extend first-order logic, yet are still tame enough to allow a useful model theory to be developed; many of the articles in [BF85] describe such logics. The most fruitful extension of first-order logic is the infinitary logic $\mathcal{L}_{\omega_1,\omega}$, which extends the formula creation rules from first-order to also allow countable conjunctions and disjunctions of formulas, subject only to the restriction that the total number of free variables remains finite. While the compactness theorem fails for $\mathcal{L}_{\omega_1,\omega}$, it is nevertheless true that many results from first-order model theory can be translated in some form to $\mathcal{L}_{\omega_1,\omega}$ - see [Kei71] for a thorough development of the model theory of $\mathcal{L}_{\omega_1,\omega}$ for discrete structures.

Many properties considered in analysis have an infinitary character. It is therefore natural to look for a logic that extends continuous first-order logic by allowing infinitary operations. In order to be useful, such a logic should still have desirable model-theoretic properties analogous to those of the discrete infinitary logic $\mathcal{L}_{\omega_1,\omega}$. There have recently been proposals for such a logic by Ben Yaacov and Iovino [BYI09], Sequeira [Seq13], and the author [Eag14]; we call these logics $\mathcal{L}^C_{\omega_1,\omega}$, $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, and $\mathcal{L}_{\omega_1,\omega}$, respectively. The superscript C is intended to emphasize the continuity of the first two of these logics, in a sense to be described below. The goal of Section 1.2 is to give an overview of some of the model-theoretic properties of each of these logics, particularly with respect to their expressive powers. Both $\mathcal{L}_{\omega_1,\omega}$ and $\mathcal{L}^C_{\omega_1,\omega}$ extend continuous first-order logic by allowing as formulas some expressions of the form $\sup_n \phi_n$, where the ϕ_n's are formulas. The main difference between $\mathcal{L}^C_{\omega_1,\omega}$ and $\mathcal{L}_{\omega_1,\omega}$ is that the former requires infinitary formulas to define *uniformly continuous* functions on all structures, while the latter does not impose any continuity requirements. Allowing discontinuous formulas provides a significant increase in expressive power, including the ability to express classical negation (Proposition 1.2.8), at the cost of a theory which is far less well-behaved with respect to metric completions (Example 1.2.7). The logic $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ is obtained by adding an additional operator ρ to $\mathcal{L}^C_{\omega_1,\omega}$, where $\rho(x, \phi)$ is interpreted as the distance from x to the zeroset of ϕ. We show in Theorem 1.2.6 that ρ can be defined in $\mathcal{L}_{\omega_1,\omega}$.

One of the most notable features of the discrete logic $\mathcal{L}_{\omega_1,\omega}$ (in a countable signature) is that for each countable structure M there is a sentence σ of $\mathcal{L}_{\omega_1,\omega}$ such that a countable structure N satisfies σ if and only if N is isomorphic to M. Such sentences are known as *Scott sentences*, having first appeared in a paper of Scott [Sco65]. In Section 1.3 we discuss some consequences of the existence of Scott sentences for complete separable metric structures. The existence of Scott sentences for complete separable metric structures was proved by Sequeira [Seq13] in $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ and Ben Yaacov, Nies, and Tsankov [BYNT14] in $\mathcal{L}^C_{\omega_1,\omega}$. Despite having shown in Section 1.2 that the three logics we are considering have different expressive powers, we use Scott sentences to prove the following in Proposition 1.3.4:

Theorem. *Let M and N be separable complete metric structures in the same countable signature. The following are equivalent:*

- $M \cong N$,
- $M \equiv N$ *in* $\mathcal{L}^C_{\omega_1,\omega}$,
- $M \equiv N$ *in* $\mathcal{L}^C_{\omega_1,\omega}(\rho)$,
- $M \equiv N$ *in* $\mathcal{L}_{\omega_1,\omega}$.

Scott's first use of his isomorphism theorem was to prove a definability result, namely that a predicate on a countable discrete structure is automorphism invariant if and only if it is definable by an $\mathcal{L}_{\omega_1,\omega}$ formula. The main new result of this note is a metric version of Scott's definability theorem (Theorem 1.4.1):

Theorem. *Let M be a separable complete metric structure, and $P : M^n \to [0,1]$ be a continuous function. The following are equivalent:*

- *P is invariant under all automorphisms of M,*
- *there is an $\mathcal{L}_{\omega_1,\omega}$ formula $\phi(\vec{x})$ such that for all $\vec{a} \in M^n$, $P(\vec{a}) = \phi^M(\vec{a})$.*

The proof of the above theorem relies heavily on replacing the constant symbols in an $\mathcal{L}_{\omega_1,\omega}$ sentence by variables to form an $\mathcal{L}_{\omega_1,\omega}$ formula; Example 1.3.5 shows that this technique cannot be used in $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, so our method does not produce a version of Scott's definability theorem in $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$.

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1.2 Infinitary logics for metric structures

Our goal is to study infinitary extensions of first-order continuous logic for metric structures. To begin, we briefly recall the definition of metric structures and the syntax of first-order continuous logic. The reader interested in an extensive treatment of continuous logic can consult the survey [BYBHU08].

Definition 1.2.1. A *metric structure* is a metric space (M, d^M) of diameter at most 1, together with:

- A set $(f_i^M)_{i\in I}$ of uniformly continuous *functions* $f_i : M^{n_i} \to M$,
- a set $(P_j^M)_{j\in J}$ of uniformly continuous *predicates* $P_j : M^{m_j} \to [0,1]$,
- a set $(c_k^M)_{k\in K}$ of distinguished *elements* of M.

We place no restrictions on the sets I, J, K, and frequently abuse notation by using the same symbol for a metric structure and its underlying metric space.

Metric structures are the semantic objects we will be studying. On the syntactic side, we have *metric signatures*. By a *modulus of continuity* for a uniformly continuous function $f : M^n \to M$ we mean a function $\delta : \mathbb{Q} \cap (0,1) \to \mathbb{Q} \cap (0,1)$ such that such that for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$ and all $\epsilon \in \mathbb{Q} \cap (0,1)$,

$$\sup_{1\leq i\leq n} d(a_i, b_i) < \delta(\epsilon) \implies d(f(a_i), f(b_i)) \leq \epsilon.$$

Similarly, δ is a modulus of continuity for $P : M^n \to [0,1]$ means that for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$,

$$\sup_{1\leq i\leq n} d(a_i, b_i) < \delta(\epsilon) \implies |P(a_i) - P(b_i)| \leq \epsilon.$$

Definition 1.2.2. A *metric signature* consists of the following information:

- A set $(f_i)_{i\in I}$ of *function symbols*, each with an associated arity and modulus of uniform continuity,
- a set $(P_j)_{j\in J}$ of *predicate symbols*, each with an associated arity and modulus of uniform continuity,
- a set $(c_k)_{k\in K}$ of *constant symbols*.

When no ambiguity can arise, we say "signature" instead of "metric signature".

When S is a metric signature and M is a metric structure, we say that M is an *S-structure* if the distinguished functions, predicates, and constants of M match the requirements imposed by S. Given a signature S, the *terms* of S are defined recursively, exactly as in the discrete case.

Definition 1.2.3. Let S be a metric signature. The *S-formulas* of continuous first-order logic are defined recursively as follows.

1. If t_1 and t_2 are terms then $d(t_1, t_2)$ is a formula.

2. If $t_1, \ldots, t_n$ are S-terms, and P is an n-ary predicate symbol, then $P(t_1, \ldots, t_n)$ is a formula.

3. If $\phi_1, \ldots, \phi_n$ are formulas, and $f : [0,1]^n \to [0,1]$ is continuous, then $f(\phi_1, \ldots, \phi_n)$ is a formula. We think of each such f as a *connective*.

4. If ϕ is a formula and x is a variable, then $\inf_x \phi$ and $\sup_x \phi$ are formulas. We think of $\sup_x$ and $\inf_x$ as *quantifiers*.

Given a metric structure M, a formula $\phi(\vec{x})$ of the appropriate signature, and a tuple $\vec{a} \in M$, we define the value of ϕ in M at $\vec{a}$, denoted $\phi^M(\vec{a})$, in the obvious recursive manner. We write $M \models \phi(\vec{a})$ to mean $\phi^M(\vec{a}) = 0$. The basic notions of model theory are then defined in the expected way by analogy to discrete first-order logic.

The only difference between our definitions and those of [BYBHU08] is that in [BYBHU08] it is assumed that the underlying metric space of each structure is complete. We do not want to make the restriction to complete metric spaces in general, so our definition of structures allows arbitrary metric spaces, and we speak of *complete metric structures* when we want to insist on completeness of the underlying metric. In first-order continuous logic there is little lost by considering only complete metric structures, since every structure is an elementary substructure of its metric completion. This is also true in $\mathcal{L}^C_{\omega_1,\omega}$ and $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, but not in $\mathcal{L}_{\omega_1,\omega}$, as Example 1.2.7 below illustrates.

In continuous logic the connectives max and min play the roles of $\wedge$ and $\vee$, respectively, in the sense that for a metric structure M, formulas $\phi(\vec{x})$ and $\psi(\vec{x})$, and a tuple $\vec{a}$, we have $M \models \max\{\phi(\vec{a}), \psi(\vec{a})\}$ if and only if $M \models \phi(\vec{a})$ and $M \models \psi(\vec{a})$, and similarly for min and disjunction. Consequently, the most direct adaptation of $\mathcal{L}_{\omega_1,\omega}$ to metric structures is to allow the formation of formulas $\sup_n \phi_n$ and $\inf_n \phi_n$, at least provided that the total number of free variables remains finite (the restriction on the number of free variables is usually assumed even in the discrete case). However, one of the important features of continuous logic is that it is a *continuous* logic, in the sense that each formula $\phi(x_1, \ldots, x_n)$ defines a continuous function $\phi^M : M^n \to [0,1]$ on each structure M. The pointwise supremum or infimum of a sequence of continuous functions is not generally continuous.

A second issue arises from the fact that one expects the metric version of $\mathcal{L}_{\omega_1,\omega}$ to have the same relationship to separable metric structures as $\mathcal{L}_{\omega_1,\omega}$ has to countable discrete structures. Separable metric structures are generally not countable, so some care is needed in arguments whose discrete version involves taking a conjunction indexed by elements of a fixed structure. For instance, one standard proof of Scott's isomorphism theorem is of this kind (see [Kei71, Theorem 1]). Closely related to the question of whether or not indexing over a dense subset is sufficient is the issue of whether the zeroset of a formula is definable.

With the above issues in mind, we present some of the infinitary logics for metric structures that have appeared in the literature. The first and third of the following logics were both called "$\mathcal{L}_{\omega_1,\omega}$" in the papers where they were introduced, and the second was called "$\mathcal{L}_{\omega_1,\omega}(\rho)$"; we add a superscript "C" to the first and second logics to emphasize that they are continuous logics.

Definition 1.2.4. The three infinitary logics for metric structures we will be considering are:

- $\mathcal{L}^C_{\omega_1,\omega}$ (Ben Yaacov-Iovino [BYI09]): Allow formulas $\sup_{n<\omega} \phi_n$ and $\inf_{n<\omega} \phi_n$ as long as the total number of free variables remains finite, and the formulas ϕ_n satisfy a common modulus of uniform continuity.
- $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ (Sequeira [Seq13]): Extend $\mathcal{L}^C_{\omega_1,\omega}$ by adding an operator $\rho(x, \phi)$, interpreted as the distance from x to the zeroset of ϕ.
- $\mathcal{L}_{\omega_1,\omega}$ (Eagle [Eag14]): Allow formulas $\sup_{n<\omega} \phi_n$ and $\inf_{n<\omega} \phi_n$ as long as the total number of free variables remains finite, without regard to continuity.

The logic $\mathcal{L}_{\omega_1,\omega}$ was further developed by Grinstead [Gri14], who in particular provided an axiomatization and proof system.

Other infinitary logics for metric structures which are not extensions of continuous first-order logic have also been studied. In a sequence of papers beginning with his thesis [Ort97], Ortiz develops a logic based on Henson's positive bounded formulas and allows infinitary formulas, but also infinite strings of quantifiers. An early version of [CL16] had infinitary formulas in a logic where the quantifiers sup and inf were replaced by category quantifiers.

Remark 1.2.5. We will often write formulas in any of the above logics in forms intended to make their meaning more transparent, but sometimes this can make it less obvious that the expressions we use are indeed valid formulas. For example, in the proof of Theorem 1.2.6 below, we will be given an $\mathcal{L}_{\omega_1,\omega}$ formula $\phi(\vec{x})$, and we will define

$$\rho_\phi(\vec{x}) = \inf_{\vec{y}} \min \left\{ \left(d(\vec{x}, \vec{y}) + \sup_{n\in\mathbb{N}} \min\{n\phi(\vec{y}), 1\} \right), 1 \right\}.$$

The preceding definition can be seen to be a valid formula of $\mathcal{L}_{\omega_1,\omega}$ as follows. For each $n \in \mathbb{N}$ define $u_n : [0,1] \to [0,1]$ by $u(z) = \min\{nz, 1\}$. Define $v : [0,1]^2 \to [0,1]$ by $v(z,w) = \min\{z + w, 1\}$. Then each u_n is continuous, as is v, and we have

$$\rho_\phi(\vec{x}) = \inf_{\vec{y}} v \left(d(\vec{x}, \vec{y}),\ \sup_{n\in\mathbb{N}} u_n(\phi(\vec{x})) \right).$$

A similar process may be used throughout the remainder of the paper to see that expressions we claim are formulas can indeed be expressed in the form of Definitions 1.2.3 and 1.2.4.

The remainder of this section explores some of the relationships between $\mathcal{L}^C_{\omega_1,\omega}$, $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, and $\mathcal{L}_{\omega_1,\omega}$. It is clear that each $\mathcal{L}^C_{\omega_1,\omega}$ formula is both an $\mathcal{L}_{\omega_1,\omega}$ formula and an $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula. The next result shows that the ρ operation is implemented by a formula of $\mathcal{L}_{\omega_1,\omega}$, so each $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula is also equivalent to an $\mathcal{L}_{\omega_1,\omega}$ formula.

Theorem 1.2.6. *For every $\mathcal{L}_{\omega_1,\omega}$ formula $\phi(\vec{x})$ there is an $\mathcal{L}_{\omega_1,\omega}$ formula $\rho_\phi(\vec{x})$ such that for every metric structure M and every $\vec{a} \in M$,*

$$\rho^M_\phi(\vec{a}) = \inf\{d(\vec{a},\vec{b}) : \phi^M(\vec{b}) = 0\}.$$

Proof. Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$ formula, and define

$$\rho_\phi(\vec{x}) = \inf_{\vec{y}} \min\left\{\left(d(\vec{x},\vec{y}) + \sup_{n\in\mathbb{N}} \min\{n\phi(\vec{y}),1\}\right), 1\right\}.$$

(See Remark 1.2.5 above for how to express this as an official $\mathcal{L}_{\omega_1,\omega}$ formula). Now consider any metric structure M, and any $\vec{y} \in M$. We have

$$\sup_{n\in\mathbb{N}} \min\{n\phi^M(\vec{y}),1\} = \begin{cases} 0 & \text{if } \phi^M(\vec{y}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore for any $\vec{a}, \vec{y} \in M$,

$$\min\left\{\left(d(\vec{a},\vec{y}) + \sup_{n\in\mathbb{N}} \min\{n\phi(\vec{y}),1\}\right), 1\right\} = \begin{cases} d(\vec{a},\vec{y}) & \text{if } \phi^M(\vec{y}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since all values are in $[0,1]$, it follows that:

$$\rho^M_\phi(\vec{a}) = \inf\left(\{d(\vec{a},\vec{y}) : \phi^M(\vec{y}) = 0\} \cup \{1\}\right) = \inf\{d(\vec{a},\vec{y}) : \phi^M(\vec{y}) = 0\}.$$

□

Each formula of $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ defines a uniformly continuous function on each structure, and just as in first-order continuous logic, the modulus of continuity of this function depends only on the signature, not the particular structure. By contrast, the functions defined by $\mathcal{L}_{\omega_1,\omega}$ formulas need not be continuous at all. The loss of continuity causes complications for the theory, especially when one is interested in *complete* metric structures, as is often the case in applications. Of particular note is the fact that, while every metric structure is an $\mathcal{L}^C_{\omega_1,\omega}$-elementary substructure of its metric completion, this is very far from being true for the logic $\mathcal{L}_{\omega_1,\omega}$:

Example 1.2.7. Let S be the signature consisting of countably many constant symbols $(q_n)_{n<\omega}$. Consider the $\mathcal{L}_{\omega_1,\omega}$ formula

$$\phi(x) = \inf_{n<\omega} \sup_{R\in\mathbb{N}} \min\{1, Rd(x,q_n)\}.$$

For any a in a metric structure M we have $M \models \phi(a)$ if and only if $a = q_n$ for some n. In particular, if M is a countable metric space which is not complete, and $(q_n)_{n<\omega}$ is interpreted as an enumeration of M, then

$$M \models \sup_x \phi(x) \qquad \text{and} \qquad \overline{M} \not\models \sup_x \phi(x).$$

In particular, $M \not\equiv_{\mathcal{L}_{\omega_1,\omega}} \overline{M}$.

While discontinuous formulas introduce complications, they also give a significant increase in expressive power. As an example, recall that continuous first-order logic lacks an exact negation connective, in the sense that there is no connective $\neg$ such that $M \models \neg\phi$ if and only if $M \not\models \phi$. Indeed there is no continuous function $\neg : [0,1] \to [0,1]$ such that $\neg(x) = 0$ if and only if $x \neq 0$, so $\mathcal{L}^C_{\omega_1,\omega}$ also lacks an exact negation connective. Similarly, the formula $\inf_n \phi_n$ is not the exact disjunction of the formulas ϕ_n, and $\inf_x$ is not an exact existential quantifier, and neither exact disjunction nor exact existential quantification is present in either continuous infinitary logic. In $\mathcal{L}_{\omega_1,\omega}$, we recover all three of these classical operations on formulas.

Proposition 1.2.8. *The logic $\mathcal{L}_{\omega_1,\omega}$ has an exact countable disjunction, an exact negation, and an exact existential quantifier.*

Proof. We first show that $\mathcal{L}_{\omega_1,\omega}$ has an exact infinitary disjunction. Suppose that $(\phi_n(\vec{x}))_{n<\omega}$ are formulas of $\mathcal{L}_{\omega_1,\omega}$. Define

$$\psi(\vec{x}) = \inf_{n<\omega} \sup_{R\in\mathbb{N}} \min\{1, R\phi_n(\vec{x})\}.$$

Then in any metric structure M, for any tuple $\vec{a}$, we have

$$M \models \psi(\vec{a}) \qquad \Longleftrightarrow \qquad M \models \phi_n(\vec{a}) \text{ for some } n.$$

Using the exact disjunction we define the exact negation. Given any formula $\phi(\vec{x})$, define

$$\neg\phi(\vec{x}) = \bigvee_{n<\omega} \left(\phi(\vec{x}) \geq \frac{1}{n}\right),$$

where $\bigvee$ is the exact disjunction described above. Then for any metric structure M, and any $\vec{a} \in M$,

$$\begin{aligned} M \models \neg\phi(\vec{a}) &\iff (\exists n < \omega) M \models \phi(\vec{a}) \geq \frac{1}{n} \\ &\iff (\exists n < \omega) \phi^M(\vec{a}) \geq \frac{1}{n} \\ &\iff \phi^M(\vec{a}) \neq 0 \\ &\iff M \not\models \phi(\vec{a}) \end{aligned}$$

Finally, with exact negation and the fact that $M \models \sup_{\vec{x}} \phi(\vec{x})$ if and only if $M \models \phi(\vec{a})$ for every $\vec{a} \in M$, we define $\exists\vec{x}\phi$ to be $\neg \sup_{\vec{x}} \neg\phi$, and have that $M \models \exists\vec{x}\phi(\vec{x})$ if and only if there is $\vec{a} \in M$ such that $M \models \phi(\vec{a})$. $\square$

Remark 1.2.9. Some caution is necessary when using the negation operation defined in Proposition 1.2.8. Consider the following properties a negation connective $\sim$ could have for all metric structures M, all tuples $\vec{a} \in M$, and all formulas $\phi(\vec{x})$. These properties mimic properties of negation in classical discrete logic:

1. $M \models {\sim}\phi(\vec{a})$ if and only if $M \not\models \phi(\vec{a})$,
2. $M \models {\sim}{\sim}\phi(\vec{a})$ if and only if $M \models \phi(\vec{a})$,
3. $({\sim}{\sim}\phi)^M(\vec{a}) = \phi^M(\vec{a})$.

Properties (1) and (3) each imply property (2). In classical $\{0, 1\}$-valued logics there is no distinction between properties (2) and (3), but these properties do not coincide for $[0, 1]$-valued logic. Property (2) is strictly weaker than property (1), since the identity connective ${\sim}\sigma = \sigma$ satisfies (2) but not (1).

The connective $\neg$ defined in the proof of Proposition 1.2.8 has properties (1) and (2), but does not have property (3), because if $\phi(\vec{a})^M > 0$ then $(\neg\neg\phi)^M(\vec{a}) = 1$. The approximate negation commonly used in continuous first-order logic, which is defined by $\sim \phi(\vec{x}) \ = 1 - \phi(\vec{x})$, satisfies properties (2) and (3), but not property (1).

In fact, there is no truth-functional connective in *any* $[0, 1]$-valued logic that satisfies both (1) and (3). Suppose that $\sim$ were such a connective. Then $\sim \ : \ [0, 1] \ \to \ [0, 1]$ would have the following two properties for all $x \in [0, 1]$, as consequences of (1) and (3), respectively:

- ${\sim}(x) = 0$ if and only if $x \neq 0$,
- ${\sim}({\sim}(x)) = x$.

The first condition implies that $\sim$ is not injective, and hence cannot satisfy the second condition.

The expressive power of $\mathcal{L}_{\omega_1,\omega}$ is sufficient to introduce a wide variety of connectives beyond those of continuous first-order logic and the specific ones described in Proposition 1.2.8.

Proposition 1.2.10. *Let* $u : [0, 1]^n \to [0, 1]$ *be a Borel function, with* $n < \omega$*, and let* $(\phi_l(\vec{x}))_{l<n}$ *be* $\mathcal{L}_{\omega_1,\omega}$*-formulas. There is an* $\mathcal{L}_{\omega_1,\omega}$*-formula* $\psi(\vec{x})$ *such that for any metric structure* M *and any* $\vec{a} \in M$*,*

$$\psi^M(\vec{a}) = u(\phi_1^M(\vec{a}), \ldots).$$

Proof. Recall that the *Baire hierarchy* of functions $f : [0, 1]^n \to [0, 1]$ is defined recursively, with f being *Baire class* 0 if it is continuous, and *Baire class* α (for an ordinal $\alpha > 0$) if it is the pointwise limit of a sequence of functions each from some Baire class $< \alpha$. The classical Lebesgue-Hausdorff theorem (see [Sri98, Proposition 3.1.32 and Theorem 3.1.36]) implies that a function $f : [0, 1]^\omega \to [0, 1]$ is Borel if and only if it is Baire class α for some $\alpha < \omega_1$.

Our proof will therefore be by induction on the Baire class α of our connective $u : [0,1]^n \to [0,1]$. The base case is $\alpha = 0$, in which case u is continuous, and hence is a connective of first-order continuous logic.

Now suppose that $u = \lim_{k\to\infty} u_k$ pointwise, with each u_k of a Baire class $\alpha_k < \alpha$. By induction, for each k let $\psi_k(\vec{x})$ be such that for every metric structure M and every $\vec{a} \in M$ $\psi_k^M(\vec{a}) = u_k(\phi_1^M(\vec{a}), \ldots, \phi_n^M(\vec{a}))$. Then we have

$$\begin{aligned} u(\phi_1^M(\vec{a}), \ldots, \phi_n^M(\vec{a})) &= \lim_{k\to\infty} u_k(\phi_1^M(\vec{a}), \ldots, \phi_n^M(\vec{a})) \\ &= \limsup_{k\to\infty} \psi_k^M(\vec{a}) \\ &= \inf_{k\geq 0} \sup_{m\geq k} \psi_m^M(\vec{a}) \end{aligned}$$

The final expression shows that the required $\mathcal{L}_{\omega_1,\omega}$ formula is $\inf_{k\geq 0} \sup_{m\geq k} \psi_m(\vec{x})$. □

Remark 1.2.11. The case of Proposition 1.2.10 for sentences appears, with a different proof, in [Gri14, Theorem 1.25].

The expressive power of continuous first-order logic is essentially unchanged if continuous functions of the form $u : [0,1]^\omega \to [0,1]$ are allowed as connectives in addition to the continuous functions on finite powers of $[0,1]$ (see [BYBHU08, Proposition 9.3]). If such infinitary continuous connectives are permitted in $\mathcal{L}_{\omega_1,\omega}$, then the same proof as above also shows that $\mathcal{L}_{\omega_1,\omega}$ implements all Borel functions $u : [0,1]^\omega \to [0,1]$.

In order to obtain the benefits of both $\mathcal{L}_{\omega_1,\omega}$ and $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, it is sometimes helpful to work in $\mathcal{L}_{\omega_1,\omega}$ and then specialize to a more restricted logic when continuity becomes relevant. A *fragment* of an infinitary metric logic $\mathcal{L}$ is a set of $\mathcal{L}$-formulas including the formulas of continuous first-order logic, closed under the connectives and quantifiers of continuous first-order logic, closed under subformulas, and closed under substituting terms for variables. In [Eag14] we defined a fragment L of $\mathcal{L}_{\omega_1,\omega}$ to be *continuous* if it has the property that every L-formula defines a continuous function on all structures. The definition of a continuous fragment ensures that if L is a continuous fragment and M is a metric structure, then $M \preceq_L \overline{M}$.

It follows immediately from the definitions that $\mathcal{L}^C_{\omega_1,\omega}$ is a continuous fragment of both $\mathcal{L}_{\omega_1,\omega}$ and $\mathcal{L}^C_{\omega_1,\omega}(\rho)$. The construction of the ρ operation as a formula of $\mathcal{L}_{\omega_1,\omega}$ in Theorem 1.2.6 uses discontinuous formulas as subformulas, so $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ is not a continuous fragment of $\mathcal{L}_{\omega_1,\omega}$, although it would be if we viewed the formula ρ_ϕ from Theorem 1.2.6 as having only ϕ as a subformula. While it is a priori possible that there are continuous fragments of $\mathcal{L}_{\omega_1,\omega}$ that are not subfragments of $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, we are not aware of any examples. It also remains unclear whether or not the ρ operation of $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ can be implemented by an $\mathcal{L}^C_{\omega_1,\omega}$ formula. We therefore ask:

Question 1.2.12. Suppose that $\phi(\vec{x})$ is an $\mathcal{L}_{\omega_1,\omega}$ formula such that for every subformula ψ of ϕ, $\psi^M : M^n \to [0,1]$ is *uniformly* continuous, with the modulus of uniform continuity not depending on M. Is ϕ equivalent to an $\mathcal{L}^C_{\omega_1,\omega}$ formula? Is $\rho(\vec{y},\phi)$ equivalent to an $\mathcal{L}^C_{\omega_1,\omega}$ formula?

A positive answer to the first part of Question 1.2.12 would imply that every continuous fragment of $\mathcal{L}_{\omega_1,\omega}$ is a fragment of $\mathcal{L}^C_{\omega_1,\omega}$. In the first part of the question the answer is negative if we only ask for ϕ to define a uniformly continuous function. For example, consider the sentence $\sigma = \sup_x \phi(x)$ from Example 1.2.7. For any M we have $\sigma^M : M^0 \to [0,1]$ is constant, yet we saw that this σ can be a witness to $M \not\equiv_{\mathcal{L}_{\omega_1,\omega}} \overline{M}$, and hence is not equivalent to any $\mathcal{L}^C_{\omega_1,\omega}$ sentence. This example can be easily modified to produce examples of $\mathcal{L}_{\omega_1,\omega}$ formulas with free variables that are uniformly continuous but not equivalent to any $\mathcal{L}^C_{\omega_1,\omega}$ sentence (for example, $\max\{\sigma, d(y,y)\}$).

1.3 Consequences of Scott's Isomorphism Theorem

The existence of Scott sentences for complete separable metric structures was first proved by Sequeira [Seq13] in $\mathcal{L}^C_{\omega_1,\omega}(\rho)$. Sequeira's proof of the existence of Scott sentences is a back-and-forth argument, generalizing the standard proof in the discrete setting. An alternative proof of the existence of Scott sentences in $\mathcal{L}^C_{\omega_1,\omega}$ goes by first proving a metric version of the López-Escobar theorem, which characterizes the isomorphism-invariant bounded Borel functions on a space of codes for structures as exactly those functions of the form $M \mapsto \sigma^M$ for an $\mathcal{L}^C_{\omega_1,\omega}$-sentence σ. Using this method Scott sentences in $\mathcal{L}^C_{\omega_1,\omega}$ were found by Coskey and Lupini [CL16] for structures whose underlying metric space is the Urysohn sphere, and such that all of the distinguished functions and predicates share a common modulus of uniform continuity. Shortly thereafter, Ben Yaacov, Nies, and Tsankov obtained the same result for all complete metric structures.

Theorem 1.3.1 ([BYNT14, Corollary 2.2])**.** *For each separable complete metric structure M in a countable signature there is an $\mathcal{L}^C_{\omega_1,\omega}$ sentence σ such that for every other separable complete metric structure N of the same signature,*

$$\sigma^N = \begin{cases} 0 & \text{if } M \cong N \\ 1 & \text{otherwise} \end{cases}$$

We note that a positive answer to Question 1.2.12 would imply that Sequeira's proof works in $\mathcal{L}^C_{\omega_1,\omega}$, and hence give a more standard back-and-forth proof of Theorem 1.3.1.

Remark 1.3.2. Even with the increased expressive power of $\mathcal{L}_{\omega_1,\omega}$ over $\mathcal{L}^C_{\omega_1,\omega}$, we cannot hope to prove the existence of Scott sentences for arbitrary (i.e., possibly incomplete) separable metric structures, because there are $2^{2^{\aleph_0}}$ pairwise non-isometric separable metric spaces ([KN51, Theorem 2.1]), but only $2^{\aleph_0}$ sentences of $\mathcal{L}_{\omega_1,\omega}$ in the empty signature.

We can easily reformulate Theorem 1.3.1 to apply to incomplete structures, but little is gained, as we only get uniqueness at the level of the metric completion.

Corollary 1.3.3. *For each separable metric structure M in a countable signature there is an $\mathcal{L}^C_{\omega_1,\omega}$ sentence σ such that for every other separable metric structure N of the same signature,*

$$\sigma^N = \begin{cases} 0 & \textit{if } \overline{M} \cong \overline{N} \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Let σ be the Scott sentence for $\overline{M}$, as in Theorem 1.3.1. Since σ is in $\mathcal{L}^C_{\omega_1,\omega}$, we have

$$\sigma^N = \sigma^{\overline{N}} = \begin{cases} 0 & \text{if } \overline{M} \cong \overline{N} \\ 1 & \text{otherwise} \end{cases}.$$

□

The following observation should be compared with Example 1.2.7 and Proposition 1.2.8, which showed that there are $\mathcal{L}_{\omega_1,\omega}$ formulas (and even sentences) that are not $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formulas.

Proposition 1.3.4. *For any separable complete metric structures M and N in the same countable signature, the following are equivalent:*

1. $M \cong N$,
2. $M \equiv_{\mathcal{L}_{\omega_1,\omega}} N$,
3. $M \equiv_{\mathcal{L}^C_{\omega_1,\omega}(\rho)} N$.
4. $M \equiv_{\mathcal{L}^C_{\omega_1,\omega}} N$,

Proof. It is clear that (1) implies (2). By Theorem 1.2.6 each $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula can be implemented as an $\mathcal{L}_{\omega_1,\omega}$ formula, so (2) implies (3). Similarly, each $\mathcal{L}^C_{\omega_1,\omega}$ formula is an $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula, so (3) implies (4). Finally, if $M \equiv_{\mathcal{L}^C_{\omega_1,\omega}} N$ then, in particular, N satisfies M's Scott sentence, and both are complete separable metric structures, so $M \cong N$ by Theorem 1.3.1. □

The formula creation rules for $\mathcal{L}_{\omega_1,\omega}$ imply that if $\phi(\vec{x})$ is an $\mathcal{L}_{\omega_1,\omega}$-formula in a signature with a constant symbol c, then the expression obtained by replacing each instance of c by a new variable y is an $\mathcal{L}_{\omega_1,\omega}$-formula $\psi(\vec{x}, y)$. In particular, the usual identification of formulas with sentences in a language with new constant symbols can be used in $\mathcal{L}_{\omega_1,\omega}$. By contrast, when this procedure is performed on an $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula, the result is not necessarily again an $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula, because it may not have the appropriate continuity property. Scott's isomorphism theorem provides a plentiful supply of examples.

Example 1.3.5. Let M be a complete separable connected metric structure such that $\mathrm{Aut}(M)$ does not act transitively on M. Pick any $a \in M$, and let $\mathcal{O}_a$ be the $\mathrm{Aut}(M)$-orbit of a. The fact that $\mathrm{Aut}(M)$ does not act transitively implies $\mathcal{O}_a \neq M$. Let $\theta_a(x)$ be the $\mathcal{L}_{\omega_1,\omega}$ formula obtained by replacing a by a new variable x in the $\mathcal{L}^C_{\omega_1,\omega}$ Scott sentence of (M,a). Then for any $b \in X$,

$$\begin{aligned} \theta_a^M(b) &= \begin{cases} 0 & \text{if } (M,a) \cong (M,b) \\ 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } b \in \mathcal{O}_a, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Since M is connected and the image of θ_a^M is $\{0,1\}$, the function θ_a^M is not continuous. Therefore θ_a is not an $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula.

The fact that the formula θ_a in the above example *is* an $\mathcal{L}_{\omega_1,\omega}$ formula will be relevant in the proof of Theorem 1.4.1 below.

1.4 Definability in $\mathcal{L}_{\omega_1,\omega}$

The original use of Scott's isomorphism theorem in [Sco65] was to prove a definability theorem. We obtain an analogous definability theorem for the metric logic $\mathcal{L}_{\omega_1,\omega}$.

Theorem 1.4.1. *Let M be a separable complete metric structure in a countable signature. For any continuous function $P : M^n \to [0,1]$, the following are equivalent:*

1. *There is an $\mathcal{L}_{\omega_1,\omega}$ formula $\phi(\vec{x})$ such that for all $\vec{a} \in M^n$, $\phi^M(\vec{a}) = P(\vec{a})$,*
2. *P is fixed by all automorphisms of M (in the sense that for all $\Phi \in \mathrm{Aut}(M)$, $P = P \circ \Phi$).*

Proof. The proof that (1) implies (2) is a routine induction on the complexity of formulas, so we only prove that (2) implies (1).

Fix a countable dense subset $D \subseteq M$. For each $\vec{a} \in D$, let $\theta_{\vec{a}}(\vec{x})$ be the formula obtained by replacing each occurrence of $\vec{a}$ in the Scott sentence of $(M,\vec{a})$ by a tuple of new variables $\vec{x}$. The Scott sentence is obtained from Theorem 1.3.1. Observe that this formula has the following property, for all

$\vec{b} \in M^n$:

$$\theta^M_{\vec{a}}(\vec{b}) = \begin{cases} 0 & \text{if there is } \Phi \in \mathrm{Aut}(M) \text{ with } \Phi(\vec{b}) = \vec{a} \\ 1 & \text{otherwise} \end{cases}$$

For each $\epsilon > 0$, define:

$$\sigma_\epsilon(\vec{x}) = \inf_{\vec{y}} \max \left\{ d(\vec{x}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\}.$$

Each $\sigma_\epsilon(\vec{x})$ is a formula of $\mathcal{L}_{\omega_1,\omega}(S)$.

Claim 1.4.1.1. *Consider any $\epsilon \in \mathbb{Q} \cap (0,1)$ and any $\vec{b} \in M^n$.*

(a) If $M \models \sigma_\epsilon(\vec{b})$ then $P(\vec{b}) \leq \epsilon$.

(b) If $P(\vec{b}) < \epsilon$ then $M \models \sigma_\epsilon(\vec{b})$.

Proof. (a) Suppose that $M \models \sigma_\epsilon(\vec{b})$. Fix $\epsilon' > 0$, and pick $0 < \delta < 1$ such that if $d(\vec{b}, \vec{y}) < \delta$ then $\left| P(\vec{b}) - P(\vec{y}) \right| < \epsilon'$. This exists because we assumed that P is continuous. Now from the definition of $M \models \sigma_\epsilon(\vec{b})$ we can find $\vec{y} \in M^n$ such that

$$\max \left\{ d(\vec{b}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\} < \delta.$$

In particular, we have that $d(\vec{b}, \vec{y}) < \delta$, so $\left| P(\vec{b}) - P(\vec{y}) \right| < \epsilon'$. On the other hand, $\inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) < \delta$, and $\theta_{\vec{a}}(\vec{y}) \in \{0, 1\}$ for all $\vec{a} \in D^n$, so in fact there is $\vec{a} \in D^n$ with $P(\vec{a}) < \epsilon$ and $\theta_{\vec{a}}(\vec{y}) = 0$. For such an $\vec{a}$ there is an automorphism of M taking $\vec{y}$ to $\vec{a}$, and hence by (2) we have that $P(\vec{y}) < \epsilon$ as well. Combining what we have,

$$\begin{aligned} P(\vec{b}) &= \left| P(\vec{b}) \right| \\ &\leq \left| P(\vec{b}) - P(\vec{y}) \right| + |P(\vec{y})| \\ &< \epsilon' + \epsilon \end{aligned}$$

Taking $\epsilon' \to 0$ we conclude $P(\vec{b}) \leq \epsilon$.

(b) Suppose that $P(\vec{b}) < \epsilon$, and again fix $\epsilon' > 0$. Using the continuity of P, find δ sufficiently small so that if $d(\vec{b}, \vec{y}) < \delta$ then $P(\vec{y}) < \epsilon$. The set D is dense in M, so we can find $\vec{y} \in D^n$ such that $d(\vec{b}, \vec{y}) < \min\{\delta, \epsilon'\}$. Then $P(\vec{y}) < \epsilon$, so choosing $\vec{a} = \vec{y}$ we have

$$\inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) = 0.$$

Therefore

$$\max \left\{ d(\vec{b}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\} = d(\vec{b}, \vec{y}) < \epsilon',$$

and so taking $\epsilon' \to 0$ shows that $M \models \sigma_\epsilon(\vec{b})$.

$\dashv$ - Claim 1.4.1.1

Consider now any $\vec{a} \in M^n$. By (a) of the claim $P(\vec{a})$ is a lower bound for $\{\epsilon \in \mathbb{Q} \cap (0,1) : M \models \sigma_\epsilon(\vec{a})\}$. If α is another lower bound, and $\alpha > P(\vec{a})$, then there is $\epsilon \in \mathbb{Q} \cap (0,1)$ such that $P(\vec{a}) < \epsilon < \alpha$. By (b) of the claim we have $M \models \sigma_\epsilon(\vec{a})$ for this ϵ, contradicting the choice of α. Therefore

$$P(\vec{a}) = \inf \{\epsilon \in \mathbb{Q} \cap (0,1) : M \models \sigma_\epsilon(\vec{a})\}.$$

Now for each $\epsilon \in \mathbb{Q} \cap (0,1)$, define a formula

$$\psi_\epsilon(\vec{x}) = \max \left\{ \epsilon, \sup_{m \in \mathbb{N}} \min \{m \sigma_\epsilon(\vec{x}), 1\} \right\}.$$

Then for any $\vec{a} \in M^n$,

$$\psi_\epsilon^M(\vec{a}) = \begin{cases} \epsilon & \text{if } \sigma_\epsilon^M(\vec{a}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\phi(\vec{x}) = \inf_{\epsilon \in \mathbb{Q} \cap (0,1)} \psi_\epsilon(\vec{x})$. Then

$$\phi^M(\vec{a}) = \inf \left\{ \epsilon : \sigma_\epsilon^M(\vec{a}) = 0 \right\} = P(\vec{a}).$$

$\square$

We also have a version where parameters are allowed in the definitions:

Corollary 1.4.2. *Let M be a separable complete metric structure in a countable signature, and fix a set $A \subseteq M$. For any continuous function $P : M^n \to [0,1]$, the following are equivalent:*

1. *There is an $\mathcal{L}_{\omega_1,\omega}$ formula $\phi(\vec{x})$ with parameters from A such that for all $\vec{a} \in M^n$,*
$$\phi^M(\vec{a}) = P(\vec{a}),$$
2. *P is fixed by all automorphisms of M that fix A pointwise,*

3. P is fixed by all automorphisms of M that fix $\overline{A}$ pointwise,

Proof. Since M is a separable metric space, there is a countable set $D \subseteq A$ such that $\overline{D} = \overline{A}$ in M. An automorphism of M fixes A pointwise if and only if it fixes D pointwise if and only if it fixes $\overline{D} = \overline{A}$ pointwise, which establishes the equivalence of (2) and (3). For the equivalence of (1) and (2), apply Theorem 1.4.1 to the structure obtained from M by adding a new constant symbol for each element of D. □

Theorem 1.4.1 does not hold, as stated, with $\mathcal{L}_{\omega_1,\omega}$ replaced by $\mathcal{L}^C_{\omega_1,\omega}$ or $\mathcal{L}^C_{\omega_1,\omega}(\rho)$, because we assumed only continuity for the function P, while formulas in $\mathcal{L}^C_{\omega_1,\omega}$ always define uniformly continuous functions. Even if P is assumed to be uniformly continuous, some intermediate steps in our proof use the formulas discussed in Example 1.3.5, as well as other possibly discontinuous formulas, and hence our argument does not directly apply to give a version of Scott's definability theorem in the other infinitary logics.

Question 1.4.3. Let M be a separable complete metric structure, and let $P : M^n \to [0,1]$ be uniformly continuous and automorphism invariant. Is P definable in M by an $\mathcal{L}^C_{\omega_1,\omega}$-formula? Is P definable in M by an $\mathcal{L}^C_{\omega_1,\omega}(\rho)$-formula?

To conclude, we give one quite simple example of definability in $\mathcal{L}_{\omega_1,\omega}$ where first-order definability fails.

Example 1.4.4. Recall that a (unital) C*-algebra is a unital Banach algebra with an involution $*$ satisfying the C*-identity $\|xx^*\| = \|x\|^2$. A formalization for treating C*-algebras as metric structures is presented in [FHS14], where it is also shown that in an appropriate language the class of C*-algebras is $\forall$-axiomatizable in continuous first-order logic. The model theory of C*-algebras has since become an active area of investigation.

A *trace* on a C*-algebra A is a bounded linear functional $\tau : A \to \mathbb{C}$ such that $\tau(1) = 1$, and for all $a, b \in A$, $\tau(a^*a) \geq 0$ and $\tau(ab) = \tau(ba)$. An appropriate way to consider traces as $[0,1]$-valued predicates on the metric structure associated to a C*-algebra is given in [FHS14]. Traces appear as important tools throughout the C*-algebra literature. In the first-order continuous model theory of C*-algebras, traces play a key role in showing that certain important C*-algebras can be constructed as Fraïssé limits [EFH+16], and traces are also related to the failure of quantifier elimination for most finite-dimensional C*-algebras [EFKV15]. Several other uses of traces in the model theory of C*-algebras can be found in [FHL+16]. Of particular interest is the case where a C*-algebra has a unique trace; such algebras are called *monotracial*.

In general, traces on C*-algebras need not be automorphism invariant. For an example, consider $C(2^\omega)$, the C*-algebra of continuous complex-valued functions on the Cantor space. Pick any $z \in 2^\omega$, and define $\tau : C(2^\omega) \to \mathbb{C}$ by $\tau(f) = f(z)$. It is straightforward to verify that τ is a trace. For any other

$z' \in 2^\omega$ there is an autohomeomorphism ϕ of 2^ω sending z to z'. The map $\Phi : f \mapsto f \circ \phi$ is then an automorphism of $C(2^\omega)$, and we have $(\tau \circ \Phi)(f) = \tau(f \circ \phi) = (f \circ \phi)(z) = f(z')$, so $\tau \circ \Phi \neq \tau$.

On the other hand, it is easily seen that if τ is a trace on A and $\Phi \in \mathrm{Aut}(A)$, then $\tau \circ \Phi$ is again a trace on A. Thus for monotracial C*-algebras the unique trace *is* automorphism invariant. The following is therefore a direct consequence of Theorem 1.4.1:

Corollary 1.4.5. *If A is a separable C*-algebra with a unique trace τ, then τ is $\mathcal{L}_{\omega_1,\omega}$-definable (without parameters) in A.*

It is natural to ask whether $\mathcal{L}_{\omega_1,\omega}$-definability in Corollary 1.4.5 can be replaced by definability in a weaker logic. Monotracial C*-algebras satisfying certain additional properties do have their traces definable in first-order continuous logic (see [FHL$^+$16]), but the additional assumptions on the C*-algebras are necessary. In [FHL$^+$16] it is shown that the separable monotracial C*-algebra constructed by Robert in [Rob15, Theorem 1.4] has the property that the trace is not definable in first-order continuous logic.

The situation for definability in $\mathcal{L}^C_{\omega_1,\omega}$ is less clear. Any trace on a C*-algebra is 1-Lipschitz, and so in particular is uniformly continuous. An interesting special case of Question 1.4.3 is then whether or not the trace on a monotracial separable C*-algebra is always $\mathcal{L}^C_{\omega_1,\omega}$-definable.

Bibliography

[BF85] J. Barwise and S. Feferman (eds.), *Model-theoretic logics*, Springer-Verlag, 1985.

[BYBHU08] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, eds.), Lecture Notes series of the London Mathematical Society, no. 350, Cambridge University Press, 2008, pp. 315–427.

[BYI09] I. Ben Yaacov and J. Iovino, *Model theoretic forcing in analysis*, Annals of Pure and Applied Logic **158** (2009), no. 3, 163–174.

[BYNT14] I. Ben Yaacov, A. Nies, and T. Tsankov, *A López-Escobar theorem for continuous logic*, preprint (2014), arXiv:1407.7102.

[BYU10] I. Ben Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. **362** (2010), 5213–5259.

[CL16] S. Coskey and M. Lupini, *A López-Escobar theorem for metric structures, and the topological Vaught conjecture*, Fund. Math., to appear (2016).

[Eag14] C. J. Eagle, *Omitting types in infinitary* $[0, 1]$*-valued logic*, Annals of Pure and Applied Logic **165** (2014), 913–932.

[Eag15] ———, *Topological aspects of real-valued logic*, Ph.D. thesis, University of Toronto, 2015.

[EFH$^+$16] C. J. Eagle, I. Farah, B. Hart, B. Kadets, V. Kalashnyk, and M. Lupini, *Fraïssé limits of* C^**-algebras*, J. Symbolic Logic, to appear (2016).

[EFKV15] C. J. Eagle, I. Farah, E. Kirchberg, and A. Vignati, *Quantifier elimination in C*-algebras*, submitted (2015), arXiv:1502.00573.

[FHL$^+$16] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, *Model theory of nuclear C*-algebras*, preprint (2016), arXiv:1602.08072.

[FHS14] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model theory*, Israel J. Math. **201** (2014), 477–505.

[Gri14] J. Grinstead, *Model existence for* $[0,1]$*-valued logic*, Master's thesis, Carnegie Mellon University, 2014.

[HI03] C. W. Henson and J. Iovino, *Ultraproducts in analysis*, Analysis and Logic, London Mathematical Society Lecture Note Series, no. 262, Camb, 2003.

[Iov14] J. Iovino, *Applications of model theory to functional analysis*, Dover, Mineola, New York, 2014.

[Kei71] H.J. Keisler, *Model theory for infinitary logic: Logic with countable conjuctions and finite quantifiers*, North-Holland, 1971.

[KN51] L.M. Kelly and E.A. Nordhaus, *Distance sets in metric spaces*, Trans. Amer. Math. Soc. **7** (1951), 440–956.

[Ort97] C. Ortiz, *Truth and approximate truth in metric spaces*, Ph.D. thesis, University of Wisconsin - Madison, 1997.

[Rob15] L. Robert, *Nuclear dimension and sums of commutators*, Indiana Univ. Math. J. **64** (2015), 559–576.

[Sco65] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, The theory of models (J. W. Addison, ed.), North-Holland, Amsterdam, 1965, pp. 329–341.

[Seq13] N. Sequeira, *Infinitary continuous model theory*, Master's thesis, McMaster University, 2013.

[Sri98] S.M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer, New York, 1998.

Part II

Model Theory of Special Classes of Structures

Chapter 2

Randomizations of Scattered Sentences

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Abstract

In 1970, Morley introduced the notion of a sentence φ of the infinitary logic $L_{\omega_1\omega}$ being scattered. He showed that if φ is scattered then the class $I(\varphi)$ of isomorphism types of countable models of φ has cardinality at most $\aleph_1$, and if φ is not scattered then $I(\varphi)$ has cardinality continuum. The absolute form of Vaught's conjecture for φ says that if φ is scattered then $I(\varphi)$ is countable. Generalizing previous work of Ben Yaacov and the author, we introduce here the notion of a separable randomization of φ, which is a separable continuous structure whose elements are random elements of countable models of φ. We improve a result by Andrews and the author, showing that if $I(\varphi)$ is countable then φ has few separable randomizations, that is, every separable randomization of φ is isomorphic to a very simple structure called a basic randomization. We also show that if φ has few separable randomizations, then φ is scattered. Hence if the absolute Vaught conjecture holds for φ, then φ has few separable randomizations if and only if $I(\varphi)$ is countable, and also if and only if φ is scattered. Moreover, assum-

ing Martin's axiom for $\aleph_1$, we show that if φ is scattered then φ has few separable randomizations.

2.1 Introduction

The notion of a scattered sentence φ of the infinitary logic $L_{\omega_1\omega}$ was introduced by Michael Morley [13] in connection with Vaught's conjecture. The notion of a randomization was introduced by the author in [10] and developed in the setting of continuous model theory by Itaï Ben Yaacov and the author in [6]. The **pure randomization theory** is a continuous theory with a sort $\mathbb{K}$ for random elements and a sort $\mathbb{E}$ for events, and a set of axioms that say that there is an event corresponding to each first order formula with random elements in its argument places, and there is an atomless probability measure on the events. By a **separable randomization** of a first order theory T we mean a separable model of the pure randomization theory in which each axiom of T has probability one.

In [1], Uri Andrews and the author showed that if T is a complete theory with at most countably many countable models up to isomorphism, then T has few separable randomizations, which means that all of its separable randomizations are very simple in a sense explained below. In this paper we generalize that result by replacing the theory T with an infinitary sentence φ, and establish relationships between sentences with countably many countable models, scattered sentences, sentences with few separable randomizations, and Vaught's conjecture.

Let φ be a sentence of $L_{\omega_1\omega}$ whose models have at least two elements, and let $I(\varphi)$ be the class of isomorphism types of countable models of φ. In [13], Morley showed that if φ is scattered then $I(\varphi)$ has cardinality at most $\aleph_1$, and if φ is not scattered then $I(\varphi)$ has cardinality continuum. The absolute form of Vaught's conjecture for φ says that if φ is scattered then $I(\varphi)$ is at most countable.

In the version of continuous model theory developed in [5], the universe of a structure is a complete metric space with distance playing the role of equality, and formulas take values in the unit interval $[0, 1]$ with 0 interpreted as true. A model is separable if its universe has a countable dense subset. The **randomization signature** L^R has two sorts, $\mathbb{K}$ for random elements and $\mathbb{E}$ for events. L^R has a function symbol $[\![\psi(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula $\psi(\vec{v})$ with n free variables. The continuous term $[\![\psi(\vec{\mathbf{f}})]\!]$ is interpreted as the event that the formula $\psi(\vec{v})$ is satisfied by the n-tuple $\vec{\mathbf{f}}$ of random elements. In the event sort $\mathbb{E}$, L^R has the Boolean operations and a predicate μ. The continuous formula $\mu(\mathsf{E})$ takes values in $[0, 1]$ and is interpreted as the probability of the event E.

In Theorem 2.5.1 we show that in any separable model of the pure randomization theory, the function $[\![\psi(\cdot)]\!]$ can be extended in a natural way from the case that $\psi(\vec{v})$ is a first order formula to the case that $\psi(\vec{v})$ is a formula of $L_{\omega_1\omega}$. We can then define a **separable randomization** of an infinitary sentence φ to be a separable model of the pure randomization theory in which $[\![\varphi]\!]$ has probability one.

A **basic randomization of** φ is a very simple kind of separable randomization of φ that is determined up to isomorphism by taking a countable subset $J \subseteq I(\varphi)$ and assigning a probability $\rho(j)$ to each $j \in J$. A basic randomization of φ has a model $\mathcal{M}_j$ of isomorphism type j for each $j \in J$, and a partition of $[0,1)$ into Borel sets B_j of measure $\rho(j)$. The events are the Borel subsets of $[0,1)$ with the usual measure, and the random elements are the Borel functions that send B_j into $\mathcal{M}_j$ for each $j \in J$.

We say that φ **has few separable randomizations** if every separable randomization of φ is isomorphic to a basic randomization of φ.

In Theorem 2.9.6, we show that if $I(\varphi)$ is countable, then φ has few separable randomizations. In Theorem 2.10.1 we show that if φ has few separable randomizations, then φ is scattered. Therefore, if the absolute form of Vaught's conjecture holds for φ, then φ has few separable randomizations if and only if $I(\varphi)$ is countable, and also if and only if φ is scattered. In Theorem 2.10.3 we show that if Martin's axiom for $\aleph_1$ holds and φ is scattered, then φ has few separable randomizations.

Section 2 reviews some results we need in the literature about scattered sentences and Vaught's conjecture. Section 3 contains a review of some previous results about randomizations. In Section 4 we introduce the basic randomizations of φ. In Section 5 we introduce the separable randomizations of φ. In Section 6 we develop a key tool for constructing separable randomizations, called a countable generator, and in Section 7 we show that every separable randomization of φ is isomorphic to one that can be constructed in that way. In Section 8 we show that every separable randomization of φ can be elementarily embedded in some basic randomization if and only if only countably many first order types are realized in countable models of φ. The methods developed in Sections 6 through 8 are used to prove our main results are in Sections 9 and 10. In Section 11 we list some open questions that are related to our results.

2.2 Scattered Sentences

We fix a countable[1] first order signature L, and all first order structures mentioned are understood to have signature L. We refer to [9] for the infinitary logic $L_{\omega_1\omega}$. Note in particular that every formula of $L_{\omega_1\omega}$ has at most finitely many free variables. By a **countable fragment** L_A of $L_{\omega_1\omega}$ we mean a countable set of formulas of $L_{\omega_1\omega}$ that contains the first order formulas and is closed under subformulas, finite Boolean combinations, quantifiers, and change of free variables.

In general, the class of countable first order structures is a proper class.

[1] In this paper, "countable" means "of cardinality at most $\aleph_0$".

To avoid this problem, let $\mathbb{M}(L)$ be the class of countable structures with signature L, whose universe is $\mathbb{N}$ or an initial segment of $\mathbb{N}$. Then $\mathbb{M}(L)$ is a set, and every countable structure is isomorphic to some element of $\mathbb{M}(L)$. We define the **isomorphism type** of a countable structure $\mathcal{M}$ to be the set of all $\mathcal{H} \in \mathbb{M}(L)$ such that $\mathcal{H}$ is isomorphic to $\mathcal{M}$.

Consider a sentence φ of $L_{\omega_1\omega}$ that has at least one model. By the Löwenheim-Skolem Theorem, φ has at least one countable model. We let $I(\varphi)$ be the set of all isomorphism types of countable models of φ. By a **Scott sentence** for a countable structure $\mathcal{M}$ we mean an $L_{\omega_1\omega}$ sentence θ such that $\mathcal{M} \models \theta$, and every countable model of θ is isomorphic to $\mathcal{M}$.

Result 2.2.1. *(Scott's Theorem, [15]) Every countable structure has a Scott sentence.*

We let I be the set of all isomorphism types of countable structures of cardinality ≥ 2. Thus $I = I((\exists x)(\exists y)x \neq y)$. For each $i \in I$, we choose once and for all a Scott sentence θ_i for the countable models of isomorphism type i. We say that two countable L-structures $\mathcal{M}, \mathcal{H}$ are α-equivalent if they satisfy the same $L_{\omega_1\omega}$-sentences of quantifier rank at most α. By Scott's theorem, $\mathcal{M}$ and $\mathcal{H}$ are isomorphic if and only if they are α-equivalent for all countable α.

Several equivalent characterizations of scattered sentences were given in [4]. We will take one of these as our definition.

Definition 2.2.2. An $L_{\omega_1\omega}$ sentence φ is **scattered** if for each countable ordinal α, there are at most countably many α-equivalence classes of countable models of φ. A first order theory T is scattered if the sentence $\bigwedge T$ is scattered.

Result 2.2.3. *(Morley [13]) If φ is scattered then $I(\varphi)$ has cardinality at most $\aleph_1$, and if φ is not scattered than $I(\varphi)$ has cardinality $2^{\aleph_0}$.*

The **Vaught conjecture for** φ ([18]) says that $I(\varphi)$ is either countable or has cardinality $2^{\aleph_0}$. The **absolute Vaught conjecture for** φ (see Steel [17]) says that if φ is scattered, then $I(\varphi)$ is countable. It is called absolute because its truth does not depend on the underlying model of ZFC. In $ZFC + GCH$ the Vaught conjecture trivially holds for all φ. In $ZFC + \neg CH$, the absolute Vaught conjecture for φ is equivalent to the Vaught conjecture for φ.

Definition 2.2.4. (Morley [13]) An **enumerated structure** $(\mathcal{M}, a)$ is a countable structure $\mathcal{M}$ with signature L together with a mapping a from $\mathbb{N}$ onto the universe M of $\mathcal{M}$.

Consider a countable fragment L_A and an enumerated structure $(\mathcal{M}, a)$. We take 2^{L_A} to be the Polish space whose elements are the functions from L_A into $\{0, 1\}$. We say that a point $t \in 2^{L_A}$ **codes** $(\mathcal{M}, a)$ if for each formula $\psi \in L_A$ with at most the free variables $v_0, \ldots, v_{n-1}$, $t(\psi) = 0$ if and only if $\mathcal{M} \models \psi(a_0, \ldots, a_{n-1})$. Note that each enumerated structure is coded by a unique $t \in 2^{L_A}$.

The lemma below is a variant of Theorem 3.3 in [4], and follows from its proof.

Lemma 2.2.5. *Let φ be a sentence of $L_{\omega_1\omega}$. The following are equivalent:*

(i) φ is not scattered.

(ii) There is a countable fragment L_A of $L_{\omega_1\omega}$ and a perfect set $P \subseteq 2^{L_A}$ such that each $t \in P$ codes an enumerated model $(\mathcal{M}(t), a(t))$ of φ, and if $s \neq t$ in P then $\mathcal{M}(s)$ and $\mathcal{M}(t)$ do not satisfy the same L_A-sentences.

2.3 Randomizations of Theories

2.3.1 Continuous Structures

We assume familiarity with the basic notions about continuous model theory as developed in [5]. We give some brief reminders here.

In continuous model theory, the universe of a structure is a complete metric space, and the universe of a pre-structure is a pseudo-metric space. A structure (or pre-structure) is said to be **separable** if its universe is a separable metric space (or pseudo-metric space). Formulas take truth values in $[0, 1]$, and are built from atomic formulas using continuous connectives on $[0, 1]$ and the quantifiers $\sup, \inf$. The value 0 in interpreted as truth, and a model of a set U of sentences is a continuous structure in which each $\Phi \in U$ has truth value 0.

We extend the notions of embedding and elementary embedding to pre-structures in the natural way. Given pre-structures $\mathcal{P}$, $\mathcal{N}$, we write $h : \mathcal{P} \prec \mathcal{N}$ (h is an **elementary embedding**) if h preserves the truth values of all formulas. If $h\colon \mathcal{P} \prec \mathcal{N}$ where h is the inclusion mapping, we write $\mathcal{P} \prec \mathcal{N}$ and say that $\mathcal{P}$ is an **elementary submodel** of $\mathcal{N}$ (leaving off the 'pre-' for brevity). If $h\colon \mathcal{P} \prec \mathcal{N}$, h preserves distance but is not necessarily one-to-one. Note that compositions of elementary embeddings are elementary embeddings. We write $h\colon \mathcal{P} \cong \mathcal{N}$ if $h : \mathcal{P} \prec \mathcal{N}$ and every element of $\mathcal{N}$ is at distance zero from some element of $h(\mathcal{P})$. We say that $\mathcal{P}$ and $\mathcal{N}$ are **isomorphic**, and write $\mathcal{P} \cong \mathcal{N}$, if $h\colon \mathcal{P} \cong \mathcal{N}$ for some h. By Remark 2.4 of [1], $\cong$ is an equivalence relation on pre-structures.

We call $\mathcal{N}$ a **reduction of** $\mathcal{P}$ if $\mathcal{N}$ is obtained from $\mathcal{P}$ by identifying elements at distance zero, and call $\mathcal{N}$ a **completion of** $\mathcal{P}$ if $\mathcal{N}$ is a structure obtained from a reduction of $\mathcal{P}$ by completing the metrics. Every pre-structure has a reduction, that is unique up to isomorphism. The mapping that identifies elements at distance zero is called the **reduction mapping**, and is an isomorphism from a pre-structure onto its reduction. Similarly, every pre-structure $\mathcal{P}$ has a completion, that is unique up to isomorphism, and the reduction map is an elementary embedding of $\mathcal{P}$ into its completion.

Following [6], we say that $\mathcal{P}$ is **pre-complete** if the metrics in a reduction of $\mathcal{P}$ are already complete. Thus if $\mathcal{P}$ is pre-complete, the reductions and completions of $\mathcal{P}$ are the same, and $\mathcal{P}$ is isomorphic to its completion.

2.3.2 Randomizations

We assume that:

- L is a countable first order signature.
- T_2 is the theory with the single axiom $(\exists x)(\exists y)x \neq y$.
- T is a theory with signature L that contains T_2.
- φ is a sentence of $L_{\omega_1\omega}$ that implies T_2.

Note that T_2 is just the theory whose models have at least two elements, and $I(\varphi) \subseteq I(T_2) = I$. The randomization theory of T is a continuous theory T^R whose signature L^R has two sorts, a sort $\mathbb{K}$ for random elements of models of T, and a sort $\mathbb{E}$ for events in an underlying probability space. The probability of the event that a first order formula holds for a tuple of random elements will be expressible by a formula of continuous logic. The signature L^R has an n-ary function symbol $[\![\theta(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula θ of L with n free variables, a $[0,1]$-valued unary predicate symbol μ of sort $\mathbb{E}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbb{E}$. The signature L^R also has distance predicates $d_{\mathbb{E}}$ of sort $\mathbb{E}$ and $d_{\mathbb{K}}$ of sort $\mathbb{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbb{E}$, and $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = 0$. For readability we write $\forall, \exists$ for $\sup, \inf$.

The axioms of T^R, which are taken from [6], are as follows:

Validity Axioms

$$\forall \vec{x}([\![\psi(\vec{x})]\!] \doteq \top)$$

where $\forall \vec{x}\, \psi(\vec{x})$ is logically valid in first order logic.

Boolean Axioms The usual Boolean algebra axioms in sort $\mathbb{E}$, and the statements

$$\forall \vec{x}([\![(\neg\theta)(\vec{x})]\!] \doteq \neg[\![\theta(\vec{x})]\!])$$

$$\forall \vec{x}([\![(\varphi \vee \psi)(\vec{x})]\!] \doteq [\![\theta(\vec{x})]\!] \sqcup [\![\psi(\vec{x})]\!])$$

$$\forall \vec{x}([\![(\theta \wedge \psi)(\vec{x})]\!] \doteq [\![\theta(\vec{x})]\!] \sqcap [\![\psi(\vec{x})]\!])$$

Distance Axioms

$$\forall x \forall y\, d_{\mathbb{K}}(x,y) = 1 - \mu[\![x = y]\!], \qquad \forall \mathsf{B} \forall \mathsf{C}\, d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \mu(\mathsf{B} \triangle \mathsf{C})$$

Fullness Axioms (or Maximal Principle)

$$\forall \vec{y} \exists x([\![\theta(x, \vec{y})]\!] \doteq [\![(\exists x \theta)(\vec{y})]\!])$$

Event Axiom

$$\forall \mathsf{B} \exists x \exists y(\mathsf{B} \doteq [\![x = y]\!])$$

Measure Axioms

$$\mu[\top] = 1 \wedge \mu[\bot] = 0$$

$$\forall \mathsf{B} \forall \mathsf{C}(\mu[\mathsf{B}] + \mu[\mathsf{C}] = \mu[\mathsf{B} \sqcup \mathsf{C}] + \mu[\mathsf{B} \sqcap \mathsf{C}])$$

Atomless Axiom

$$\forall \mathsf{B} \exists \mathsf{C} (\mu[\mathsf{B} \sqcap \mathsf{C}] = \mu[\mathsf{B}]/2)$$

Transfer Axioms

$$[\![\theta]\!] \doteq \top$$

where $\theta \in T$.

By a **separable randomization of** T we mean a separable pre-model of T^R. In this paper we will focus on the **pure randomization theory** T_2^R. T_2^R has the single transfer axiom $[\![(\exists x)(\exists y) x \neq y]\!] \doteq \top$. Note that for any theory $T \supseteq T_2$, any model of T^R is a model of the pure randomization theory. By a **separable randomization** we mean a separable randomization of T_2^R. A separable randomization is called **complete** if it is a model of T_2^R, and **pre-complete** if it is a pre-complete model of T_2^R.

We will use $\mathcal{M}, \mathcal{H}$ to denote models of T_2 with signature L, and use $\mathcal{N}$ and $\mathcal{P}$ to denote models or pre-models of T_2^R with signature L^R. The universe of $\mathcal{M}$ will be denoted by M. A pre-model of T_2^R will be a pair $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ where $\mathcal{K}$ is the part of sort $\mathbb{K}$ and $\mathcal{E}$ is the part of sort $\mathbb{E}$. We write $[\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ for the interpretation of $[\![\theta(\vec{v})]\!]$ in a pre-structure $\mathcal{N}$ at a tuple $\vec{\mathbf{f}}$, and write $[\![\theta(\vec{\mathbf{f}})]\!]$ for $[\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ when $\mathcal{N}$ is clear from the context.

Result 2.3.1. *([6], Theorem 2.7) Every model or pre-complete model* $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ *of* T_2^R *has perfect witnesses, i.e.,*

(i) for each first order formula $\theta(x, \vec{y})$ *and each* $\vec{\mathbf{g}}$ *in* $\mathcal{K}^n$ *there exists* $\mathbf{f} \in \mathcal{K}$ *such that*

$$[\![\theta(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists x \theta)(\vec{\mathbf{g}})]\!];$$

(ii) for each $\mathsf{B} \in \mathcal{E}$ *there exist* $\mathbf{f}, \mathbf{g} \in \mathcal{K}$ *such that* $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$.

We let $\mathcal{L}$ be the family of Borel subsets of $[0,1)$, and let $([0,1), \mathcal{L}, \lambda)$ be the usual probability space, where λ is the restriction of Lebesgue measure to $\mathcal{L}$. We let $\mathcal{M}^{[0,1)}$ be the set of functions with countable range from $[0,1)$ into M such that the inverse image of any element of M belongs to $\mathcal{L}$. The elements of $\mathcal{M}^{[0,1)}$ are called **random elements of** $\mathcal{M}$.

Definition 2.3.2. The **Borel randomization of** $\mathcal{M}$ is the pre-structure $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for L^R whose universe of sort $\mathbb{K}$ is $\mathcal{M}^{[0,1)}$, whose universe of sort $\mathbb{E}$ is $\mathcal{L}$, whose measure μ is given by $\mu(\mathsf{B}) = \lambda(\mathsf{B})$ for each $\mathsf{B} \in \mathcal{L}$, and whose $[\![\psi(\cdot)]\!]$ functions are

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in [0,1) : \mathcal{M} \models \psi(\vec{\mathbf{f}}(t))\}.$$

(So $[\![\psi(\vec{\mathbf{f}})]\!] \in \mathcal{L}$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$). Its distance predicates are defined by

$$d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \mu(\mathsf{B} \triangle \mathsf{C}), \quad d_{\mathbb{K}}(\mathbf{f}, \mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!]),$$

where $\triangle$ is the symmetric difference operation.

Result 2.3.3. *([6], Corollary 3.6) Every Borel randomization of a countable model of T_2 is a pre-complete separable randomization (in other words, a pre-complete separable model of T_2^R).*

Result 2.3.4. *([1], Theorem 4.5) Suppose $\mathcal{N}$ is pre-complete and elementarily embeddable in the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ of a countable model of T_2. Then $\mathcal{N}$ is isomorphic to an elementary submodel of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ whose event sort is all of $\mathcal{L}$.*

2.4 Basic Randomizations

Basic randomizations are generalizations of Borel randomizations. They are very simple continuous pre-structures of sort L^R. Intuitively, a basic randomization is a combination of countably many Borel randomizations of first order structures. [1] considered basic randomizations that are combinations of Borel randomizations of models of a single complete theory T, and called them called product randomizations.

Definition 2.4.1. Suppose that

- J is a countable subset of I;
- $[0,1) = \bigcup_{j\in J} \mathsf{B}_j$ is a partition of $[0,1)$ into Borel sets of positive measure;
- for each $j \in J$, $\mathcal{M}_j$ has isomorphism type j;
- $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$ is the set of all functions $\mathbf{f}\colon [0,1) \to \bigcup_{j\in J} M_j$ such that for all $j \in J$,

$$(\forall t \in \mathsf{B}_j)\mathbf{f}(t) \in M_j \text{ and } (\forall a \in M_j)\{t \in \mathsf{B}_j : \mathbf{f}(t) = a\} \in \mathcal{L};$$

- $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is the pre-structure for L^R whose whose measure and distance functions are as in Definition 2.3.2. and $[\![\psi(\cdot)]\!]$ functions are

$$[\![\psi(\vec{\mathbf{f}})]\!] = \bigcup_{j\in J}\{t \in \mathsf{B}_j : \mathcal{M}_j \models \psi(\vec{\mathbf{f}}(t))\},$$

$(\prod_{i\in J} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L})$ is called a **basic randomization**. Given a basic randomization, we let $\mathcal{M}_t = \mathcal{M}_j$ whenever $j \in J$ and $t \in \mathsf{B}_j$. By a **basic randomization of** φ we mean a basic randomization such that $\mathcal{M}_j \models \varphi$ for each $j \in J$.

Remark 2.4.2.

1. In a basic randomization, the set $\bigcup_{j\in J} M_j$ is countable, so each $\mathbf{f} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$ has countable range.
2. If $\mathcal{M}_j \cong \mathcal{H}_j$ for each $j \in J$, then $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L}) \cong (\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$.

3. Every basic randomization $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is isomorphic to a basic randomization $(\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$ such that for each $j \in J$, $\mathcal{H}_j \in \mathbb{M}(L)$ (so the universe of $\mathcal{H}_j$ is $\mathbb{N}$ or an initial segment of $\mathbb{N}$).

4. If $\mathcal{M}_j \prec \mathcal{H}_j$ for each $j \in J$, then $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L}) \prec (\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$. (In this part we do not require that $\mathcal{H}_j$ has isomorphism type j).

Lemma 2.4.3. *Every basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is a pre-model of the pure randomization theory.*

Proof. All of the axioms for T_2^R except the Fullness Axioms hold trivially. Therefore $\mathcal{P}$ is a pseudo-metric space in both sorts. By Result 2.3.3, $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$ satisfies the Fullness Axioms for each $j \in J$, and it follows easily that $\mathcal{P}$ also satisfies the Fullness Axioms, and thus is a pre-model of T_2^R. $\blacksquare_{2.4.3}$

We next introduce useful mappings from a basic randomization $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ to the Borel randomizations $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$.

Definition 2.4.4. Suppose $\mathsf{B} \in \mathcal{L}$ and $\lambda(\mathsf{B}) > 0$. We say that a mapping ℓ **stretches** B to $[0,1)$ if ℓ is a Borel bijection from B to $[0,1)$, ℓ^{-1} is also Borel, and for each Borel set $\mathsf{A} \subseteq \mathsf{B}$, $\lambda(\ell(\mathsf{A})) = \lambda(\mathsf{A})/\lambda(\mathsf{B})$.

Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization, and for each $j \in J$, choose an ℓ_j that stretches B_j to $[0,1)$. Define the mapping $\ell_j \colon \mathcal{P} \to (\mathcal{M}_j^{[0,1)}, \mathcal{L})$ by

$$(\ell_j(\mathbf{f}))(t) = \mathbf{f}(\ell_j^{-1}(t)), \quad \ell_j(\mathsf{E}) = \ell_j(\mathsf{B}_j \cap \mathsf{E}).$$

Remark 2.4.5. Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization.

1. For each $j \in J$, there exists a mapping ℓ_j that stretches B_j to $[0,1)$.

2. ℓ_j maps $\mathcal{P}$ onto $\mathcal{P}_j := (\mathcal{M}_j^{[0,1)}, \mathcal{L})$.

3. For each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ of elements of $\mathcal{P}$ of sort $\mathbb{K}$.
$$\lambda([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{P}}) = \sum_{j\in J} \lambda(\mathsf{B}_j)\lambda([\![\psi(\ell_j\vec{\mathbf{f}})]\!]^{\mathcal{P}_j}).$$

4. $d_{\mathbb{K}}^{\mathcal{P}}(\mathbf{f}, \mathbf{g}) = \sum_{j\in J} \lambda(\mathsf{B}_j) d_{\mathbb{K}}^{\mathcal{P}_j}(\ell_j(\mathbf{f}), \ell_j(\mathbf{g}))$.

Proof. Since $\nu(\mathsf{A}) = \lambda(\mathsf{A})/\lambda(\mathsf{B}_j)$ is a probability measure on B_j, (1) follows from Theorem 17.41 in [8]. (2)–(4) are clear $\blacksquare_{2.4.5}$

The following result is a generalization of Theorem 7.3 of [1], but the proof we give here is different.

Theorem 2.4.6. *Every basic randomization is pre-complete and separable.*

Proof. Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization. By Result 2.3.3, $\mathcal{P}$ is separable and pre-complete in the event sort. For each $j \in J$, pick a mapping ℓ_j that stretches B_j to $[0,1)$. Pick an element $\mathbf{a} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.

Separability in sort $\mathbb{K}$: By 2.3.3, for each $j \in J$, there is a countable set C_j that is dense in $\mathcal{M}_j^{[0,1)}$. For each finite $F \subseteq J$, let D_F be the set of all $\mathbf{f}$ such that for all $j \in F$, $\mathbf{f}$ agrees with some element of $\ell_j^{-1} C_j$ on B_j, and $\mathbf{f}$ agrees with $\mathbf{a}$ on $[0,1) \setminus \bigcup_{i\in F} \mathsf{B}_j$. Then $D = \bigcup_F D_F$ is a countable subset of $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$. For each $\varepsilon > 0$, there is a finite $F \subseteq J$ such that $\sum_{j\in F} \mu(\mathsf{B}_j) \geq 1-\varepsilon$. It follows that for each $\mathbf{g} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, there exists $\mathbf{f} \in D_F$ such that for each $j \in F$, $d_{\mathbb{K}}(\ell_j(\mathbf{f}), \ell_j(\mathbf{g})) < \varepsilon/(|F|+1)$, and therefore by Remark 2.4.5, $d_{\mathbb{K}}(\mathbf{f},\mathbf{g}) < 2\varepsilon$. Hence D is dense in $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.

Pre-completeness in sort $\mathbb{K}$: Suppose that $\langle \mathbf{f}_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence of sort $\mathbb{K}$. By Remark 2.4.5, for each $j \in J$, $\langle \ell_j(\mathbf{f}_n)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_j^{[0,1)}$. By Result 2.3.3, $\mathcal{M}_j^{[0,1)}$ is pre-complete, so there exists $\mathbf{g}_j$ in $\mathcal{M}_j^{[0,1)}$ such that $\lim_{n\to\infty} d_{\mathbb{K}}(\ell_j(\mathbf{f}_n), \mathbf{g}_j) = 0$. Let $\mathbf{g}$ be the function that agrees with $\ell_j^{-1}\mathbf{g}_j$ on B_j for each $j \in J$. Then $\mathbf{g}_j = \ell_j(\mathbf{g}))$ for each $j \in J$, so $\lim_{n\to\infty} d_{\mathbb{K}}(\ell_j(\mathbf{f}_n), \ell_j(\mathbf{g})) = 0$. By Remark 2.4.5, $\lim_{n\to\infty} d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{g}) = 0$ in $\mathcal{P}$. $\blacksquare_{2.4.6}$

Definition 2.4.7. By a **probability density function** on I we mean a function $\rho\colon I \to [0,1]$ such that $\rho(i) = 0$ for all but countably many $i \in I$, and $\sum_i \rho(i) = 1$.

For each basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$, the function $\rho(i) = \lambda(\mathsf{B}_i)$ for $i \in J$, and $\rho(i) = 0$ for $i \in I \setminus J$, is called the **density function of** $\mathcal{P}$.

Remark 2.4.8. It easily seen that ρ is a probability density function on I if and only if ρ is the density function of some basic randomization.

The following result is a generalization of Theorem 7.5 of [1], and is proved in the same way.

Theorem 2.4.9. *Two basic randomizations are isomorphic if and only if they have the same density function.*

If a continuous structure $\mathcal{N}$ is isomorphic to a basic randomization $\mathcal{P}$, the density function of $\mathcal{P}$ is also called a density function of $\mathcal{N}$. Thus such an $\mathcal{N}$ has a unique density function, which characterizes $\mathcal{N}$ up to isomorphism.

2.5 Events Defined by Infinitary Formulas

In this section we consider arbitrary complete separable randomizations. By definition, each complete separable randomization has an event function $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula $\psi(\vec{v})$ with n free variables.

The following theorem extends this to the case where $\psi(\vec{v})$ is a formula of the infinitary logic $L_{\omega_1\omega}$.

Theorem 2.5.1. *Let $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ be a complete separable randomization, and let Ψ_n be the class of $L_{\omega_1\omega}$ formulas with n free variables. There is a unique family of functions $[\![\psi(\cdot)]\!]^{\mathcal{N}}$, $\psi \in \bigcup_n \Psi_n$, such that:*

(i) *When $\psi \in \Psi_n$, $[\![\psi(\cdot)]\!]^{\mathcal{N}} : \mathcal{K}^n \to \mathcal{E}$.*

(ii) *When ψ is a first order formula, $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ is the usual event function for the structure $\mathcal{N}$.*

(iii) $[\![\neg\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \neg[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.

(iv) $[\![(\psi_1 \vee \psi_2)(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = [\![\psi_1(\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcup [\![\psi_2(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.

(v) $[\![\bigvee_k \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.

(vi) $[\![(\exists u)\theta(u, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_{\mathbf{g}\in\mathcal{K}} [\![\theta(\mathbf{g}, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$.

Moreover, for each $\psi \in \Psi_n$, the function $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ is Lipschitz continuous with bound one, that is, for any pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{h}} \in \mathcal{K}^n$ we have

$$d_{\mathbb{E}}([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) \leq \sum_{m<n} d_{\mathbb{K}}(\mathbf{f}_m, \mathbf{h}_m).$$

Proof. We argue by induction on the complexity of formulas. Assume that the result holds for all subformulas of ψ. If ψ is a first order formula or a negation or finite disjunction, it is clear that the result holds for ψ.

Suppose $\psi = \bigvee_k \psi_k$. We show that the supremum exists. For each $m \in \mathbb{N}$ we have

$$[\![\bigvee_{k=0}^{m} \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \bigsqcup_{k=0}^{m} [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}.$$

This is increasing in k, so by the completeness of the metric $d_{\mathbb{E}}$ on $\mathcal{E}$, $\lim_{k\to\infty}[\![\bigvee_{j=0}^{k} \psi_j(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ exists and is equal to $\sup_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$. By hypothesis, the Lipschitz condition holds for each ψ_k. It follows that the Lipschitz condition also holds for ψ.

Now suppose $\psi(\vec{v}) = (\exists u)\theta(u, \vec{v})$. We again show first that the supremum exists. By separability, there is a countable dense subset $D = \{\mathbf{d}_k : k \in \mathbb{N}\}$ of $\mathcal{K}$. It follows from the axioms of T_2^R that there is a sequence $\langle \mathbf{g}_k \rangle_{k\in\mathbb{N}}$ of elements of $\mathcal{K}$ such that $\mathbf{g}_0 = \mathbf{d}_0$ and for each k, $\mathbf{g}_{k+1}$ agrees with $\mathbf{g}_k$ on the event $[\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$ and agrees with $\mathbf{d}_k$ elsewhere. Then for each $m \in \mathbb{N}$ we have

$$[\![\theta(\mathbf{g}_m, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \bigsqcup_{k=0}^{m} [\![\theta(\mathbf{d}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}.$$

So whenever $k \leq m$, we have

$$[\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq [\![\theta(\mathbf{g}_m, \vec{\mathbf{f}})]\!]^{\mathcal{N}},$$

and hence

$$\mathsf{E} := \lim_{k\to\infty} [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_{k\in\mathbb{N}} [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$$

exists in $\mathcal{E}$.

Consider any $\mathbf{h} \in \mathcal{K}$. To show that the supremum $\sup_{\mathbf{h}\in\mathcal{K}} [\![\theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$ exists in $\mathcal{E}$, it suffices to show that $[\![\theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}$, because it will then follow that E is the desired supremum. Let $\varepsilon > 0$. For some $k \in \mathbb{N}$ we have $d_{\mathbb{K}}(\mathbf{d}_k, \mathbf{h}) < \varepsilon$. Moreover,

$$[\![\mathbf{d}_k = \mathbf{h} \wedge \theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = [\![\mathbf{d}_k = \mathbf{h} \wedge \theta(\mathbf{d}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}.$$

Then

$$[\![\theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcap \neg\mathsf{E} \sqsubseteq [\![\mathbf{d}_k \neq \mathbf{h}]\!]^{\mathcal{N}},$$

so

$$\mu([\![\theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcap \neg\mathsf{E}) \le \mu([\![\mathbf{d}_k \neq \mathbf{h}]\!]^{\mathcal{N}}) = d_{\mathbb{K}}(\mathbf{d}_k, \mathbf{h}) < \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have $[\![\theta(\mathbf{h}, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}$.

To prove the Lipschitz condition for ψ, we consider a pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{h}} \in \mathcal{K}^n$. By the preceding paragraph we have

$$[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \lim_{k\to\infty} [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}, \quad [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}} = \lim_{k\to\infty} [\![\theta(\mathbf{g}_k, \vec{\mathbf{h}})]\!]^{\mathcal{N}}.$$

Therefore for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d_{\mathbb{E}}([\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) < \varepsilon, \quad d_{\mathbb{E}}([\![\theta(\mathbf{g}_k, \vec{\mathbf{h}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) < \varepsilon.$$

By the Lipschitz condition for $\theta(u, \vec{v})$ we have

$$d_{\mathbb{E}}([\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\theta(\mathbf{g}_k, \vec{\mathbf{h}})]\!]^{\mathcal{N}}) \le \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j, \mathbf{h}_j).$$

Then by the triangle inequality, for every $\varepsilon > 0$ we have

$$d_{\mathbb{E}}(([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, d_{\mathbb{E}}([\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) < \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j, \mathbf{h}_j) + 2\varepsilon,$$

so

$$d_{\mathbb{E}}(([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, d_{\mathbb{E}}([\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) \le \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j, \mathbf{h}_j).$$

$\blacksquare_{2.5.1}$

Remark 2.5.2. The proof of Theorem 2.5.1 only used the metric completeness of the sort $\mathbb{E}$ part of $\mathcal{N}$. Hence the result also holds in the case that $\mathcal{N}$ is a separable randomization that has a metric in sort $\mathbb{K}$ and a complete metric in sort $\mathbb{E}$.

Corollary 2.5.3. *Suppose that $\mathcal{N}, \mathcal{P}$ are complete separable randomizations and $h : \mathcal{N} \cong \mathcal{P}$. Then for every $L_{\omega_1\omega}$ formula $\psi(\vec{v})$ and every tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{N}$, we have $h([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) = [\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{P}}$.*

Proof. By Theorem 2.5.1 and an easy induction on the complexity of ψ. $\blacksquare_{2.5.3}$

When $\mathcal{P}$ is a pre-complete separable randomization, h is the reduction map from $\mathcal{P}$ onto its completion $\mathcal{N}$, and $\psi(\vec{v})$ is a formula of $L_{\omega_1\omega}$, then $[\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ is uniquely defined by Theorem 2.5.1. In that case, we will sometimes abuse notation and write $\mu([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{P}})$ for $\mu([\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}})$.

We can now define the notion of a separable randomization of φ.

Definition 2.5.4. We say that $\mathcal{N}$ is a **complete separable randomization of** φ if $\mathcal{N}$ is a complete separable randomization such that $[\![\varphi]\!]^{\mathcal{N}}$ is the true event $\top$. We call $\mathcal{P}$ a **separable randomization of** φ if the completion of $\mathcal{P}$ is a complete separable randomization of φ. We say that φ **has few separable randomizations** if every complete separable randomization of φ is isomorphic to a basic randomization.

Thus when φ has few separable randomizations, each complete separable randomization $\mathcal{N}$ of φ has a unique density function ρ, and ρ characterizes $\mathcal{N}$ up to isomorphism.

Corollary 2.5.5. *Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization with completion $\mathcal{N}$, and let $h\colon \mathcal{P} \cong \mathcal{N}$ be the reduction map. For each $L_{\omega_1\omega}$ formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ in $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, $[\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ is the reduction of the event*

$$\bigcup_{j\in J}\{t \in \mathsf{B}_j : \mathcal{M}_j \models \psi(\vec{\mathbf{f}}(t))\}.$$

Hence $\mathcal{P}$ is a basic randomization of φ if and only if $\mathcal{P}$ is a basic randomization and $\mathcal{P}$ is a separable randomization of φ.

Proof. In the case that $\psi(\vec{v})$ is an atomic formula, the result holds by definition. A routine induction on the complexity of formulas gives the result for arbitrary $L_{\omega_1\omega}$ formulas. $\blacksquare_{2.5.5}$

Note that the complete separable randomizations of the sentence $\bigwedge T$ are exactly the separable models of the continuous theory T^R. With more overhead, we could have taken an alternative approach in which the complete separable randomizations of an $L_{\omega_1\omega}$ sentence φ are exactly the separable models of a theory φ^R in an infinitary continuous logic such as the logic in [7]. The idea would be to consider a countable fragment L_A of $L_{\omega_1\omega}$, and have the randomization signature $(L_A)^R$ contain a function symbol $[\![\psi(\cdot)]\!]$ for each formula $\psi(\vec{v})$ of L_A. Then Theorem 2.5.1 shows that every separable randomization can be expanded in a unique way to a model with the signature $(L_A)^R$ that satisfies the infinitary sentences corresponding to the conditions (i)–(v). In this approach, φ^R would be the theory in infinitary continuous logic with the axioms of the pure randomization theory plus the above infinitary sentences and an axiom stating that $[\![\varphi]\!] \doteq \top$.

2.6 Countable Generators of Randomizations

In this section we give a general method of constructing pre-complete separable randomizations. In the next section we will show that every pre-complete separable randomization is isomorphic to one that can be constructed in that way.

Definition 2.6.1. Assume that $(\Omega, \mathcal{E}, \nu)$ is an atomless probability space such that the metric space $(\mathcal{E}, d_{\mathbb{E}})$ is separable, and for each $t \in \Omega$, $\mathcal{M}_t$ is a countable model of T_2.

A **countable generator** (in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$) is a countable set C of elements $\mathbf{c} \in \prod_{t\in\Omega} M_t$ such that:

(a) $M_t = \{\mathbf{c}(t) \colon \mathbf{c} \in C\}$ for each $t \in \Omega$, and

(b) For every first order atomic formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{b}}$ in C,

$$\{t \in \Omega : \mathcal{M}_t \models \psi(\vec{\mathbf{b}}(t))\} \in \mathcal{E}.$$

Theorem 2.6.2. *Let C be a countable generator in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$. There is a unique pre-structure $\mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$ such that:*

(c) $\mathcal{K}$ is the set of all $\mathbf{f} \in \prod_{t\in\Omega} M_t$ such that $\{t \in \Omega \colon \mathbf{f}(t) = \mathbf{c}(t)\} \in \mathcal{E}$ for each $\mathbf{c} \in C$;

(d) $\top, \bot, \sqcup, \sqcap, \neg$ are the usual Boolean operations on $\mathcal{E}$, and μ is the measure ν;

(e) for each first order formula $\psi(\vec{x})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, we have

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in \Omega : \mathcal{M}_t \models \psi(\mathbf{f}(t))\};$$

(f) $d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \nu(\mathsf{B} \triangle \mathsf{C})$, $\quad d_{\mathbb{K}}(\mathbf{f}, \mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!])$.

Moreover, $\mathcal{P}(C)$ is a pre-complete separable randomization.

Proof of Theorem 2.6.2. It is clear that $\mathcal{P}(C)$ is unique. We first show by induction on the complexity of formulas that condition (b) holds for all first order formulas ψ. The steps for logical connectives are trivial. For the quantifier step, suppose (b) holds for $\psi(u, \vec{v})$. Then by (a) and (c)–(f),

$$[\![(\exists u)\psi(u, \vec{\mathbf{b}})]\!] = \{t \colon \mathcal{M}_t \models (\exists u)\psi(u, \vec{\mathbf{b}}(t))\} = \{t \colon (\exists c \in M_t)\mathcal{M}_t \models \psi(c, \vec{\mathbf{b}}(t))\} =$$

$$= \{t \colon (\exists \mathbf{c} \in C)\mathcal{M}_t \models \psi(\mathbf{c}(t), \vec{\mathbf{b}}(t))\} = \bigcup_{\mathbf{c}\in C} [\![\psi(\mathbf{c}, \vec{\mathbf{b}})]\!] \in \mathcal{E},$$

so (b) holds for $(\exists u)\psi(u, \vec{v})$. By the definition of $\mathcal{K}$, for each tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$ and $\vec{\mathbf{b}}$ in C, we have $[\![\vec{\mathbf{g}} = \vec{\mathbf{b}}]\!] \in \mathcal{E}$. Then for every first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$,

$$[\![\psi(\vec{\mathbf{g}})]\!] = \bigcup \{[\![\psi(\vec{\mathbf{b}}) \wedge \vec{\mathbf{g}} = \vec{\mathbf{b}}]\!] \colon \vec{\mathbf{b}} \text{ is a tuple in } C\}.$$

We therefore have

(b') For each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$, $[\![\psi(\vec{\mathbf{g}})]\!] \in \mathcal{E}$.

This shows that $\mathcal{P}(C)$ is a pre-structure with signature L^R.

It is easily seen that $\mathcal{P}(C)$ satisfies all the axioms of T_2^R except possibly the Fullness and Event Axioms. We next show that $\mathcal{P}(C)$ has perfect witnesses. Once this is done, it follows at once that $\mathcal{P}(C)$ also satisfies the Fullness and Event Axioms, and hence is a pre-model of T_2^R.

Consider a first order formula $\psi(u,\vec{v})$ and a tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$. For each $t \in \Omega$, there is a least $n(t) \in \mathbb{N}$ such that $\mathcal{M}_t \models (\exists u)\psi(u,\vec{\mathbf{g}}(t)) \to \psi(\mathbf{c}_{n(t)}(t),\vec{\mathbf{g}}(t))$. Since (b') holds and $C \subseteq \mathcal{K}$, the function $\mathbf{f}$ such that $\mathbf{f}(t) := \mathbf{c}_{n(t)}(t)$ belongs to $\mathcal{K}$. Therefore

$$[\![\psi(\mathbf{f},\vec{\mathbf{g}})]\!] \doteq [\![(\exists u)\psi(u,\vec{\mathbf{g}})]\!].$$

Now consider an event $\mathsf{E} \in \mathcal{E}$. Since each $\mathcal{M}_t \models T_2$, we have $[\![(\exists u)u \neq \mathbf{c}_0]\!] \doteq \top$. Therefore there exists $\mathbf{f} \in \mathcal{K}$ such that $[\![\mathbf{f} \neq \mathbf{c}_0]\!] \doteq \top$. Then the function $\mathbf{g}$ such that $\mathbf{g}(t) = \mathbf{f}(t)$ for $t \in \mathsf{E}$ and $\mathbf{g}(t) = \mathbf{c}_0(t)$ for $t \notin \mathsf{E}$ belongs to $\mathcal{K}$, and $[\![\mathbf{f} = \mathbf{g}]\!] \doteq \mathsf{E}$. This shows that $\mathcal{P}(C)$ has perfect witnesses, so $\mathcal{P}(C)$ is a pre-model of T_2^R.

We now show that $\mathcal{P}(C)$ is pre-complete. This means that when d is either $d_{\mathbb{K}}$ or $d_{\mathbb{E}}$, for every Cauchy sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ with respect to d, there exists x such that $d(x_n,x) \to 0$ as $n \to \infty$. This is clear for $d_{\mathbb{E}}$ because $(\Omega,\mathcal{E},\nu)$ is countably additive. Suppose $\langle \mathbf{f}_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for $d_{\mathbb{K}}$. Let $C = \{\mathbf{c}_k \colon k \in \mathbb{N}\}$, and $C_m = \{\mathbf{c}_0,\ldots,\mathbf{c}_m\}$. For each $k \in \mathbb{N}$, $\langle [\![\mathbf{f}_n = \mathbf{c}_k]\!]\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $d_{\mathbb{E}}$. Therefore there exists $\mathsf{B}_k \in \mathcal{E}$ such that $\lim_{n\to\infty} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!],\mathsf{B}_k) = 0$. Then $\mu(\mathsf{B}_k) = \lim_{n\to\infty}\mu([\![\mathbf{f}_n = \mathbf{c}_k]\!])$. We now cut the sets B_k down to disjoint sets with the same unions. Let $\mathsf{A}_0 = \mathsf{B}_0$, and for each m, let $\mathsf{A}_{m+1} = \mathsf{B}_{m+1} \setminus \bigcup_{k=0}^m \mathsf{B}_k$. Note that for all m,

$$\bigcup_{k=0}^m \mathsf{A}_k = \bigcup_{k=0}^m \mathsf{B}_k, \quad \mathsf{A}_k \subseteq \mathsf{B}_k, \quad (\forall k < m)\mathsf{A}_k \cap \mathsf{A}_m = \emptyset.$$

Claim. $\mu(\bigcup_{k=0}^\infty \mathsf{A}_k) = 1$.

Proof of Claim: Fix an $\varepsilon > 0$. We show that there exists m such that $\mu(\bigcup_{k=0}^m \mathsf{B}_k) > 1-\varepsilon$. Note that for each m,

$$\mu(\bigcup_{k=0}^m \mathsf{B}_k) = \lim_{n\to\infty}\mu([\![\mathbf{f}_n \in C_m]\!]).$$

Therefore it suffices to show that

$$(\exists m)(\forall n)\mu([\![\mathbf{f}_n \in C_m]\!]) > 1-\varepsilon.$$

Suppose this is not true. Then

$$(\forall m)(\exists n)\,\mu([\![\mathbf{f}_n \notin C_m]\!]) \geq \varepsilon.$$

Since $C = \bigcup_m C_m$,

$$(\forall n)(\exists h)\,\mu([\![\mathbf{f}_n \in C_h]\!]) \geq 1-\varepsilon/2,$$

so

$$(\forall m)(\exists n)(\exists h)\mu([\![\mathbf{f}_n \in (C_h \setminus C_m)]\!]) \geq \varepsilon/2.$$

It follows that there are sequences $n_0 < n_1 < \ldots$ and $m_0 < m_1 < \ldots$ such that

$$(\forall k)\mu([\![\mathbf{f}_{n_k} \in (C_{m_{k+1}} \setminus C_{m_k})]\!]) \geq \varepsilon/2.$$

Therefore

$$(\forall k)(\forall h > k) d_{\mathbb{K}}(\mathbf{f}_{n_k}, \mathbf{f}_{n_h}) \geq \varepsilon/2.$$

This contradicts the fact that $\langle \mathbf{f}_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence, and the Claim is proved.

By Condition (c), there is an $\mathbf{f}$ in $\mathcal{P}(C)$ such that $\mathbf{f}$ agrees with $\mathbf{c}_k$ on A_k for each $k \in \mathbb{N}$. For each n and h we have

$$d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) = \mu([\![\mathbf{f}_n \neq \mathbf{f}]\!]) = \sum_{k=0}^{\infty} \mu([\![\mathbf{f}_n \neq \mathbf{f}]\!] \cap \mathsf{A}_k) = \sum_{k=0}^{\infty} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{A}_k) \leq$$

$$\leq \sum_{k=0}^{h} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{A}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k) \leq \sum_{k=0}^{h} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{B}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k)$$

$$\leq \sum_{k=0}^{h} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!], \mathsf{B}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k).$$

By the Claim, for each $\varepsilon > 0$ we may take h such that $\mu(\bigcup_{k>h} \mathsf{A}_k) < \varepsilon/2$. For all sufficiently large n we have

$$\sum_{k=0}^{h} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!], \mathsf{B}_k) < \varepsilon/2,$$

and hence $d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) < \varepsilon$. It follows that $\lim_{n \to \infty} d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) = 0$, so $\mathcal{P}(C)$ is pre-complete.

We have not yet used the hypothesis that $(\mathcal{E}, d_{\mathbb{E}})$ is separable. We use it now to show that $\mathcal{P}(C)$ is separable. The Boolean algebra $\mathcal{E}$ has a countable subalgebra $\mathcal{E}_0$ such that $\mathcal{E}_0$ is dense with respect to $d_{\mathbb{E}}$, and $[\![\psi(\vec{\mathbf{b}})]\!] \in \mathcal{E}_0$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{b}}$ in C. Let D be the set of all $\mathbf{f} \in \mathcal{K}$ such that for some $k \in \mathbb{N}$, $[\![\mathbf{f} \in C_k]\!] = \top$ and $[\![\mathbf{f} = \mathbf{c}_n]\!] \in \mathcal{E}_0$ for all $n \leq k$. Then D is countable and D is dense in $\mathcal{K}$ with respect to $d_{\mathbb{K}}$, so $\mathcal{P}(C)$ is separable. $\blacksquare_{2.6.2}$

Remark 2.6.3. Suppose C is a countable generator in $\langle \mathcal{M}_t \rangle_{t \in \Omega}$ over $(\Omega, \mathcal{E}, \nu)$, and let $\mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$. Then:

1. $C \subseteq \mathcal{K}$.
2. If $C \subseteq D \subseteq \mathcal{K}$ and D is countable, then D is a countable generator in $\langle \mathcal{M}_t \rangle_{t \in \Omega}$.
3. For each $t \in \Omega$, $M_t = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{K}\}$.

4. If $\mathcal{M}_t \cong \mathcal{H}_t$ for all t, then there is a countable generator D in $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ such that $\mathcal{P}(D) \cong \mathcal{P}(C)$.

Proof. We prove (4). For each t, choose an isomorphism $h_t : \mathcal{M}_t \cong \mathcal{H}_t$. For each $\mathbf{c} \in C$, define $h\mathbf{c}$ by $(h\mathbf{c})(t) = h_t(\mathbf{c}(t))$ and let $D = \{h\mathbf{c} : \mathbf{c} \in C\}$. Then D is a countable generator in $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ and $\mathcal{P}(D) \cong \mathcal{P}(C)$. $\blacksquare_{2.6.3}$

The next corollary connects countable generators to basic randomizations.

Corollary 2.6.4. *Let $\mathcal{N} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization.*

(i) There is a countable generator C in $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$ such that $C \subseteq \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.

(ii) If C is as in (i), then $\mathcal{P}(C) = \mathcal{N}$.

(iii) If C is a countable generator in $\langle \mathcal{H}_t \rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$, $C \subseteq \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, and $\mathcal{H}_t \prec \mathcal{M}_t$ for all t, then $\mathcal{P}(C) \prec \mathcal{N}$.

Proof. (i) For each $j \in J$, choose an enumerated structure $(\mathcal{M}_j, a_{j,0}, a_{j,1}, \ldots)$. Let $C = \{\mathbf{c}_n : n \in \mathbb{N}\}$ where $\mathbf{c}_n(t) = a_{j,n}$ whenever $j \in J$ and $t \in \mathsf{B}_j$. C has the required properties.

(ii) Let $\mathcal{P}(C) = (\mathcal{K}, \mathcal{L})$. Since $C \subseteq \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, for all $j \in J, a \in M_j$, and $\mathbf{c} \in C$ we have $\{t \in \mathsf{B}_j : \mathbf{c}(t) = a\} \in \mathcal{L}$. It follows that for each $j \in J$ and $\mathbf{f}$,

$$(\forall a \in M_j)\{t \in \mathsf{B}_j : \mathbf{f}(t) = a\} \in \mathcal{L} \Leftrightarrow (\forall \mathbf{c} \in C)\{t \in \mathsf{B}_j : \mathbf{f}(t) = \mathbf{c}(t)\} \in \mathcal{L}.$$

Therefore $\mathcal{K} = \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, and (ii) holds.

(iii) Let $\mathcal{P}(C) = (\mathcal{K}, \mathcal{L})$. For each $\mathbf{f} \in \mathcal{K}$ we have $[0,1) = \bigcup_{\mathbf{c}\in C}\{t : \mathbf{f}(t) = \mathbf{c}(t)\}$, and $\{t : \mathbf{f}(t) = \mathbf{c}(t)\} \in \mathcal{L}$ for all $\mathbf{c} \in C$. Therefore $\mathcal{K} \subseteq \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$. Since $\mathcal{H}_t \prec \mathcal{M}_t$, $[\![\psi(\cdot)]\!]$ has the same interpretation in $\mathcal{P}(C)$ as in $\mathcal{N}$ for every first order formula $\psi(\vec{v})$. Therefore $(\mathcal{K}, \mathcal{L})$ is a pre-substructure of $\mathcal{N}$. By quantifier elimination (Theorem 2.9 of [6]) we have $\mathcal{P}(C) \prec \mathcal{N}$. $\blacksquare_{2.6.4}$

The next result gives a very useful "pointwise" characterization of the event corresponding to an infinitary formula in a complete separable randomization that is isomorphic to $\mathcal{P}(C)$.

Proposition 2.6.5. *Suppose $\mathcal{N}$ is a complete separable randomization, C is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$, and $h : \mathcal{P}(C) \cong \mathcal{N}$. Then for every $L_{\omega_1\omega}$ formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{P}(C)$, we have*

$$\{t : \mathcal{M}_t \models \psi(\vec{\mathbf{f}}(t))\} \in \mathcal{E}, \qquad [\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}} = h(\{t : \mathcal{M}_t \models \psi(\vec{\mathbf{f}}(t))\}).$$

Moreover, $\mathcal{N}$ is a separable randomization of φ if and only if $\mu(\{t : \mathcal{M}_t \models \varphi\}) = 1$.

Proof. This is proved by a straightforward induction on the complexity of $\psi(\vec{v})$ using Theorems 2.5.1 and 2.6.2. $\blacksquare_{2.6.5}$

2.7 A Representation Theorem

In this section we show that every complete separable randomization of φ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in countable models of φ.

We will use the following result, which is a consequence of Theorem 3.11 of [3], and generalizes Proposition 2.1.10 of [2].

Proposition 2.7.1. *For every pre-complete model $\mathcal{N}'$ of T^R, there is an atomless probability space $(\Omega, \mathcal{E}, \nu)$ and a family of models $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ of T such that $\mathcal{N}'$ is isomorphic to a pre-complete model $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ of T^R such that $\mathcal{K} \subseteq \prod_{t\in\Omega} M_t$ and $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 2.6.2.*

Proof. Proposition 2.1.10 of [2] gives this result in the case that T is a complete theory, with the additional conclusion that there is a single model $\mathcal{M}$ of T such that $\mathcal{M}_t \prec \mathcal{M}$ for all $t \in \Omega$ [2]. The same argument works in the general case, but without the model $\mathcal{M}$. $\blacksquare_{2.7.1}$

Proposition 2.7.2. *Suppose $\mathcal{N}'$ is pre-complete and elementarily embeddable in a basic randomization. Then $\mathcal{N}'$ is isomorphic to a pre-complete elementary submodel $\mathcal{N}$ of a basic randomization $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ such that the event sort of $\mathcal{N}$ is all of $\mathcal{L}$. Moreover, Conditions (d), (e), and (f) of Theorem 2.6.2 hold for $\mathcal{N} = (\mathcal{K}, \mathcal{L})$ and $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$.*

Proof. Suppose $\mathcal{N}' \cong \mathcal{N}'' \prec (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$. For each $j \in J$, let ℓ_j be a mapping that stretches B_j to $[0, 1)$. Then ℓ_j maps $\mathcal{N}''$ onto a pre-complete elementary submodel $\mathcal{N}_j$ of $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$. By Result 2.3.4, $\mathcal{N}_j$ is isomorphic to a pre-complete elementary submodel of $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$ with event sort $\mathcal{L}$. Using the inverse mappings ℓ_j^{-1}, it follows that $\mathcal{N}''$ is isomorphic to a pre-complete elementary submodel $\mathcal{N} = (\mathcal{K}, \mathcal{L}) \prec (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ with event sort $\mathcal{L}$. It is easily checked that $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 2.6.2. $\blacksquare_{2.7.2}$

Theorem 2.7.3. *(Representation Theorem) Every pre-complete separable randomization $\mathcal{N}$ of φ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family of countable models of φ. Moreover, if $\mathcal{N}$ is elementarily embeddable in some basic randomization, then C can be taken to be over the probability space $([0, 1), \mathcal{L}, \lambda)$.*

Proof. Let $\mathcal{N}'$ be a pre-complete separable randomization of φ. By Proposition 2.7.1, there is an atomless probability space $(\Omega, \mathcal{E}, \nu)$ and a family of models $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ such that $\mathcal{N}'$ is isomorphic to a pre-complete model $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ of T_2^R where $\mathcal{K} \subseteq \prod_{t\in\Omega} M_t$ and $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 2.6.2. If $\mathcal{N}'$ is elementarily embeddable in a basic randomization, then by Proposition 2.7.2, we may take $(\Omega, \mathcal{E}, \nu) = ([0, 1), \mathcal{L}, \lambda)$.

Since $\mathcal{N}$ is separable, there is a countable pre-structure $(\mathcal{J}_0, \mathcal{A}_0) \prec \mathcal{N}$ that

[2] In [2], $\mathcal{P}$ is called a neat randomization of $\mathcal{M}$

is dense in $\mathcal{N}$. We will use an argument similar to the proofs of Lemmas 4.7 and 4.8 of [1]. By Result 2.3.1, $\mathcal{N}$ has perfect witnesses. Hence by listing the first order formulas, we can construct a chain of countable pre-structures $(\mathcal{J}_n, \mathcal{A}_n), n \in \mathbb{N}$ such that for each n:

- $(\mathcal{J}_n, \mathcal{A}_n) \subseteq (\mathcal{J}_{n+1}, \mathcal{A}_{n+1}) \subseteq \mathcal{N}$;
- for each first order formula $\theta(u, \vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{J}_n$ there exists $\mathbf{f} \in \mathcal{J}_{n+1}$ such that
$$[\![\theta(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists u\theta)(\vec{\mathbf{g}})]\!];$$
- For each $\mathsf{B} \in \mathcal{A}_n$ there exist $\mathbf{f}, \mathbf{g} \in \mathcal{J}_{n+1}$ such that $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$.

The union
$$\mathcal{P} = (\mathcal{J}, \mathcal{A}) = \bigcup_n (\mathcal{J}_n, \mathcal{A}_n)$$
is a countable dense elementary submodel of $\mathcal{N}$ that has perfect witnesses. Therefore for each first order formula $\theta(u, \vec{v})$ and each tuple $\vec{\mathbf{g}}$ in $\mathcal{J}$, there exists $\mathbf{f} \in \mathcal{J}$ such that
$$[\![(\exists u)\theta(u, \vec{\mathbf{g}})]\!]^{\mathcal{N}} = [\![(\exists u)\theta(u, \vec{\mathbf{g}})]\!]^{\mathcal{P}} \doteq [\![\theta(\mathbf{f}, \vec{\mathbf{g}})]\!]^{\mathcal{P}} = [\![\theta(\mathbf{f}, \vec{\mathbf{g}})]\!]^{\mathcal{N}}.$$
Since $\mathcal{J}$ is countable, there is an event $\mathsf{E} \in \mathcal{E}$ such that $\nu(\mathsf{E}) = 1$ and for every tuple $\vec{\mathbf{g}}$ in $\mathcal{J}$ there exists $\mathbf{f} \in \mathcal{J}$ so that
$$(\forall t \in \mathsf{E})\mathcal{M}_t \models [(\exists u)\theta(u, \vec{\mathbf{g}}(t)) \leftrightarrow \theta(\mathbf{f}(t), \vec{\mathbf{g}}(t))].$$
For each $t \in \Omega$ let $\mathcal{H}_t = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{J}\}$. By the Tarski-Vaught test, we have $\mathcal{H}_t \prec \mathcal{M}_t$, and hence $\mathcal{H}_t \models T_2$, for each $t \in \mathsf{E}$.

Pick a countable model $\mathcal{H}$ of φ. For any set $\mathsf{D} \subseteq \mathsf{E}$ such that $\mathsf{D} \in \mathcal{E}$ and $\nu(\mathsf{D}) = 1$, let C^{D} be the set of all functions that agree with an element of $\mathcal{J}$ on D and take a constant value in $\mathcal{H}$ on $\Omega \setminus \mathsf{D}$. Let $\mathcal{H}_t^{\mathsf{D}} = \mathcal{H}_t$ for $t \in \mathsf{D}$, and $\mathcal{H}_t^{\mathsf{D}} = \mathcal{H}$ for $t \in \Omega \setminus \mathsf{D}$. Then $\mathcal{H}_t^{\mathsf{D}}$ is a model of T_2 for each $t \in \Omega$, and C^{D} is a countable generator in $\langle \mathcal{H}_t^{\mathsf{D}} \rangle_{t \in \Omega}$. By Theorem 2.6.2, $\mathcal{P}(C^{\mathsf{D}})$ is a pre-complete separable randomization. The reduction of $(\mathcal{J}, \mathcal{A})$ is dense in the reductions of $\mathcal{N}$ and of $\mathcal{P}(C^{\mathsf{D}})$, and both $\mathcal{N}$ and $\mathcal{P}(C^{\mathsf{D}})$ are pre-complete. Therefore $\mathcal{N} \cong \mathcal{P}(C^{\mathsf{D}})$.

In particular, C^{E} is a countable generator in $\langle \mathcal{H}_t^{\mathsf{E}} \rangle_{t \in \Omega}$, and $\mathcal{N} \cong \mathcal{P}(C^{\mathsf{E}})$. Now let $\mathsf{D} = \{t \in \mathsf{E} \colon \mathcal{H}_t^{\mathsf{E}} \models \varphi\}$. Since $\mathcal{N}$ is a pre-complete randomization of φ, we see from Proposition 2.6.5 that $\mu(\mathsf{D}) = 1$. Then $\mathcal{H}_t^{\mathsf{D}} \models \varphi$ for all $t \in \Omega$, $\mathcal{P}(C^{\mathsf{D}}) \cong \mathcal{N}$, and C^{D} is a countable generator in a family of countable models of φ. $\blacksquare_{2.7.3}$

2.8 Elementary Embeddability in a Basic Randomization

Let $S_n(T)$ be the set of first order n-types realized in countable models of T, and $S_n(\varphi)$ be the set of first order types realized in countable models of φ. Note that $S_0(\varphi) = \{Th(\mathcal{M}) \colon \mathcal{M} \models \varphi\}$.

Theorem 3.12 in [6] and Proposition 5.7 in [1] show that:

Result 2.8.1. *Let T be complete. The following are equivalent:*

(i) $\bigcup_n S_n(T)$ *is countable.*

(ii) Every complete separable randomization of T is elementarily embeddable in the Borel randomization of a countable model of T.

(iii) For every complete separable randomization $\mathcal{N}$ of T, $n \in \mathbb{N}$, and n-tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{N}$, there is a type $p \in S_n(T)$ such that $\mu([\![\bigwedge p(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) > 0$.

In Theorem 2.8.3 below, we generalize this result by replacing a complete theory T and a Borel randomization by an arbitrary $L_{\omega_1\omega}$ sentence φ and a basic randomization.

We will use Proposition 6.2 of [1], which can be formulated as follows.

Result 2.8.2. *Let T be complete. The following are equivalent:*

(i) $\mathcal{N}$ is a complete separable randomization of T and for each n and each n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, $\sum_{q\in S_n(T)} \mu([\![\bigwedge q(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) = 1$.

(ii) $\mathcal{N}$ is elementarily embeddable in the Borel randomization of a countable model of T.

Theorem 2.8.3. *The following are equivalent:*

(i) $\bigcup_n S_n(\varphi)$ *is countable.*

(ii) Every complete separable randomization of φ is elementarily embeddable in a basic randomization.

(iii) For every complete separable randomization $\mathcal{N}$ of φ, $n \in \mathbb{N}$, and n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, there is a type $p \in S_n(\varphi)$ such that $\mu([\![\bigwedge p(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) > 0$.

In (ii), we do not know whether the basic randomization can be taken to be a basic randomization of φ.

Proof of Theorem 2.8.3. We first assume (i) and prove (ii). Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 2.7.3, there is a countable generator C in a family of countable models $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ of φ over an atomless probability space $(\Omega, \mathcal{E}, \nu)$, such that $\mathcal{N} \cong \mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$. For each $t \in \Omega$, $\mathcal{M}_t$ is a countable model of φ, so $Th(\mathcal{M}_t) \in S_0(\varphi)$. By (i), $S_0(\varphi)$ is countable. Let $\mathsf{B}_T = \{t \in \Omega \colon \mathcal{M}_t \models T\}$. By Proposition 2.6.5, $\mathsf{B}_T \in \mathcal{E}$. Let $G = \{T \in S_0(\varphi) \colon \nu(\mathsf{B}_T) > 0\}$, and consider any $T \in G$. Let ν_T be the atomless probability measure on $(\Omega, \mathcal{E})$ such that $\nu_T(\mathsf{E}) = \nu(\mathsf{E} \cap \mathsf{B}_T)/\nu(\mathsf{B}_T)$. (Note that ν_T is the conditional probability of ν with respect to B_T.) Let $\mathcal{N}_T$ be the structure $(\mathcal{K}, \mathcal{E})$ with the probability measure ν_T instead of ν. Then $\mathcal{N}_T$ is a pre-complete separable randomization of both φ and T. Let $S_n = S_n(T) \cap S_n(\varphi)$. Since $S_n(\varphi)$ is countable, $(\forall \vec{v}) \bigvee_{q\in S_n} \bigwedge q(\vec{v})$ is a sentence of $L_{\omega_1\omega}$ and is a consequence of φ. Therefore

$$\nu_T([\![(\forall \vec{v}) \bigvee_{q\in S_n} \bigwedge q(\vec{v})]\!]) = 1.$$

Then for every n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, $\sum_{q\in S_n(T)} \nu_T([\![\bigwedge q(\vec{\mathbf{f}})]\!]) = 1$. Hence by Result 2.8.2, there is a countable model $\mathcal{H}_T$ of T and an elementary embedding

$$h_T\colon \mathcal{N}_T \prec (\mathcal{H}_T^{[0,1)}, \mathcal{L}).$$

Let $\{\mathsf{A}_T : T \in G\}$ be a Borel partition of $[0,1)$ such that $\lambda(\mathsf{A}_T) = \nu(\mathsf{B}_T)$ for each T. Let J be the set of isomorphism types of the models $\{\mathcal{H}_T : T \in G\}$. For each $T \in G$ let $\mathcal{H}_j = \mathcal{H}_T$, $h_j = h_T$, and $\mathsf{A}_j = \mathsf{A}_T$ where j is the isomorphism type of $\mathcal{H}_T$. Then $\mathcal{P} = (\prod_{j\in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$ is a basic randomization. For each $j \in J$, let ℓ_j be a mapping that stretches A_j to $[0,1)$, and let $\ell_j\colon \mathcal{P} \to (\mathcal{H}_j^{[0,1)}, \mathcal{L})$ be the mapping defined in Definition 2.4.4. We then get an elementary embedding of $\mathcal{N}$ into $\mathcal{P}$ by sending each $\mathsf{E} \in \mathcal{E}$ to the set $\bigcup_{j\in J} \ell_j^{-1}(h_j(\mathsf{E}))$, and sending each $\mathbf{f} \in \mathcal{K}$ to the function that agrees with $\ell_j^{-1}(h_j(\mathbf{f}))$ on A_j for each $j \in J$.

We next assume (ii) and prove (iii). Let $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ be a complete separable randomization of φ, and let $\vec{\mathbf{f}}$ be an n-tuple in $\mathcal{K}$. By (ii), there is an elementary embedding h from $\mathcal{N}$ into a basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$. Then there is a $j \in J$ and a set $\mathsf{B} \subseteq \mathsf{A}_j$ such that $\lambda(\mathsf{B}) > 0$ and $(h\vec{\mathbf{f}})$ is constant on B. Let $r = \lambda(\mathsf{B})$. Let p be the type of $h(\vec{\mathbf{f}})$ in $\mathcal{H}_j$. Then for each $\theta(\vec{v}) \in p$ we have $\mathcal{P} \models \mu([\![\theta(h\vec{\mathbf{f}})]\!]) \geq r$. Since h is an elementary embedding, for each $\theta \in p$ we have $\mathcal{N} \models \mu([\![\theta(\vec{\mathbf{f}})]\!]) \geq r$. Therefore

$$\mu([\![\bigwedge p(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) = \inf_{\theta\in p} \mu([\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) \geq r > 0,$$

and (iii) is proved.

Finally, we assume that (i) fails and prove that (iii) fails. Since (i) fails, there exists n such that $S_n(\varphi)$ is uncountable. We introduce some notation. Let L_0 be the set of all atomic first order formulas. Let 2^{L_0} be the Polish space whose elements are the functions $s\colon L_0 \to \{0,1\}$. As in Section 2, we say that a point $t \in 2^{L_0}$ **codes** an enumerated structure $(\mathcal{M}, a)$ if for each formula $\theta(v_0, \ldots, v_{n-1}) \in L_0$, $t(\theta) = 0$ if and only if $\mathcal{M} \models \theta[a_0, \ldots, a_{n-1}]$. We note for each $t \in 2^{L_0}$, any two enumerated structures that are coded by t are isomorphic. When t codes an enumerated structure, we choose one and denote it by $(\mathcal{M}(t), a(t))$. For each $L_{\omega_1\omega}$ formula $\psi(v_0, \ldots, v_{n-1})$, let $[\psi]$ be the set of all $t \in 2^{L_0}$ such that $(\mathcal{M}(t), a(t))$ exists and $\mathcal{M}(t) \models \psi[a_0(t), \ldots, a_{n-1}(t)]$.

Claim. There is a perfect set $P \subseteq [\varphi]$ such that for all s, t in P, we have

$$(\mathcal{M}(s), a_0(s), \ldots, a_{n-1}(s)) \equiv (\mathcal{M}(t), a_0(t), \ldots, a_{n-1}(t))$$

if and only if $s = t$.

Proof of Claim: By Proposition 16.7 in [8], for each $L_{\omega_1\omega}$ formula ψ, $[\psi(\vec{v})]$ is a Borel subset of 2^{L_0}. In particular, $[\varphi]$ is Borel. Let E be the set of pairs $(s,t) \in [\varphi] \times [\varphi]$ such that

$$(\mathcal{M}(s), a_0(s), \ldots, a_{n-1}(s)) \equiv (\mathcal{M}(t), a_0(t), \ldots, a_{n-1}(t)).$$

E is obviously an equivalence relation on $[\varphi]$. Since $S_n(\varphi)$ is uncountable, E

has uncountably many equivalence classes. We show that E is Borel. Let F be the set of all first order formulas $\theta(v_0, \ldots, v_{n-1})$. For each $\theta \in F$, let

$$E_\theta = \{(s,t) \in [\varphi] \times [\varphi] \colon s \in [\theta] \leftrightarrow t \in [\theta]\}.$$

Since $[\varphi]$ and $[\theta]$ are Borel, E_θ is Borel. Moreover, F is countable, and $E = \bigcap_{\theta \in F} E_\theta$. Therefore E is a Borel equivalence relation. By Silver's theorem in [14], there is a perfect set $P \subseteq [\varphi]$ such that whenever $s,t \in [\varphi]$, we have $(s,t) \in E$ if and only if $s = t$, as required in the Claim.

By Theorem 6.2 in [8], P has cardinality $2^{\aleph_0}$. By the Borel Isomorphism Theorem (15.6 in [8]), there is a Borel bijection β from $[0,1)$ onto P whose inverse is also Borel. Each $s \in P$ codes an enumerated model $(\mathcal{M}(s), a(s))$ of φ. For each $t \in [0,1)$ and $n \in \mathbb{N}$, $a_n(\beta(t)) \in M(\beta(t))$, so for each n the composition $\mathbf{c}_n = a_n \circ \beta$ is a function such that $\mathbf{c}_n(t) \in M(\beta(t))$. Let $C = \{\mathbf{c}_n \colon n \in \mathbb{N}\}$. Then for each t, we have

$$\{\mathbf{c}(t) \colon \mathbf{c} \in C\} = \{a_n(\beta(t)) \colon n \in \mathbb{N}\} = M(\beta(t)),$$

so C satisfies Condition (a) of Definition 2.6.1.

We next show that C is a countable generator. We will then show that the completion of $\mathcal{P}(C)$ is a separable randomization of φ that is not elementarily embeddable in a basic randomization.

For each $\theta \in L_0$, the set

$$P \cap [\theta] = \{s \in P \colon \mathcal{M}(s) \models \theta[a_0(s), \ldots, a_{n-1}(s)]\}$$

is Borel. Since β and its inverse are Borel functions, it follows that

$$\{t \in [0,1) \colon \mathcal{M}(\beta(t)) \models \theta(\mathbf{c}_0(t), \ldots, \mathbf{c}_{n-1}(t))\} \in \mathcal{L}.$$

Thus C satisfies condition (b) of Definition 2.6.1, and hence is a countable generator in the family $\langle \mathcal{M}(\beta(t))\rangle_{t \in [0,1)}$ of countable models of φ over the probability space $([0,1), \mathcal{L}, \lambda)$.

By Theorem 2.6.2 and Proposition 2.6.5, $\mathcal{P}(C)$ is a pre-complete separable randomization of φ. Then the completion $\mathcal{N}$ of $\mathcal{P}(C)$ is a complete separable randomization of φ. By the properties of P, for each first-order n-type p, there is at most one $t \in [0,1)$ such that $(\mathbf{c}_0(t), \ldots, \mathbf{c}_{n-1}(t))$ realizes p in $\mathcal{M}(\beta(t))$. Then

$$\mu([\![\bigwedge p(\mathbf{c}_0, \ldots, \mathbf{c}_{n-1})]\!]^{\mathcal{N}}) = 0.$$

Therefore $\mathcal{N}$ cannot be elementarily embeddable in a basic randomization. This shows that (iii) fails, and completes the proof. $\blacksquare_{2.8.3}$

2.9 Sentences with Few Separable Randomizations

In this section we show that any infinitary sentence that has only countably many countable models has few separable randomizations (Theorem 2.9.6 below). We begin by stating a result from [1].

Result 2.9.1. *([1], Theorem 6.3). If T is complete and $I(T)$ is countable, then T has few separable randomizations.*

Theorem 2.9.6 below will generalize this result by replacing the complete theory T by an arbitrary $L_{\omega_1\omega}$ sentence φ.

The following lemma is a consequence of Theorem 7.6 in [1]. The underlying definitions are somewhat different in [1], so for completeness we give a direct proof here.

Lemma 2.9.2. *Let $\mathcal{N} = (\mathcal{H}^{[0,1)}, \mathcal{L})$ be the Borel randomization of a countable model $\mathcal{H}$ of T_2. Suppose $\mathcal{M}_t \cong \mathcal{H}$ for each $t \in [0,1)$, and C is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$. Then $\mathcal{P}(C) \cong \mathcal{N}$.*

Remark 2.9.3. In the special case that $\mathcal{M}_t = \mathcal{M}$ for all $t \in [0,1)$ and $C \subseteq \mathcal{M}^{[0,1)}$, Corollary 2.6.4 and Remark 2.4.2 (ii) immediately give

$$\mathcal{P}(C) = (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{N}.$$

This argument does not work in the general case, where the structures $\mathcal{M}_t$ may vary with t and there is no measurability requirement on the elements of C.

Proof of Lemma 2.9.2. Let $\mathcal{P}(C) = (\mathcal{J}, \mathcal{L})$. Let H denote the universe of $\mathcal{H}$. Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots\}$ and $\{\mathbf{g}'_1, \mathbf{g}'_2, \ldots\}$ be countable dense subsets of $\mathcal{J}$ and $\mathcal{H}^{[0,1)}$ respectively.

Claim. There is a sequence $\langle \mathbf{g}_1, \mathbf{g}_2, \ldots \rangle$ in $\mathcal{J}$, and a sequence $\langle \mathbf{f}'_1, \mathbf{f}'_2, \ldots \rangle$ in $\mathcal{H}^{[0,1)}$, such that the following statement $S(n)$ holds for each $n \in \mathbb{N}$:

For all $t \in [0,1)$,

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{g}_1, \ldots, \mathbf{g}_n)(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_n, \mathbf{g}'_1, \ldots, \mathbf{g}'_n)(t)).$$

Once the Claim is proved, it follows that for each first order formula $\psi(\vec{u}, \vec{v})$,

$$[\![\psi(\vec{\mathbf{f}}, \vec{\mathbf{g}})]\!]^{\mathcal{P}(C)} = [\![\psi(\vec{\mathbf{f}}', \vec{\mathbf{g}}')]\!]^{\mathcal{N}},$$

and hence there is an isomorphism $h\colon \mathcal{P}(C) \cong \mathcal{N}$ such that $h(\mathsf{E}) = \mathsf{E}$ for all $\mathsf{E} \in \mathcal{L}$, and $h(\mathbf{f}_n) = \mathbf{f}'_n$ and $h(\mathbf{g}_n) = \mathbf{g}'_n$ for all n.

Proof of Claim: Note that the statement $S(0)$ just says that $\mathcal{M}_t \cong \mathcal{H}$ for all $t \in [0,1)$, and is true by hypothesis. Let $n \in \mathbb{N}$ and assume that we already have functions $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$ in $\mathcal{J}$ and $\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}$ in $\mathcal{H}^{[0,1)}$ such that the statement $S(n-1)$ holds. Thus for each $t \in [0,1)$, there is an isomorphism

$$h_t : (\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t)).$$

We will find functions $\mathbf{g}_n \in \mathcal{J}, \mathbf{f}'_n \in \mathcal{H}^{[0,1)}$ such that $S(n)$ holds.

Let Z be the set of all isomorphism types of structures

$$(\mathcal{H}, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, a, b),$$

and for each $z \in Z$ let θ_z be a Scott sentence for structures of isomorphism type z. Since H is countable, Z is countable. For each $a \in H$ and $t \in [0,1)$ let $z(a,t)$ be the isomorphism type of

$$(\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t), a, \mathbf{g}'_n(t)).$$

Then $z(a,t) \in Z$.

For each $a \in H$ and $c \in C$, let $\mathsf{B}(a,c)$ be the set of all $t \in [0,1)$ such that

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t), \mathbf{f}_n(t), c(t)) \models \theta_{z(a,t)}.$$

By Proposition 2.6.5, each of the sets $\mathsf{B}(a,c)$ is Borel. By taking $a \in H$ such that $a = h_t(\mathbf{f}_n(t))$, and $c \in C$ such that $c(t) = h_t^{-1}(\mathbf{g}'_n(t))$, we see that for every $t \in [0,1)$ there exist $a \in H$ and $c \in C$ with $t \in \mathsf{B}(a,c)$. Thus

$$[0,1) = \bigcup\{\mathsf{B}(a,c) \colon a \in H, c \in C\}.$$

Every countable family of Borel sets with union $[0,1)$ can be cut down to a countable partition of $[0,1)$ into Borel sets. Thus there is a partition

$$\langle \mathsf{D}(a,c) \colon a \in H, c \in C \rangle$$

of $[0,1)$ into Borel sets $\mathsf{D}(a,c) \subseteq \mathsf{B}(a,c)$.

Let $\mathbf{f}'_n$ be the function that has the constant value a on each set $\mathsf{D}(a,c)$, and let $\mathbf{g}_n$ be the function that agrees with c on each set $\mathsf{D}(a,c)$. Then $\mathbf{f}'_n$ is Borel and thus belong to $\mathcal{H}^{[0,1)}$, and $\mathbf{g}_n$ belongs to $\mathcal{J}$. Moreover, whenever $t \in \mathsf{D}(a,c)$ we have $t \in \mathsf{B}(a,c)$ and hence

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{g}_1, \ldots, \mathbf{g}_n)(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_n, \mathbf{g}'_1, \ldots, \mathbf{g}'_n)(t)).$$

So the functions $\mathbf{f}'_n$ and $\mathbf{g}_n$ satisfy the condition $S(n)$. This completes the proof of the Claim and of Lemma 2.9.2. $\blacksquare_{2.9.3}$

Recall that for each $i \in I$, θ_i is a Scott sentence for structures of isomorphism type i.

Lemma 2.9.4. *Let $\mathcal{P} = (\prod_{j \in J} (\mathcal{H}_j)^{\mathsf{A}_j}, \mathcal{L})$ be a basic randomization. Then for each complete separable randomization $\mathcal{N}$, the following are equivalent:*

(i) $\mathcal{N}$ is isomorphic to $\mathcal{P}$.

(ii) $\mu([\![\theta_j]\!]^{\mathcal{N}}) = \lambda(\mathsf{A}_j)$ for each $j \in J$.

Proof. Assume (i) and let $h : \mathcal{P} \cong \mathcal{N}$. By Corollary 2.6.4, $\mathcal{P} = \mathcal{P}(C)$ for some countable generator C in $\langle \mathcal{H}_t \rangle_{t \in [0,1)}$ over $([0,1), \mathcal{L}, \lambda)$. By Proposition 2.6.5, for each $j \in J$ we have

$$[\![\theta_j]\!]^{\mathcal{N}} = h(\{t \in [0,1) \colon \mathcal{H}_t \models \theta_j\}) = h(\mathsf{A}_j),$$

so (ii) holds.

We now assume (ii) and prove (i). Since the events $\mathsf{A}_j, j \in J$ form a partition of $[0,1)$, $\sum_{j \in J} \lambda(\mathsf{A}_j) = 1$, so by (ii) we have $\sum_{j \in J} \mu([\![\theta_j]\!]^{\mathcal{N}}) = 1$. Therefore $[\![\bigvee_{j \in J} \theta_j]\!]^{\mathcal{N}} = \top$, so $\mathcal{N}$ is a randomization of the sentence $\varphi =$

$\bigvee_{j\in J}\theta_j$. Since $I(\varphi)$ is countable, $\bigcup_n S_n(\varphi)$ is countable. Then by Theorem 2.8.3, $\mathcal{N}$ is elementarily embeddable in a basic randomization. By Theorem 2.7.3, $\mathcal{N}$ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family $\langle\mathcal{M}_t\rangle_{t\in[0,1)}$ of countable models of φ over the probability space $([0,1),\mathcal{L},\lambda)$. By Proposition 2.6.5, for each $j\in J$ the set $\mathsf{B}_j=\{t\in[0,1)\colon \mathcal{M}_t\models\theta_j\}\in\mathcal{L}$ and $\lambda(\mathsf{B}_j)=\mu(\llbracket\theta_j\rrbracket^{\mathcal{N}})=\lambda(\mathsf{A}_j)$. By Theorem 2.4.9, $\mathcal{P}\cong\mathcal{P}'=(\prod_{j\in J}(\mathcal{H}_j)^{\mathsf{B}_j},\mathcal{L})$. For each $j\in J$, let ℓ_j be a mapping that stretches B_j to $[0,1)$.

Our plan is to use Lemma 2.9.2 to show that the images of $\mathcal{P}(C)$ and $\mathcal{P}'$ under ℓ_j are isomorphic for each j. Intuitively, this shows that for each j, the part of $\mathcal{P}(C)$ on B_j is isomorphic to the part of $\mathcal{P}'$ on A_j. The isomorphisms on these parts can then be combined to get an isomorphism from $\mathcal{P}(C)$ to $\mathcal{P}'$.

Here are the details. For each j, $\mathcal{P}_j=(\mathcal{H}_j^{[0,1)},\mathcal{L})$ is the Borel randomization of $\mathcal{H}_j$, and ℓ_j maps $\mathcal{P}'$ to $\mathcal{P}_j$ and maps C to a countable generator $\ell_j(C)$ in $\langle\mathcal{M}'_t\rangle_{t\in[0,1)}$ over $([0,1),\mathcal{L},\lambda)$, where $\mathcal{M}'_t=\mathcal{M}_{\ell_j^{-1}(t)}$. Note that for each $j\in J$ and $t\in\ell_j(\mathsf{B}_j)$, we have $\mathcal{M}'_t\cong\mathcal{H}_j$. Therefore by Lemma 2.9.2, we have an isomorphism $h_j\colon\mathcal{P}(\ell_j(C))\cong\mathcal{P}_j$ for each $j\in J$. By pulling these isomorphisms back we get an isomorphism $h\colon\mathcal{P}(C)\cong\mathcal{P}'$ as follows. For an element $\mathbf{f}$ of $\mathcal{P}(C)$ of sort $\mathbb{K}$, $h(\mathbf{f})$ is the element of $\mathcal{P}'$ that agrees with $\ell_j^{-1}(h_j(\ell_j(\mathbf{f})))$ on the set B_j for each j. Since $\mathcal{N}\cong\mathcal{P}(C)$ and $\mathcal{P}'\cong\mathcal{P}$, (i) holds. $\blacksquare_{2.9.4}$

Lemma 2.9.5. *The following are equivalent.*

(i) φ has few separable randomizations.

(ii) For every complete separable randomization $\mathcal{N}$ of φ, there is a countable set $J\subseteq I$ such that $\llbracket\bigvee_{j\in J}\theta_j\rrbracket^{\mathcal{N}}=\top$.

(iii) For every complete separable model $\mathcal{N}$ of φ, $\mu(\llbracket\theta_i\rrbracket^{\mathcal{N}})>0$ for some $i\in I$.

Proof. It follows from Lemma 2.9.4 that (i) implies (ii). It is trivial that (ii) implies (iii).

We now assume (ii) and prove (i). Let $\mathcal{N}$ be a complete separable randomization of φ and let J be as in (ii). By removing j from J when $\llbracket\theta_j\rrbracket^{\mathcal{N}}=\perp$, we may assume that $\mu(\llbracket\theta_j\rrbracket^{\mathcal{N}})>0$ for each $j\in J$. We also have

$$\sum_{j\in J}\mu(\llbracket\theta_j\rrbracket^{\mathcal{N}})=\mu(\llbracket\bigvee_{j\in J}\theta_j\rrbracket^{\mathcal{N}})=1.$$

For each $j\in J$, choose $\mathcal{H}_j\in j$. Choose a partition $\{\mathsf{A}_j\colon j\in J\}$ of $[0,1)$ such that $\mathsf{A}_j\in\mathcal{L}$ and $\lambda(\mathsf{A}_j)=\mu(\llbracket\theta_j\rrbracket^{\mathcal{N}})$ for each $j\in J$. Then by Lemma 2.9.4, $\mathcal{N}$ is isomorphic to the basic randomization $(\prod_{j\in J}\mathcal{H}_j^{\mathsf{A}_j},\mathcal{L})$. Therefore (i) holds.

We assume that (ii) fails and prove that (iii) fails. Since (ii) fails, there is a complete separable randomization $\mathcal{N}$ of φ such that for every countable set $J\subseteq I$, $\mu(\llbracket\bigvee_{i\in I}\theta_j\rrbracket^{\mathcal{N}})<1$. The set $J=\{i\in I\colon\mu(\llbracket\theta_j\rrbracket^{\mathcal{N}})>0\}$ is countable. By Theorem 2.7.3, $\mathcal{N}$ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family $\langle\mathcal{M}_t\rangle_{t\in\Omega}$ of countable models of φ over a probability space $(\Omega,\mathcal{E},\nu)$. By Proposition 2.6.5, the set $\mathsf{E}=\{t\colon\mathcal{M}_t\models\bigvee_{j\in J}\theta_j\}$ belongs to $\mathcal{E}$, and $\nu(\mathsf{E})=\mu(\llbracket\bigvee_{j\in J}\theta_j\rrbracket^{\mathcal{N}})<1$. Let $\mathcal{P}'$ be the pre-structure $\mathcal{P}(C)$ but with the measure ν replaced by the measure υ defined by $\upsilon(\mathsf{D})=\nu(\mathsf{D}\setminus\mathsf{E})/\nu(\Omega\setminus\mathsf{E})$.

This is the conditional probability of D given $\Omega \setminus \mathsf{E}$. Then the completion $\mathcal{N}'$ of $\mathcal{P}'$ is a separable randomization of φ such that $\mu(\llbracket \theta_i \rrbracket^{\mathcal{N}'}) = 0$ for every $i \in I$, so (iii) fails. $\blacksquare_{2.9.5}$

Here is our generalization of Result 2.9.1.

Theorem 2.9.6. *If $I(\varphi)$ is countable, then φ has few separable randomizations.*

Proof. Suppose $J = I(\varphi)$ is countable. Then φ has the same countable models as the sentence $\bigvee_{j \in J} \theta_j$. Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 2.7.3, $\mathcal{N} \cong \mathcal{P}(C)$ for some countable generator C in a family of $\langle \mathcal{M}_t \rangle_{t \in \Omega}$ countable models of φ. By Proposition 2.6.5,

$$\mu(\llbracket \bigvee_{j \in J} \theta_j \rrbracket^{\mathcal{N}}) = \mu(\llbracket \bigvee_{j \in J} \theta_j \rrbracket^{\mathcal{P}(C)}) = \mu(\{t \colon \mathcal{M}_t \models \bigvee_{j \in J} \theta_j\}) = \mu(\{t \colon \mathcal{M}_t \models \varphi\}) = 1.$$

Therefore $\llbracket \bigvee_{j \in J} \theta_j \rrbracket^{\mathcal{N}} = \top$, so φ satisfies Condition (ii) of Lemma 2.9.5. By Lemma 2.9.5, φ has few separable randomizations. $\blacksquare_{2.9.6}$

2.10 Few Separable Randomizations Versus Scattered

In this section we prove two main results. First, any infinitary sentence with few separable randomizations is scattered. Second, Martin's axiom for $\aleph_1$ implies that every scattered infinitary sentence has few separable randomizations. We also discuss the connection between these results and the absolute Vaught conjecture.

Theorem 2.10.1. *If φ has few separable randomizations, then φ is scattered.*

Proof. Suppose φ is not scattered. By Lemma 2.2.5, there is a countable fragment L_A of $L_{\omega_1 \omega}$ and a perfect set $P \subseteq 2^{L_A}$ such that:

- Each $s \in P$ codes an enumerated model $(\mathcal{M}(s), a(s))$ of φ, and
- If $s \neq t$ in P then $\mathcal{M}(s)$ and $\mathcal{M}(t)$ do not satisfy the same L_A-sentences.

By Theorem 6.2 in [8], P has cardinality $2^{\aleph_0}$. By the Borel Isomorphism Theorem (15.6 in [8]), there is a Borel bijection β from $[0,1)$ onto P whose inverse is also Borel. For each $s \in P$, $(\mathcal{M}(s), a(s))$ can be written as $(\mathcal{M}(s), a_0(s), a_1(s), \ldots)$. For each $t \in [0,1)$, let $\mathcal{M}_t = \mathcal{M}(\beta(t))$. It follows that:

(i) $\mathcal{M}_t \models \varphi$ for each $t \in [0,1)$, and

(ii) If $s \neq t$ in P then $\mathcal{M}_s$ and $\mathcal{M}_t$ do not satisfy the same L_A-sentences.

For each $n \in \mathbb{N}$, the composition $\mathbf{c}_n = a_n \circ \beta$ belongs to the Cartesian product $\prod_{t\in[0,1)} M_t$. For each $t \in [0,1)$, we have

$$\{\mathbf{c}_n(t) \colon n \in \mathbb{N}\} = \{a_n(\beta(t)) \colon n \in \mathbb{N}\} = M(\beta(t)) = M_t.$$

Consider an atomic formula $\psi(\vec{v})$ and a tuple $(\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_n}) \in C$. ψ belongs to the fragment L_A. The set

$$\{s \in P \colon \mathcal{M}(s) \models \psi(a_{i_1}(s), \dots, a_{i_n}(s))\} = \{s \in P \colon s(\psi(v_{i_1}, \dots, v_{i_n})) = 0\}$$

is Borel in P. Since β and its inverse are Borel functions, it follows that

$$\{t \in [0,1) \colon \mathcal{M}_t \models \psi(\mathbf{c}_{i_1}(t), \dots, \mathbf{c}_{i_n}(t)) \in \mathcal{L}.$$

Thus C satisfies conditions (a) and (b) of Definition 2.6.1, and hence is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$.

By (ii), for each $i \in I$, there is at most one $t \in [0,1)$ such that $\mathcal{M}_t \models \theta_i$. By Theorem 2.6.2 and Proposition 2.6.5, the randomization $\mathcal{N} = \mathcal{P}(C)$ generated by C is a separable pre-complete randomization of φ. The event sort of $\mathcal{N}$ is $([0,1), \mathcal{L}, \lambda)$. Therefore, for each $i \in I$, the event $[\![\theta_i]\!]^{\mathcal{N}}$ is either a singleton or empty, and thus has measure zero. So by Lemma 2.9.5, φ does not have few separable randomizations. $\blacksquare_{2.10.1}$

Corollary 2.10.2. *Assume that the absolute Vaught conjecture holds for the $L_{\omega_1\omega}$ sentence φ. Then the following are equivalent:*

(i) $I(\varphi)$ is countable;

(ii) φ has few separable randomizations;

(iii) φ is scattered.

Proof. (i) implies (ii) by Result 2.9.1. (ii) implies (iii) by Theorem 2.10.1. The absolute Vaught conjecture for φ says that (iii) implies (i). $\blacksquare_{2.10.2}$

Our next theorem will show that if ZFC is consistent, then the converse of Theorem 2.10.1 is consistent with ZFC.

The Lebesgue measure is said to be $\aleph_1$-additive if the union of $\aleph_1$ sets of Lebesgue measure zero has Lebesgue measure zero. Note that the continuum hypothesis implies that Lebesgue measure is not $\aleph_1$-additive. Solovay and Tennenbaum [16] proved the relative consistency of Martin's axiom $MA(\aleph_1)$, and Martin and Solovay [12] proved that $MA(\aleph_1)$ implies that the Lebesgue measure is $\aleph_1$-additive. Hence if ZFC is consistent, then so is ZFC plus the Lebesgue measure is $\aleph_1$-additive. See [11] for an exposition.

Theorem 2.10.3. *Assume that the Lebesgue measure is $\aleph_1$-additive. If φ is scattered, then φ has few separable randomizations.*

Proof. Suppose φ is scattered. Then there are at most countably many ω-equivalence classes of countable models of φ, so there are at most countably many first order types that are realized in countable models of φ. Thus $\bigcup_n S_n(\varphi)$ is countable.

Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 2.8.3, $\mathcal{N}$ is elementarily embeddable in some basic randomization. By Theorem 2.7.3, there is a countable generator C in a family $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ of countable models of φ over $([0,1), \mathcal{L}, \lambda)$ such that $\mathcal{N} \cong \mathcal{P}(C)$. By Proposition 2.6.5, for each $i \in I(\varphi)$ we have $\mathsf{B}_i := \{t \colon \mathcal{M}_t \models \theta_i\} \in \mathcal{L}$. Moreover, the events B_i are pairwise disjoint and their union is $[0,1)$. By Result 2.2.3, $I(\varphi)$ has cardinality at most $\aleph_1$.

Let $J := \{i \in I(\varphi) \colon \lambda(\mathsf{B}_i) > 0\}$. Then J is countable. The set $I(\varphi) \setminus J$ has cardinality at most $\aleph_1$, so by hypothesis we have $\lambda(\bigcup_{j\in J} \mathsf{B}_j) = 1$. Pick an element $j_0 \in J$. For $j \neq j_0$ let $\mathsf{A}_j = \mathsf{B}_j$. Let A_{j_0} contain the other elements of $[0,1)$, so $\mathsf{A}_{j_0} = \mathsf{B}_{j_0} \cup ([0,1) \setminus \bigcup_{j\in J} \mathsf{B}_j)$. Then $\langle \mathsf{A}_j \rangle_{j\in J}$ is a partition of $[0,1)$. For each $j \in J$, choose a model $\mathcal{H}_j$ of isomorphism type j. Then $\mathcal{P} = (\prod_{j\in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$ is a basic randomization of φ. For each $j \in J$ we have $\lambda(\llbracket \theta_j \rrbracket^{\mathcal{N}}) = \lambda(\mathsf{A}_j)$, so by Lemma 2.9.4, $\mathcal{N}$ is isomorphic to $\mathcal{P}$. This shows that φ has few separable randomizations. $\blacksquare_{2.10.3}$

Corollary 2.10.4. *Assume that the Lebesgue measure is $\aleph_1$-additive. Then the following are equivalent.*

(i) For every φ, the absolute Vaught conjecture holds.

(ii) For every φ, if φ has few separabable randomiztions then $I(\varphi)$ is countable.

Proof. Corollary 2.10.2 shows that (i) implies (ii).

Assume that (i) fails. Then there is a scattered sentence φ such that $|I(\varphi)| = \aleph_1$. By Theorem 2.10.3, φ has few separable randomizations. Therefore (ii) fails. $\blacksquare_{2.10.4}$

2.11 Some Open Questions

Question 2.11.1. Suppose $\mathcal{N}$ and $\mathcal{P}$ are complete separable randomizations. If

$$\mu(\llbracket \varphi \rrbracket^{\mathcal{N}}) = \mu(\llbracket \varphi \rrbracket^{\mathcal{P}})$$

for every $L_{\omega_1\omega}$ sentence φ, must $\mathcal{N}$ be isomorphic to $\mathcal{P}$?

Question 2.11.2. Suppose C and D are countable generators in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$, $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ over the same probability space $(\Omega, \mathcal{E}, \nu)$. If $\mathcal{M}_t \cong \mathcal{H}_t$ for ν-almost all $t \in \Omega$, must $\mathcal{P}(C)$ be isomorphic to $\mathcal{P}(D)$?

Question 2.11.3. (Possible improvement of Theorem 2.8.3.) If $\bigcup_n S_n(\varphi)$ is countable, must every complete separable randomization of φ be elementarily embeddable in a basic randomization of φ?

Question 2.11.4. Can Theorem 2.10.3 be proved in ZFC (without the hypothesis that the Lebesgue measure is $\aleph_1$-additive)?

Bibliography

[1] Uri Andrews and H. Jerome Keisler. Separable Models of Randomizations. Journal of Symbolic Logic 80 (2015), 1149-1181.

[2] Uri Andrews, Isaac Goldbring, and H. Jerome Keisler. Definable Closure in Randomizations. Annals of Pure and Applied Logic 166 (2015), pp. 325-341.

[3] Itaï Ben Yaacov. On Theories of Random Variables. Israel J. Math 194 (2013), 957-1012.

[4] John Baldwin, Sy Friedman, Martin Koerwein, and Michael Laskowski. Three Red Herrings around Vaught's Conjecture. To appear, Transactions of the American Mathematical Society.

[5] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson and Alexander Usvyatsov. Model Theory for Metric Structures. In Model Theory with Applications to Algebra and Analysis, vol. 2, London Math. Society Lecture Note Series, vol. 350 (2008), 315-427.

[6] Itaï Ben Yaacov and H. Jerome Keisler. Randomizations of Models as Metric Structures. Confluentes Mathematici 1 (2009), pp. 197-223.

[7] C.J. Eagle. Omitting Types in Infinitary [0,1]-valued Logic. Annals of Pure and Applied Logic 165 (2014), 913-932.

[8] Alexander Kechris. Classical Descriptive Set Theory. Springer-Verlag (1995).

[9] H. Jerome Keisler. Model Theory for Infinitary Logic. North-Holland 1971.

[10] H. Jerome Keisler. Randomizing a Model. Advances in Mathematics 143 (1999), 124-158.

[11] Kenneth Kunen. Set Theory. Studies in logic 34, London:College Publications (2011).

[12] Donald Martin and Robert Solovay. Internal Cohen Extensions. Annals of Mathematical Logic 2 (1970), 143-178.

[13] Michael Morley. The Number of Countable Models. Jounal of Symbolic logic 35 (1970), 14-18.

[14] Jack H. Silver. Counting the Number of Equivalence Classes of Borel and Coanalytic Equivalence Relations. Ann. Math. Logic 18 (1980), 1-28.

[15] Dana Scott. Logic with Denumerably Long Formulas and Finite Strings of Quantifiers. In *The Theory of Models*, ed. by J. Addison et. al., North-Holland 1965, 329-341.

[16] Robert Solovay and Stanley Tennenbaum. Iterated Cohen Extensions and Souslin's Problem. Annals of Mathematics 94 (1971), 201-245.

[17] John R. Steel. On Vaught's conjecture. In Cabal Seminar 76-77, 193-208, Lecture Notes in Math., 689, Springer (1978).

[18] Robert L. Vaught. Denumerable Models of Complete Theories. Pages 303-321 in Infinitistic Methods, Warsaw 1961.

Chapter 3

Analytic Zariski structures and non-elementary categoricity

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Abstract

We study analytic Zariski structures from the point of view of non-elementary model theory. We show how to associate an abstract elementary class with a one-dimensional analytic Zariski structure and prove that the class is stable, quasi-minimal and homogeneous over models. We also demonstrate how Hrushovski's predimension arises in this general context as a natural geometric notion and use it as one of our main tools.

The notion of an analytic Zariski structure was introduced in [1] by the author and N.Peatfield in a form slightly different from the one presented here and then in [4], Ch.6 in the current form. Analytic Zariski generalises the previously known notion of a Zariski structure. The latter has been defined as a structure M with a Noetherian topology on all cartesian powers M^n of the universe, the closed sets of which are given by positive quantifier-free formulas. Any closed set is assigned a dimension which behaves in a certain way (modelled on algebraic geometry) with regards to projection maps $M^n \to M^m$, see the *addition formula* (AF) and the *fibre condition* (FC) in section 3.1 below.

In the definition of analytic Zariski structures we drop the requirement of Noetherianity. This leads to a considerably more flexible and broader notion at the cost of a longer list of assumptions modelled on the properties of analytic subsets of complex manifolds.

In [1] we assumed that the Zariski structure is compact (or compactifiable), here we drop this assumption, which may be too restrictive in applications.

We remark that in the broad setting it is appropriate to consider the notion of a Zariski structure as belonging to *positive model theory* in the sense of Ben-Yaacov [5].

The class of analytic Zariski structures is much broader and geometrically richer than the class of Noetherian Zariski structures. The main examples come from two sources:

(i) structures which are constructed in terms of complex analytic functions and relations;

(ii) "new stable structures" introduced by Hrushovski's construction; in many cases these objects exhibit properties similar to those of class (i).

However, although there are concrete examples for both (i) and (ii), in many cases we lack the technology to prove that the structure is analytic Zariski. In particular, despite some attempts the conjecture that $\mathbb{C}_{\exp}$ is analytic Zariski, assuming it satisfies axioms of pseudo-exponentiation (see [17]), is still open.

The aim of this paper is to carry out a model-theoretic analysis of analytic Zariski structures in the appropriate language. Recall that if M is a Noetherian Zariski structure the relevant key model-theoretic result states that its first-order theory allows elimination of quantifiers and is ω-stable of finite Morley rank. In particular, it is strongly minimal (and so uncountably categorical) if $\dim M = 1$ and M is irreducible.

For analytic Zariski 1-dimensional M we carry out a model theoretic study it in the spirit of the theory of *abstract elementary classes*. We start by introducing a suitable countable fragment of the family of basic Zariski relations and a correspondent substructure of constants over which all the further analysis is carried out. Then we proceed to the analysis of the notion of dimension of Zariski closed sets and define more delicate notions of the *predimension* and *dimension* of a tuple in M. In fact by doing this we reinterpret dimensions which are present in every analytic structure in terms familiar to many from Hrushovski's construction, thus establishing once again conceptual links between classes (i) and (ii).

Our main results are proved under assumption that M is one-dimensional (as an analytic Zariski structure) and irreducible. No assumption on presmoothness is needed. We prove for such an M, in the terminology of [16]:

(1) M *is a quasi-minimal pregeometry structure with regards to a closure operator* cl *associated with the predimension;*

(2) M *has quantifier-elimination to the level of* $\exists$*-formulas in the following sense: every two tuples which are (first-order)* $\exists$*-equivalent over a countable submodel, are* $L_{\infty,\omega}$ *equivalent;*

(3) *The abstract elementary class associated with* M *is categorical in uncountable cardinals and is excellent.*

In fact, (3) is a corollary of (1) using the main result of [16], so the main work is in proving (1) which involves (2) as an intermediate step.

Note that the class of 1-dimensional Noetherian Zariski structures is essentially classifiable by the main result of [2], and in particular the class contains no instances of structures obtained by the proper Hrushovski construction. The class of analytic Zariski structures, in contrast, is consistent with Hrushovski's construction and at the same time, by the result above, has excellent model-theoretic properties. This gives a hope for a classification theory based on the relevant notions.

However, it must be mentioned that some natural questions in this context are widely open. In particular, we have no classification for presmooth analytic Zariski groups (with the graph of multiplication analytic). It is not known if a 1-dimensional irreducible presmooth analytic group has to be abelian. See related analysis of groups in [9].

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3.1 Analytic L-Zariski structures

Let $\mathrm{M} = (M; L)$ be a structure with primitives (basic relations) L. We use also the extension $L(M_0)$ of the language L with names for points of a subset M_0 of M.

We introduce a topology on M^n, for all $n \geq 1$, by declaring a subset $P \subseteq M^n$ closed if there is an n-type p consisting of quantifier-free positive formulas with parameters in M such that

$$P = \{a \in M^n : \ \mathrm{M} \vDash p(a)\}.$$

In other words, the sets defined by atomic $L(M)$-formulae form a basis for the topology.

We say P is L-closed ($L(M_0)$-closed) if p is over $\emptyset$ (over M_0).

3.1.1 Remark

Note that it follows that projections

$$\mathrm{pr}_{i_1,\ldots,i_m} : M^n \to M^m, \ \ \langle x_1, \ldots, x_n\rangle \mapsto \langle x_{i_1}, \ldots, x_{i_m}\rangle$$

are continuous in the sense that the inverse image of a closed set under a projection is closed. Indeed, $\mathrm{pr}^{-1}_{i_1,\ldots,i_m} S \ = S \times M^{n-m}$.

We will drop the subscript in $\mathrm{pr}_{i_1,\ldots,i_m}$ when it is clear from the context.

We write $X \subseteq_{op} V$ to say that X is open in V and $X \subseteq_{\mathrm{cl}} V$ to say it is closed. The latter means that $X = V \cap S$, for some $S \subseteq M^n$ closed in M^n. The former, that $X = V \setminus S$.

We say that $P \subseteq M^n$ is **constructible** if P is a finite union of some sets S, such that $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$.

A subset $P \subseteq M^n$ will be called **projective** if P is a union of finitely many sets of the form $\mathrm{pr}\, S$, for some $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$.

We say that P is L-constructible or L-projective if P is defined over L.

Note that any set S such that $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^{n+k}$, is constructible, a projection of a constructible set is projective and that any constructible set is projective.

3.1.2 Dimension

To any nonempty projective S a non-negative integer $\dim S$, called **the dimension of** S, is attached.

We assume:

(SI) (**strong irreducibility**) for an irreducible set $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ (that is S is not a proper union of two closed in S subsets) and any closed subset $S' \subseteq_{\mathrm{cl}} S$,

$$\dim S' = \dim S \Rightarrow S' = S;$$

(DP) (**dimension of points**) for a nonempty projective S, $\dim S = 0$ if and only if S is at most countable.

(CU) (**countable unions**) If $S = \bigcup_{i\in\mathbb{N}} S_i$, all projective, then $\dim S = \max_{i\in\mathbb{N}} \dim S_i$;

(WP) (**weak properness**) given an irreducible $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ and $F \subseteq_{\mathrm{cl}} V \subseteq_{op} M^{n+k}$ with the projection $\mathrm{pr} : M^{n+k} \to M^n$ such that $\mathrm{pr}\, F \subseteq S$ and $\dim \mathrm{pr}\, F = \dim S$, there exists $D \subseteq_{op} S$ such that $D \subseteq \mathrm{pr}\, F$.

3.1.3 Remark

(CU) in the presence of the descending chain condition implies the *essential uncountability property* (EU) usually assumed for Noetherian Zariski structures.

We postulate further, for an irreducible $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^{n+k}$, a projection $\mathrm{pr} : M^{n+k} \to M^n$ and its fibres $S_u := \mathrm{pr}^{-1}(u) \cap S$ on S over $u \in \mathrm{pr} S$:

(AF) $\dim \mathrm{pr}\, S = \dim S - \min_{u\in \mathrm{pr}\, S} \dim S_u$;

(FC) The set $\{a \in \mathrm{pr}\, S : \dim S_a \geq m\}$ is of the form $T \cap \mathrm{pr}\, S$ for some constructible T, and there exists an open set V such that $V \cap \mathrm{pr}\, S \neq \emptyset$ and

$$\min_{a\in \mathrm{pr}\, S} \dim S_a = \dim S_v, \text{ for any } v \in V \cap \mathrm{pr}\,(S).$$

The following helps to understand the dimension of projective sets.

3.1.4 Lemma

Let $P = \mathrm{pr}\, S \subseteq M^n$, for S irreducible constructible, and $U \subseteq_{op} M^n$ with $P \cap U \neq \emptyset$. Then

$$\dim P \cap U = \dim P.$$

Proof. We can write $P \cap U = \mathrm{pr}\, S' = P'$, where $S' = S \cap \mathrm{pr}^{-1} U$ constructible irreducible, $\dim S' = \dim S$ by (SI). By (FC), there is $V \subseteq_{op} M^n$ such that for all $c \in V \cap P$,

$$\dim S_c = \min_{a \in P} \dim S_a = \dim S - \dim P.$$

Note that $\mathrm{pr}^{-1} U \cap \mathrm{pr}^{-1} V \cap S \neq \emptyset$, since S is irreducible. Taking $s \in \mathrm{pr}^{-1} U \cap \mathrm{pr}^{-1} V \cap S$ and $c = \mathrm{pr}\, s$ we get, using (AF) for S',

$$\dim S'_c = \dim S_c = \min_{a \in P'} \dim S_a = \dim S - \dim P'.$$

So, $\dim P' = \dim P$. $\square$

3.1.5 Analytic subsets

A subset S, $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, is called **analytic in** U if for every $a \in S$ there is an open $V_a \subseteq_{op} U$ such that $a \in V_a$ and $S \cap V_a$ is the union of finitely many closed in V_a irreducible subsets. We write $S \subseteq_{an} U$ accordingly.

We postulate the following properties:

(INT) (**Intersections**) If $S_1, S_2 \subseteq_{an} U$ are irreducible then $S_1 \cap S_2$ is analytic in U;

(CMP) (**Components**) If $S \subseteq_{an} U$ and $a \in S$, a closed point, then there is $S_a \subseteq_{an} U$, a finite union of irreducible analytic subsets of U, and some $S'_a \subseteq_{an} U$ such that $a \in S_a \setminus S'_a$ and $S = S_a \cup S'_a$;

Each of the irreducible subsets of S_a above is called an **irreducible component of** S **containing** a.

(CC) (**Countability of the number of components**) Any $S \subseteq_{an} U$ is a union of at most countably many irreducible components.

3.1.6 Remark

For S analytic and $a \in \mathrm{pr}\, S$, the fibre S_a is analytic.

3.1.7 Lemma

If $S \subseteq_{an} U$ is irreducible, V open, then $S \cap V$ is an irreducible analytic subset of V and, if non-empty, $\dim S \cap V = \dim S$.

Proof. Immediate. □

3.1.8 Lemma

(i) $\emptyset$, any singleton and U are analytic in U;

(ii) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cup S_2$ is analytic in U;

(iii) If $S_1 \subseteq_{an} U_1$ and $S_2 \subseteq_{an} U_2$, then $S_1 \times S_2$ is analytic in $U_1 \times U_2$;

(iv) If $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$;

(v) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cap S_2$ is analytic in U.

Proof. Immediate. □

3.1.9 Definition

Given a subset $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ we define the notion of the **analytic rank** of S in U, $\mathrm{ark}_U(S)$, which is a natural number satisfying

1. $\mathrm{ark}_U(S) = 0$ iff $S = \emptyset$;
2. $\mathrm{ark}_U(S) \le k+1$ iff there is a set $S' \subseteq_{\mathrm{cl}} S$ such that $\mathrm{ark}_U(S') \le k$ and with the set $S^0 = S \setminus S'$ being analytic in $U \setminus S'$.

Obviously, any nonempty analytic subset of U has analytic rank 1.

The next assumption guarantees that the class of analytic subsets explicitly determines the class of closed subsets in M.

(AS) [**Analytic stratification**] For any $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, $\mathrm{ark}_U S$ is defined and is finite.

We will justify this non obvious property later in 3.3.10 and 3.3.11.

3.1.10 Lemma

For any $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$,

$$\dim \mathrm{pr}\, S + \min_{a \in \mathrm{pr}\, S} \dim S_a \ge \dim S.$$

Proof. We use (AS) and prove the statement by induction on $\mathrm{ark}_U S \ge 1$.

For $\mathrm{ark}_U S = 1$, S is analytic in U and so by (CC) is the union of countably many irreducibles $S^{(i)}$. By (AF)

$$\dim \mathrm{pr}\, S^{(i)} + \min_{a \in \mathrm{pr}\, S^{(i)}} \dim S^{(i)}_a \geq \dim S^{(i)}$$

and so by (CU) lemma follows. □

3.1.11 Presmoothness

The following property (which we are not going to use in the context of the present paper) is relevant.

(PS) **[Presmoothness]** If $S_1, S_2 \subseteq_{an} U \subseteq_{op} M^n$ and S_1, S_2 and U irreducible, then for any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

3.1.12 Definition

An L-structure M is said to be **analytic L-Zariski** if

- M satisfies (SI), (WP), (CU), (INT), (CMP),(CC), (AS);
- the expansion $\mathrm{M}^\sharp$ of M to the language $L(M)$ (names for points in M added) satisfies all the above with the dimension extending the one for M;
- $\mathrm{M}^\sharp$ also satisfies (AF) and (FC) with V in (FC) being L-definable whenever S is.

An analytic Zariski structure will be called **presmooth** if it has the presmoothness property (PS).

3.2 Model theory of analytic Zariski structures

For the rest of the section we assume that M be analytic L-Zariski and assume L is countable.

3.2.1 Lemma

There is a countable $\mathrm{M}_0 \preccurlyeq \mathrm{M}$ such that for any $L(M_0)$-closed set S any irreducible component P of S is $L(M_0)$-closed.

Proof. Use the standard Löwenheim - Skolem downward arguments. □

We call such M_0 a **core substructure (subset) of** M.

3.2.2 Assumption

By extending L to $L(M_0)$ we assume that the set of L-closed points is the core subset.

For finite subset X of M of size n we denote $\vec{X}$ an n-tuple with range X.

3.2.3 Definition

For finite $X \subseteq M$ we define the **predimension**

$$\delta(X) = \min\{\dim S : \ \vec{X} \in S, \ S \subseteq_{an} U \subseteq_{op} M^n, \ S \text{ is } L\text{-constructible}\}, \quad (3.1)$$

relative predimension for finite $X, Y \subseteq M$

$$\delta(X/Y) = \min\{\dim S : \ \vec{X} \in S, \ S \subseteq_{an} U \subseteq_{op} M^n, \ S \text{ is } L(Y)\text{-constructible}\}, \quad (3.2)$$

and **dimension of** X

$$\partial(X) = \min\{\delta(XY) : \quad \text{finite } Y \subset M\}.$$

(Here and below XY means $X \cup Y$ and $Xy = X \cup \{y\}$).

We call a minimal S as in (3.2) an **analytic locus of** X **over** Y.

For $X \subseteq M$ finite, we say that X is **self-sufficient** and write $X \leq M$, if $\partial(X) = \delta(X)$.

For infinite $A \subseteq M$ we say $A \leq M$ if for any finite $X \subseteq A$ there is a finite $X \subseteq X' \subseteq A$ such that $X' \leq M$.

3.2.4

For the rest of the paper we assume that $\dim M = 1$ and M is irreducible. This is an analogue of an analytic curve.

Note that we then have

$$0 \leq \delta(Xy) \leq \delta(X) + 1, \text{ for any } y \in M,$$

since $\vec{X}y \in S \times M$.

3.2.5 Lemma

Given $F \subseteq_{an} U \subseteq_{op} M^k$, $\dim F > 0$, there is $i \leq k$ such that for $\mathrm{pr}_i : (x_1, \ldots, x_k) \mapsto x_i$,

$$\dim \mathrm{pr}_i F > 0.$$

Proof. Use (AF) and induction on k. $\square$

3.2.6 Proposition

Let $P = \mathrm{pr}\, S$, for some L-constructible $S \subseteq_{an} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$. Then

$$\dim P = \max\{\partial(x) : \ x \in P(M)\}. \tag{3.3}$$

Moreover, this formula is true when $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^{n+k}$.

Proof. We use induction on $\dim S$.

We first note that by induction on $\mathrm{ark}_U S$, if (3.3) holds for all analytic S of dimension less or equal to k then it holds for all closed S of dimension less or equal to k.

The statement is obvious for $\dim S = 0$ and so we assume that $\dim S > 0$ and for all analytic S' of lower dimension the statement is true.

By (CU) and (CMP) we may assume that S is irreducible. Then by (AF)

$$\dim P = \dim S - \dim S_c \tag{3.4}$$

for any $c \in P \cap V$ (such that S_c is of minimal dimension) for some open L-constructible V.

Claim 1. It suffices to prove the statement of the proposition for the projective set $P \cap V'$, for some L-open $V' \subseteq_{op} M^n$.

Indeed,

$$P \cap V' = \mathrm{pr}(S \cap \mathrm{pr}^{-1} V'), \quad S \cap \mathrm{pr}^{-1} V' \subseteq_{\mathrm{cl}} \mathrm{pr}^{-1} V' \cap U \subseteq_{op} M^{n+k}.$$

And $P \setminus V' = \mathrm{pr}(S \cap T)$, $T = \mathrm{pr}^{-1}(M^n \setminus V') \in L$. So, $P \setminus V'$ is the projection of a proper analytic subset, of lower dimension. By induction, for $x \in P \setminus V'$, $\partial(x) \leq \dim P \setminus V' \leq \dim P$ and hence, using 3.1.4,

$$\dim P \cap V' = \max\{\partial(x) : \ x \in P \cap V'\} \Rightarrow \dim P = \max\{\partial(x) : \ x \in P\}.$$

Claim 2. The statement of the proposition holds if $\dim S_c = 0$ in (3.4).

Proof. Given $x \in P$ choose a tuple $y \in M^k$ such that $S(x^\frown y)$ holds. Then $\delta(x^\frown y) \leq \dim S$. So we have $\partial(x) \leq \delta(x^\frown y) \leq \dim S = \dim P$.

It remains to notice that there exists $x \in P$ such that $\partial(x) \geq \dim P$.

Consider the L-type

$$x \in P \ \&\{x \notin R : \ \dim R \cap P < \dim P \text{ and } R \text{ is projective}\}.$$

This is realised in M, since otherwise $P = \bigcup_R (P \cap R)$ which would contradict (CU).

For such an x let y be a tuple in M such that $\delta(x^\frown y) = \partial(x)$. By definition there exist $S' \subseteq_{an} U' \subseteq_{op} M^m$ such that $\dim S' = \delta(x^\frown y)$. Let $P' = \mathrm{pr}\, S'$, the projection into M^n. By our choice of x, $\dim P' \geq \dim P$. But $\dim S' \geq \dim P'$. Hence, $\partial(x) \geq \dim P$. Claim proved.

Claim 3. There is a L-constructible $R \subseteq_{an} S$ such that all the fibres R_c of the projection map $R \to \mathrm{pr}\, R$ are 0-dimensional and $\dim \mathrm{pr}\, R = \dim P$.

Proof. We have by construction $S_c \subseteq M^k$. Assuming $\dim S_c > 0$ on every open subset we show that there is a $b \in M_0$ such that (up to the order of coordinates) $\dim S_c \cap \{b\} \times M^{k-1} < \dim S_c$, for all $c \in P \cap V' \neq \emptyset$, for some open $V' \subseteq V$ and $\dim \mathrm{pr}\, S_c \cap \{b\} \times M^{k-1} = \dim P$. By induction on $\dim S$ this will prove the claim.

To find such a b choose $a \in P \cap V$ and note that by 3.2.5, up to the order of coordinates, $\dim \mathrm{pr}_1 S(a, M) > 0$, where $\mathrm{pr}_1 : M^k \to M$ is the projection on the first coordinate.

Consider the projection $\mathrm{pr}_{M^n,1} : M^{n+k} \to M^{n+1}$ and the set $\mathrm{pr}_{M^n,1} S$. By (AF) we have

$$\dim \mathrm{pr}_{M^n,1} S = \dim P + \dim \mathrm{pr}_1 S_a = \dim P + 1.$$

Using (AF) again for the projection $\mathrm{pr}_1 : M^{n+1} \to M$ with the fibres $M^n \times \{b\}$, we get, for all b in some open subset of M,

$$1 \geq \dim \mathrm{pr}_1 \mathrm{pr}_{M^n,1} S = \dim \mathrm{pr}_{M^n,1} S - \dim[\mathrm{pr}_{M^n,1} S] \cap [M^n \times \{b\}] =$$

$$= \dim P + 1 - \dim[\mathrm{pr}_{M^n,1} S] \cap [M^n \times \{b\}].$$

Hence $\dim[\mathrm{pr}_{M^n,1} S] \cap [M^n \times \{b\}] \geq \dim P$, for all such b, which means that the projection of the set $S_b = S \cap (M^n \times \{b\} \times M^{k-1})$ on M^n is of dimension $\dim P$, which finishes the proof if $b \in M_0$. But $\dim S_b = \dim S - 1$ for all $b \in M \cap V'$, some L-open V', so for any $b \in M_0 \cap V'$. The latter is not empty since (M_0, L) is a core substructure. This proves the claim.

Claim 4. Given R satisfying Claim 3,

$$P \setminus \mathrm{pr}\, R \subseteq \mathrm{pr}\, S', \text{ for some } S' \subseteq_{\mathrm{cl}} S, \ \dim S' < \dim S.$$

Proof. Consider the cartesian power

$$M^{n+2k} = \{x^\frown y^\frown z : x \in M^n, \ y \in M^k, \ z \in M^k\}$$

and its L-constructible subset

$$R\&S := \{x^\frown y^\frown z : \quad x^\frown z \in R \ \& \ x^\frown y \in S\}.$$

Clearly $R\&S \subseteq_{an} W \subseteq_{op} M^{n+2k}$, for an appropriate L-constructible W.

Now notice that the fibres of the projection $\mathrm{pr}_{xy} : x^\frown y^\frown z \mapsto x^\frown y$ over $\mathrm{pr}_{xy} R\&S$ are 0-dimensional and so, for some irreducible component $(R\&S)^0$ of the analytic set $R\&S$, $\dim \mathrm{pr}_{xy}(R\&S)^0 = \dim S$. Since $\mathrm{pr}_{xy} R\&S \subseteq S$ and S irreducible, we get by (WP) $D \subseteq \mathrm{pr}_{xy} R\&S$ for some $D \subseteq_{op} S$. Clearly

$$\mathrm{pr}\, R = \mathrm{pr}\, \mathrm{pr}_{xy} R\&S \supseteq \mathrm{pr}\, D$$

and $S' = S \setminus D$ satisfies the requirement of the claim.

Now we complete the proof of the proposition: By Claims 2 and 3

$$\dim P = \max_{x \in \mathrm{pr}\, R} \partial(x).$$

By induction on $\dim S$, using Claim 4, for all $x \in P \setminus \mathrm{pr}\, R$,

$$\partial(x) \leq \dim \mathrm{pr}\, S' \leq \dim P.$$

The statement of the proposition follows. □

In what follows a L-substructure of M is a L-structure on a subset $N \supseteq M_0$. Recall that L is purely relational.

Recall the following well-known fact, see [10].

3.2.7 Karp's characterisation of $\equiv_{\infty,\omega}$

Given $a, a' \in M^n$ the $L_{\infty,\omega}(L)$-types of the two n-tuples in M are equal if and only if they are *back and forth equivalent* that is there is a nonempty set I of isomorphisms of L-substructures of M such that $a \in \mathrm{Dom}\, f_0$ and $a' \in \mathrm{Range}\, f_0$, for some $f_0 \in I$, and

(forth) for every $f \in I$ and $b \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Dom}\, g$;

(back) For every $f \in I$ and $b' \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b' \in \mathrm{Range}\, g$.

3.2.8 Definition

For $a \in M^n$, the **projective type of** a **over** M is

$$\{P(x) : \ a \in P, \ P \text{ is a projective set over } L\} \cup$$

$$\cup \{\neg P(x) : \ a \notin P, \ P \text{ is a projective set over } L\}.$$

3.2.9 Lemma

Suppose $X \leq M$, $X' \leq M$ and the (first-order) quantifier-free L-type of X is equal to that of X'. Then the $L_{\infty,\omega}(L)$-types of X and X' are equal.

Proof. We are going to construct a back-and-forth system for X and X'.

Let $S_X \subseteq_{an} V \subseteq_{op} M^n$, S_X irreducible, all L-constructible, and such that $X \in S_X(M)$ and $\dim S_X = \delta(X)$.

Claim 1. The quantifier-free L-type of X (and X') is determined by formulas equivalent to $S_X \cap V'$, for V' open such that $X \in V'(M)$.

Proof. Use the stratification of closed sets (AS) to choose L-constructible $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ such that $X \in S$ and $\mathrm{ark}_U S$ is minimal. Obviously then $\mathrm{ark}_U S = 0$, that is $S \subseteq_{an} U \subseteq_{op} M^n$. Now S can be decomposed into irreducible components, so we may choose S to be irreducible. Among all such

S choose one which is of minimal possible dimension. Obviously $\dim S = \dim S_X$, that is we may assume that $S = S_X$. Now clearly any constructible set $S' \subseteq_{\rm cl} U' \subseteq_{op} M^n$ containing X must satisfy $\dim S' \cap S_X \geq \dim S_X$, and this condition is also sufficient for $X \in S'$.

Let y be an element of M. We want to find a finite Y containing y and an Y' such that the quantifier-free type of XY is equal to that of $X'Y'$ and both are self-sufficient in M (recall that $XY := X \cup Y$). This, of course, extends the partial isomorphism $X \to X'$ to $XY \to X'Y'$ and will prove the lemma.

We choose Y to be a minimal set containing y and such that $\delta(XY)$ is also minimal, that is

$$1 + \delta(X) \geq \delta(Xy) \geq \delta(XY) = \partial(XY)$$

and $XY \leq M$.

We have two cases: $\delta(XY) = \partial(X)+1$ and $\delta(XY) = \partial(X)$. In the first case $Y = \{y\}$. By Claim 1 the quantifier-free L-type r_{Xy} of Xy is determined by the formulas of the form $(S_X \times M)\setminus T$, $T \subseteq_{cl} M^{n+1}$, $T \in L$, $\dim T < \dim(S_X \times M)$.

Consider

$$r_{Xy}(X', M) = \{z \in M : X'z \in (S_X \times M) \setminus T,\ \dim T < \dim S_X,\ \text{all } T\}.$$

We claim that $r_{Xy}(X', M) \neq \emptyset$. Indeed, otherwise M is the union of countably many sets of the form $T(X', M)$. But the fibres $T(X', M)$ of T are of dimension 0 (since otherwise $\dim T = \dim S_X + 1$, contradicting the definition of the T). This is impossible, by (CU).

Now we choose $y' \in r_{Xy}(X', M)$ and this is as required.

In the second case, by definition, there is an irreducible $R \subseteq_{an} U \subseteq_{op} M^{n+k}$, $n = |X|, k = |Y|$, such that $XY \in R(M)$ and $\dim R = \delta(XY) = \partial(X)$. We may assume $U \subseteq V \times M^k$.

Let $P = \operatorname{pr} R$, the projection into M^n. Then $\dim P \leq \dim R$. But also $\dim P \geq \partial(X)$, by 3.2.6. Hence, $\dim R = \dim P$. On the other hand, $P \subseteq S_X$ and $\dim S_X = \delta(X) = \dim P$. By axiom (WP) we have $S_X \cap V' \subseteq P$ for some L-constructible open V'.

Hence $X' \in S_X \cap V' \subseteq P(M)$, for P the projection of an irreducible analytic set R in the L-type of XY. By Claim 1 the quantifier-free L-type of XY is of the form

$$r_{XY} = \{R \setminus T : T \subseteq_{\rm cl} R,\ \dim T < \dim R\}.$$

Consider

$$r_{XY}(X', M) = \{Z \in M^k : X'Z \in R \setminus T,\ T \subseteq_{\rm cl} R,\ \dim T < \dim R\}.$$

We claim again that $r_{XY}(X', M) \neq \emptyset$. Otherwise the set $R(X', M) = \{X'Z : R(X'Z)\}$ is the union of countably many subsets of the form $T(X', M)$. But $\dim T(X', M) < \dim R(X', M)$ as above, by (AF).

Again, an $Y' \in r_{XY}(X', M)$ is as required. $\square$

3.2.10 Corollary

There are at most countably many $L_{\infty,\omega}(L)$-types of tuples $X \leq M$.

Indeed, any such type is determined uniquely by the choice of a L-constructible $S_X \subseteq_{an} U \subseteq_{op} M^n$ such that $\dim S_X = \partial(X)$.

3.2.11 Lemma

Suppose, for finite $X, X' \subseteq M$, the projective L-types of X and X' coincide. Then the $L_{\infty,\omega}(L)$-types of the tuples are equal.

Proof. Choose finite Y such that $\partial(X) = \delta(XY)$. Then $XY \leq M$. Let $XY \in S \subseteq_{an} U \subseteq_{op} M^n$ be L-constructible and such that $\dim S$ is minimal possible, that is $\dim S = \delta(XY)$. We may assume that S is irreducible. Notice that for every proper closed L-constructible $T \subseteq_{\rm cl} U$, $XY \notin T$ by dimension considerations.

By assumptions of the lemma $X'Y' \in S$, for some Y' in M. We also have $X'Y' \notin T$, for any T as above, since otherwise a projective formula would imply that $XY'' \in T$ for some Y'', contradicting that $\partial(X) > \dim T$.

We also have $\delta(X'Y') = \dim S$. But for no finite Z' it is possible that $\delta(X'Z') < \dim S$, for then again a projective formula will imply that $\delta(XZ) < \dim S$, for some Z.

It follows that $X'Y' \leq M$ and the quantifier-free types of XY and $X'Y'$ coincide, hence the $L_{\infty,\omega}(L)$-types are equal, by 3.2.9. □

3.2.12 Definition

Set, for finite $X \subseteq M$,

$$\mathrm{cl}_L(X) = \{y \in M : \ \partial(Xy) = \partial(X)\}.$$

We fix L and omit the subscript below.

3.2.13 Lemma

The following two conditions are equivalent

(a) $b \in \mathrm{cl}(A)$, for $\vec{A} \in M^n$;

(b) $b \in P(\vec{A}, M)$ for some projective first-order $P \subseteq M^{n+1}$ such that $P(\vec{A}, M)$ is at most countable.

In particular, $\mathrm{cl}(A)$ is countable for any finite A.

Proof. Let $d = \partial(A) = \delta(AV)$, and $\delta(AV)$ is minimal for all possible finite $V \subseteq M$. So by definition $d = \dim S_0$, some analytic irreducible S_0 such that

$\vec{A}\vec{V} \in S_0$ and S_0 of minimal dimension. This corresponds to a L-definable relation $S_0(x, v)$, where x, v strings of variables of length n, m

First assume (b), thais is that b belongs to a countable $P(\vec{A}, M)$. By definition

$$P(x, y) \equiv \exists w\, S(x, y, w),$$

for some analytic $S \subseteq M^{n+1+k}$, some tuples x, y, w of variables of length $n, 1$ and k respectively, and the fibre $S(\vec{A}, b, M^k)$ is nonempty. We also assume that P and S are of minimal dimension, answering this description. By (FC), (AS) and minimality we may choose S so that $\dim S(\vec{A}, b, M^k)$ is minimal among all the fibres $S(\vec{A}', b', M^k)$.

Consider the analytic set $S^\sharp \subseteq M^{n+m+1+k}$ given by $S_0(x, v) \,\&\, S(x, y, w)$. By (AF), considering the projection of the set on (x, v)-coordinates,

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, M, M^k),$$

since $S(\vec{A}, M, M^k)$ is a fibre of the projection. Now we note that by countability $\dim S(\vec{A}, M, M^k) = \dim S(\vec{A}, b, M^k)$, so

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, b, M^k).$$

Now the projection $\mathrm{pr}_w S^\sharp$ along w (corresponding to $\exists w\, S^\sharp$) has fibres of the form $S(\vec{X}, y, M^k)$, so by (AF)

$$\dim \mathrm{pr}_w S^\sharp \leq \dim S_0 = d.$$

Projecting further along v we get $\dim \mathrm{pr}_v \mathrm{pr}_w S^\sharp \leq d$, but $\vec{A}b \in \mathrm{pr}_v \mathrm{pr}_w S^\sharp$ so by Proposition 3.2.6 $\partial(\vec{A}b) \leq d$. The inverse inequality holds by definition, so the equality holds. This proves that $b \in \mathrm{cl}(A)$.

Now assume (a), that is $b \in \mathrm{cl}(A)$. So, $\partial(\vec{A}b) = \partial(\vec{A}) = d$. By definition there is a projective set P containing $\vec{A}b$, defined by the formula $\exists w\, S(x, y, w)$ for some analytic S, $\dim S = d$. Now $\vec{A}$ belongs to the projective set $\mathrm{pr}_y P$ (defined by the formula $\exists y \exists w\, S(x, y, w)$) so by Proposition 3.2.6 $d \leq \dim \mathrm{pr}_y P$, but $\dim \mathrm{pr}_y P \leq \dim P \leq \dim S = d$. Hence all the dimensions are equal and so, the dimension of the generic fibre is 0. We may assume, as above, without loss of generality that all fibres are of minimal dimension, so

$$\dim S(\vec{A}, M, M^k) = 0.$$

Hence, b belongs to a 0-dimensional set $\exists w\, S(\vec{A}, y, w)$, which is projective and countable. $\square$

3.2.14 Lemma

Suppose $b \in \mathrm{cl}(A)$ and the projective type of $\vec{A}b$ is equal to that of $\vec{A}'b'$. Then $b' \in \mathrm{cl}(A')$.

Proof. First note that, by (FC) and (AS), for analytic $R(u,v)$ and its fibre $R(a,v)$ of minimal dimension one has

$$\mathrm{tp}(a) = \mathrm{tp}(a') \Rightarrow \dim R(a,v) = \dim R(a',v).$$

By the second part of the proof of 3.2.13 the assumption of the lemma implies that for some analytic S we have $\vDash \exists w S(\vec{A}, b, w)$ and $\dim S(\vec{A}, M, M^k) = 0$. Hence $\vDash \exists w S(\vec{A'}, b', w)$ and $\dim S(\vec{A'}, M, M^k) = 0$. But this immediately implies $b' \in \mathrm{cl}(A')$. □

3.2.15 Lemma

(i)
$$\mathrm{cl}(\emptyset) = \mathrm{cl}(M_0) = M_0.$$

(ii) Given finite $X \subseteq M$, $y, z \in M$,

$$z \in \mathrm{cl}(X, y) \setminus \mathrm{cl}(X) \Rightarrow y \in \mathrm{cl}(X, z).$$

(iii)
$$\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X).$$

Proof. (i) Clearly $M_0 \subseteq \mathrm{cl}(\emptyset)$, by definition.

We need to show the converse, that is if $\partial(y) = 0$, for $y \in M$, then $y \in M_0$. By definition $\partial(y) = \partial(\emptyset) = \min\{\delta(Y) : y \in Y \subset M\} = 0$. So, $y \in Y$, $\vec{Y} \in S \subseteq_{an} U \subseteq_{op} M^n$, $\dim S = 0$. The irreducible components of S are closed points (singletons) and $\{\vec{Y}\}$ is one of them, so must be in M_0, hence $y \in M_0$.

(ii) Assuming the left-hand side of (ii), $\partial(Xyz) = \partial(Xy) > \partial(X)$ and $\partial(Xz) > \partial(X)$. By the definition of ∂ then,

$$\partial(Xy) = \partial(X) + 1 = \partial(Xz),$$

so $\partial(Xzy) = \partial(Xz)$, $y \in \mathrm{cl}(Xz)$.

(iii) Immediate by 3.2.13. □

Below, if not stated otherwise, we use the language $L^{\exists}$ the primitives of which correspond to relations $\exists$-definable in M. Also, we call a **submodel of** M any $L^{\exists}$-substructure closed under cl.

3.2.16 Theorem

(i) Every $L_{\infty,\omega}(L)$-type realised in M is equivalent to a projective type, that is a type consisting of existential (first-order) formulas and the negations of existential formulas.

(ii) There are only countably many $L_{\infty,\omega}(L)$-types realised in M.

(iii) $(M, L^{\exists})$ is quasi minimal ω-homogeneous over countable submodels, that is the following hold:

(a) for any countable (or empty) submodel G and any n-tuples X and X', both cl-independent over G, a bijection $\phi : X \to X'$ is a G-monomorphism;

(b) given any G-monomorphism $\phi : Y \to Y'$ for finite tuples Y, Y' in M and given a $z \in M$ we can extend ϕ so that $z \in \operatorname{Dom}\phi$.

Proof. (i) Immediate from 3.2.11.

(ii) By 3.2.10 there are only countably many types of finite tuples $Z \leq M$. Let $N \subseteq M_0$ be a countable subset of M such that any finite $Z \leq M$ is $L_{\infty,\omega}(L)$-equivalent to some tuple in N. Every finite tuple $X \subset M$ can be extended to $XY \leq M$, so there is a $L_{\infty,\omega}(L)$-monomorphism $XY \to N$. This monomorphism identifies the $L_{\infty,\omega}(L)$-type of X with one of a tuple in N, hence there are no more than countably many such types.

(iii) Lemma 3.2.15 proves that cl defines a pregeometry on M.

Consider first (a). Note that $GX \leq M$ and $GX' \leq M$ and so the types of X and X' over G are L-quantifier-free. But there is no proper L-closed subset $S \subseteq_{\mathrm{cl}} M^n$ such that $\vec{X} \in S$ or $\vec{X}' \in S$. Hence the types are equal.

For (b) just use the fact that the G-monomorphism by our definition preserves $\exists$-formulas, so by 3.2.11 complete $L_{\infty,\omega}(L(G))$-types of X and X' coincide, so by 3.2.7 ϕ can be extended. $\square$

3.2.17 Theorem

M is a quasiminimal pregeometry structure (see [16]). In other words, the following properties of M hold:

(QM1) The pregeometry cl is determined by the language. That is, if $\operatorname{tp}(x, Y) = \operatorname{tp}(x', Y')$, then $x \in \operatorname{cl}(Y)$ if and only $x' \in \operatorname{cl}(Y')$. (Here the types are first order).

(QM2) The structure M is infinite-dimensional with respect to cl.

(QM3) (Countable closure property). If $X \subset M$ is finite, then $\operatorname{cl}(X)$ is countable.

(QM4) (Uniqueness of the generic type). Suppose that $H, H' \subset M$ are countable closed subsets, enumerated such that $\operatorname{tp}(H) = \operatorname{tp}(H')$. If $y \in M \setminus H$ and $y' \in M \setminus H'$, then $\operatorname{tp}(H, y) = \operatorname{tp}(H', y')$.

(QM5) (ω-homogeneity over closed sets and the empty set). Let$H, H' \subset M$ be countable closed subsets or empty, enumerated such that $\operatorname{tp}(H) = \operatorname{tp}(H')$, and let Y, Y' be finite tuples from M such that $\operatorname{tp}(H, Y) = \operatorname{tp}(H', Y')$, and let $z \in \operatorname{cl}(H, Y)$. Then there is $z' \in M$ such that $\operatorname{tp}(H, Y, z) = \operatorname{tp}(H', Y', z')$.

Proof. (M, cl) is a pregeometry by 3.2.15. (QM1) is proved in 3.2.14. (QM3) is 3.2.13 and (QM2) follows from (QM3) and (CU). (QM4)&(QM5) is 3.2.16(iii). $\square$

Now we define an *abstract elementary class* $\mathcal{C}$ associated with M. We follow [4], Ch.6 for this construction. Similar construction was used in [16].

Set

$$\mathcal{C}_0(\mathrm{M}) = \{ \text{ countable } L^{\exists}\text{-structures N} : \ \mathrm{N} \cong \mathrm{N}' \subseteq \mathrm{M}, \ \mathrm{cl}(N') = N' \}$$

and define embedding $\mathrm{N}_1 \preccurlyeq \mathrm{N}_2$ in the class as an $L^{\exists}$-embedding $f : \mathrm{N}_1 \to \mathrm{N}_2$ such that there are isomorphisms $g_i : \mathrm{N}_i \to \mathrm{N}_i'$, $\mathrm{N}_1' \subseteq \mathrm{N}_2' \subseteq \mathrm{M}$, all embeddings commuting and $\mathrm{cl}(N_i') = N_i'$.

Now define $\mathcal{C}(\mathrm{M})$ to be the class of L-structures H with cl_L defined with respect to H and satisfying:
(i) $\mathcal{C}_0(\mathrm{H}) \subseteq \mathcal{C}_0(\mathrm{M})$ as classes with embeddings
and
(ii) for every finite $X \subseteq H$ there is $\mathrm{N} \in \mathcal{C}_0(\mathrm{H})$, such that $X \subseteq N$.

Given $\mathrm{H}_1 \subseteq \mathrm{H}_2$, $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{C}(\mathrm{M})$, we define $\mathrm{H}_1 \preccurlyeq \mathrm{H}_2$ to hold in the class, if for every finite $X \subseteq H_1$, $\mathrm{cl}(X)$ is the same in H_1 and H_2. More generally, for $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{C}(\mathrm{M})$ we define $\mathrm{H}_1 \preccurlyeq_f \mathrm{H}_2$ to be an embedding f such that there are isomorphisms $\mathrm{H}_1 \cong \mathrm{H}_1'$, $\mathrm{H}_2 \cong \mathrm{H}_2'$ such that $\mathrm{H}_1' \subseteq \mathrm{H}_2'$, all embeddings commute, and $\mathrm{H}_1 \preccurlyeq \mathrm{H}_2$.

3.2.18 Lemma

$\mathcal{C}(\mathrm{M})$ is closed under the unions of ascending $\preccurlyeq$-chains.

Proof. Immediate from the fact that for infinite $Y \subseteq M$,

$$\mathrm{cl}(Y) = \bigcup \{ \mathrm{cl}(X) : \ X \subseteq_{\text{finite}} Y \}.$$

$\square$

3.2.19 Theorem

The class $\mathcal{C}(\mathrm{M})$ contains structures of any infinite cardinality and is categorical in uncountable cardinals.

Proof. This follows from 3.2.17 by the main result of [16].

$\square$

3.2.20 Proposition

Any uncountable $\mathrm{H} \in \mathcal{C}(\mathrm{M})$ is an analytic 1-dimensional irreducible Zariski structure in the language L. Also H is presmooth if M is.

Proof. We define $\mathcal{C}(\mathrm{H})$ to consist of the subsets of H^n of the form

$$P_a(H) := \{ x \in H^n : \mathrm{H} \vDash P(a^\frown x) \},$$

for $P \in L$ of arity $k + n$, $a \in H^k$. The assumption (L) is obviously satisfied.

Now note that the constructible and projective sets in $\mathcal{C}$(H) are also of the form $P_a(H)$ for some L-constructible or L-projective P.

Define

$\dim P_a(H) := d$ if $\dim P_b(M) = d$, for some $b \in M^k$ such that the $L^{\exists}$-quantifier-free types of a and b are equal.

This is well-defined by (FC) and the fact that any $L^{\exists}$-quantifier-free type realised in H is also realised in M. Moreover, we have the following.

Claim. The set of $L^{\exists}$-quantifier-free types realised in H is equal to that realised in M.

Indeed, this is immediate from the definition of the class $\mathcal{C}$(M), stability of $\mathcal{C}$(M) and the fact that the class is categorical in uncountable cardinalities.

The definition of dimension immediately implies (DP), (CU),(AF) and (FC) for H.

(SI): if $P'_{a_1}(H) \subseteq_{\mathrm{cl}} P_{a_0}(H)$, $\dim P'_{a_1}(H) = \dim P_{a_0}(H)$ and the two sets are not equal, then the same holds for $P'_{b_1}(M)$ and $P_{b_0}(M)$ for equivalent b_0, b_1 in M. Then, $P_{b_0}(M)$ is reducible, that is for some proper $P''_{b_2}(M) \subset_{cl} P_{b_0}(M)$ we have $P_{b_0}(M) = P'_{b_1}(M) \cup P''_{b_2}(M)$. Now, by homogeneity we can choose a_2 in H such that $P_{a_0}(H) = P'_{a_1}(H) \cup P''_{a_2}(H)$, a reducible representation.

This also shows that the notion of irreducibility is preserved by equivalent substitution of parameters. Then the same is true for the notion of analytic subset. Hence (INT), (CMP),(CC) and (PS) follow. For the same reason (AS) holds. Next we notice that the axioms (WP) follows by the homogeneity argument. □

3.3 Some examples

3.3.1 Universal covers of semiabelian varieties

Let $\mathbb{A}$ be a semiabelian variety of dimension d, e.g. $d = 1$ and $\mathbb{A}$ the algebraic torus $\mathbb{C}^{\times}$. Let V be the universal cover of $\mathbb{A}$, which classically can be identified as a complex manifold $\mathbb{C}^d$.

We define a structure with a (formal) topology on V and show that this is analytic Zariski.

By definition of universal cover there is a covering holomorphic map

$$\exp : V \to \mathbb{A}$$

(a generalisation of the usual exp on $\mathbb{C}$).

We will assume that $\mathcal{C}$ has no proper semiabelian subvarieties (is simple) and no complex multiplication.

We consider the two sorted structure $(V, \mathbb{A})$ in the language that has all Zariski closed subsets of $\mathbb{A}^n$, all n, the addition $+$ on V and the map exp as the primitives.

This case was first looked at model-theoretically in [14] and the special case $\mathbb{A} = \mathbb{C}^\times$ in [12], [13] and in the DPhil thesis [15] of Lucy Smith.

Our aim here is to show that the structure on the sort V with a naturally given formal topology is analytic Zariski.

The positive quantifier-free definable subsets of V^n, $n = 1, 2, \ldots$ form a base of a topology which we call the PQF-topology. In other words

3.3.2 Definition

A PQF-closed subset of V^n is defined as a finite union of sets of the form

$$L \cap m \cdot \ln W \tag{3.1}$$

where $W \subseteq \mathbb{A}^n$, an algebraic subvariety, and L is a $\mathbb{Q}$-linear subspace of V^n, that is defined by a system of equations of the form $m_1 x_1 + \ldots + m_n x_n = a$, $m_i \in \mathbb{Z},\ a \in V^n$.

The relations on V which correspond to PQF_ω-closed sets are the primitives of our language L.

PQF-closed subsets form a base for a topology on the cartesian powers of V which will underlie the analytic Zariski structure on V.

Remark. Among closed sets of the topology we have sets of the form

$$\cup_{a \in I}(S + a)$$

where S is of the form (3.1) and I a subset of $(\ker \exp)^n$.

Slightly rephrasing the quantifier-elimination statement proved in [14] Corollary 2 of section 3, we have the following result.

3.3.3 Proposition

(i) Projection of a PQF-closed set is PQF-constructible, that is a boolean combination of PQF-closed sets.

(ii) The image of a constructible set under exponentiation is a Zariski-constructible (algebraic) subset of $\mathbb{A}^n$. The image of the set of the form (3.1) is Zariski closed.

We assign **dimension** to a closed set of the form (3.1)

$$\dim L \cap m \cdot \ln W := \dim \exp\left(L \cap m \cdot \ln W\right).$$

using the fact that the object on the right hand side is an algebraic variety. We extend this to an arbitrary closed set assuming (CU), that is that the dimension of a countable union is the maximum dimension of its members.This immediately gives (DP). Using 3.3.3 we also get (WP).

The analysis of irreducibility below is more involved. Since $\exp L$ is definably and topologically isomorphic to $\mathbb{A}^k$, some $k \geq 1$, we can always reduce the analysis of a closed set of the form (3.1) to a one of the form $\ln W$ with $W \subseteq \mathbb{A}^k$ not contained in a coset of a proper algebraic subgroup.

For such a W consider its m-th "root"

$$W^{\frac{1}{m}} = \{\langle x_1, \dots, x_n \rangle \in \mathbb{A}^k : \langle x_1^m, \dots, x_k^m \rangle \in W\}.$$

Let $d = d_W(m)$ be the number of irreducible components of $W^{\frac{1}{m}}$.

It is easy to see that if $d > 1$, irreducible components $W_i^{\frac{1}{m}}$, $i = 1, \dots, d$, of $W^{\frac{1}{m}}$ are shifts of each other by m-th roots of unity, and $m \cdot \ln W_i^{\frac{1}{m}}$ are proper closed subsets of $\ln W$ of the same dimension. It follows that $\ln W$ *is irreducible (in the sense of* (SI)*) if and only if* $d_W(m) = 1$ *for all* $m \geq 1$. In [18] W satisfying this condition is called **Kummer generic**. If $W \subset \exp L$ for some $\mathbb{Q}$-linear subspace $L \subset V^n$, then one uses the relative version of Kummer genericity.

We say that the sequence $W^{\frac{1}{m}}$, $m \in \mathbb{N}$, stops branching if the sequence $d_W(m)$ is eventually constant, that is if $W^{\frac{1}{m}}$ is Kummer generic for some $m \geq 1$.

The following is proved for $\mathbb{A} = \mathbb{C}^\times$ in [12], (Theorem 2, case $n = 1$ and its Corollary) and in general in [18].

3.3.4 Theorem

The sequence $W^{\frac{1}{m}}$ stops branching if and only if W is not contained in a coset of a proper algebraic subgroup of $\mathbb{A}^k$.

3.3.5 Corollary

Any irreducible closed subset of V^n is of the form $L \cap \ln W$, for W Kummer generic in $\exp L$.

Any closed subset of V^n is analytic in V^n.

It is easy now to check that the following.

3.3.6 Corollary

The structure $(V; L)$ is analytic Zariski and presmooth.

The reader may notice that the analysis above treats only *formal* notion of analyticity on the cover $\mathbb{C}$ of $\mathbb{C}^\times$ but does not address the classical one. In particular, the following question is order: *is the formal analytic decomposition as described by 3.3.5 the same as the actual complex analytic one?* In a private communication F.Campana answered this question in positive, using a cohomological argument. M.Gavrilovich proved this and much more general statement in his thesis (see [11], III.1.2) by a similar argument.

3.3.7 Covers in positive characteristic

Now we look into yet another version of a cover structure which is proven to be analytic Zariski, a **cover of the one-dimensional algebraic torus over an algebraically closed field of a positive characteristic.**

Let $(V, +)$ be a divisible torsion free abelian group and K an algebraically closed field of a positive characteristic p. We assume that V and K are both of the same uncountable cardinality. Under these assumptions it is easy to construct a surjective homomorphism

$$\mathrm{ex} : V \to K^{\times}.$$

The kernel of such a homomorphism must be a subgroup which is p-divisible but not q-divisible for each q coprime with p. One can easily construct ex so that

$$\ker \mathrm{ex} \cong \mathbb{Z}[\frac{1}{p}],$$

the additive group (which is also a ring) of rationals of the form $\frac{m}{p^n}$, $m, n \in \mathbb{Z}$, $n \geq 0$. In fact in this case it is convenient to view V and $\ker \mathrm{ex}$ as $\mathbb{Z}[\frac{1}{p}]$-modules.

In this new situation Lemma 3.3.3 is still true, with obvious alterations, and we can use the definition 3.3.2 to introduce a topology and the family L as above. The necessary version of Theorem 3.3.4 is proved in [18]. Hence the corresponding versions of 3.3.5 follows.

3.3.8 Remark

In all the above examples the analytic rank of any nonempty closed subset is 1, that is any closed subset is analytic.

3.3.9 $\mathbb{C}_{\exp}$ and other pseudo-analytic structures

$\mathbb{C}_{\exp}$, the structure $(\mathbb{C}; +, \cdot, \exp)$, was a prototype of **the field with pseudo-exponentiation** studied by the current author in [17]. It was proved (with later corrections, see [19]) that this structure is quasi-minimal and its (explicitly written) $L_{\omega_1,\omega}(Q)$-axioms are categorical in all uncountable cardinality. This result has been generalised to many other structures of analytic origin in [19], in particular to the the formal analogue of $\mathbb{C}_{\mathfrak{P}} = (\mathbb{C}; +, \cdot, \mathfrak{P})$, where $\mathfrak{P} = \mathfrak{P}(\tau, z)$ is the Weierstrass function of variable z with parameter τ. We call these structures **pseudo-analytic.**

It is a reasonable conjecture to assume that the pseudo-analytic structures of cardinality continuum are isomorphic to their complex prototypes. Nevertheless, even under this conjecture it is not known whether $\mathbb{C}_{\exp}$, $\mathbb{C}_{\mathfrak{P}}$ or any of the other pseudo-analytic structures (which do not satisfy 3.3.8) are analytic Zariski. One may start by defining the family of (formal) closed sets in the structure to coincide with the family of definable subsets which are closed in the metric topology of the complex manifold. The problem then is to conveniently classify such subsets. A suggestion for such a classification may come from the following notion.

3.3.10 Generalised analytic sets

In [6] we have discussed the following notion of generalised analytic subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and, more generally, of $[\mathbf{P}^1(K)]^n$ for K algebraically closed complete valued field.

Let $F \subseteq \mathbb{C}^2$ be a graph of an entire analytic function and $\bar{F}$ its closure in $[\mathbf{P}^1(\mathbb{C})]^2$. It follows from Picard's Theorem that $\bar{F} = F \cup \{\infty\} \times \mathbf{P}^1(\mathbb{C})$, in particular $\bar{F}$ has analytic rank 2.

Generalised analytic sets are defined as the subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ for all n, obtained from classical (algebraic) Zariski closed subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and some number of sets of the form $\bar{F}$ by applying the positive operations: Cartesian products, finite intersections, unions and projections. It is clear by definition that the complex generalised analytic sets are closed (but not obvious for the case of K, algebraically closed complete non-Archimedean valued field).

3.3.11 Theorem (see [6])

Any generalised analytic set is of finite analytic rank.

Bibliography

[1] N.Peatfield and B.Zilber, *Analytic Zariski structures and the Hrushovski construction,* Annals of Pure and Applied Logic, Vol 132/2-3 (2004), 127-180

[2] E.Hrushovski and B.Zilber, *Zariski Geometries*, Journal of AMS, 9(1996), 1-56

[3] B.Zilber, *Model Theory and Algebraic Geometry.* In **Seminarberichte Humboldt Universitat zu Berlin**, Nr 93-1, Berlin 1993, 202–222

[4] B.Zilber, **Zariski Geometries**, CUP, 2010

[5] I. Ben-Yaacov, *Positive model theory and compact abstract theories,* Journal of Mathematical Logic 3 (2003), no. 1, 85–118

[6] B.Zilber, *Generalized Analytic Sets* Algebra i Logika, Novosibirsk, v.36, N 4 (1997), 361 - 380 (translation on the author's web-page and published by Kluwer as 'Algebra and Logic')

[7] B.Zilber, *A categoricity theorem for quasi-minimal excellent classes.* In: **Logic and its Applications** eds. Blass and Zhang, Cont.Maths, v.380, 2005, pp.297-306

[8] J.Baldwin, **Categoricity**, University Lecture Series, v.50, AMS, Providence, 2009

[9] T. Hyttinen, O. Lessmann, and S. Shelah, *Interpreting groups and fields in some nonelementary classes,* J. Math. Log., 5(1):1–47, 2005

[10] D. Kueker, *Back-and-forth arguments in infinitary languages*, In **Infinitary Logic: In Memoriam Carol Karp**, D. Kueker ed., Lecture

Notes in Math 72, Springer-Verlag 1975

[11] M.Gavrilovich, **Model Theory of the Universal Covering Spaces of Complex Algebraic Varieties**, DPhil Thesis, Oxford 2006, http://misha.uploads.net.ru/misha-thesis.pdf

[12] B.Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero* J. London Math. Soc. (2), 74(1):41–58, 2006

[13] M.Bays and B.Zilber, *Covers of Multiplicative Groups of Algebraically Closed Fields of Arbitrary Characteristic*, Bull. London Math. Soc. (2011) 43 (4), 689–702

[14] B.Zilber, *Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety* In: **Model Theory and Applications**, ed. L.Belair and el. (Proc. of Conf. in Ravello 2000) Quaderni di Matematica, v.11, Series edited by Seconda Univli, Caserta, 2002

[15] L.Smith, **Toric Varieties as Analytic Zariski Structures**, DPhil Thesis, Oxford 2008

[16] Bays, M., Hart, B., Hyttinen, T., Kesala, M., Kirby, J. *Quasiminimal structures and excellence*, Bulletin of the London Mathematical Society, 46. (2014) pp. 155-163

[17] B.Zilber, *Pseudo-exponentiation on algebraically closed fields of characteristic zero*, Ann. Pure Appl. Logic, v.132, 2005, pp.67–95

[18] M.Bays, M.Gavrilovich and M.Hils, *Some Definability Results in Abstract Kummer Theory* Int. Math. Res. Not. IMRN, (14):39754000, 2014

[19] , M.Bays and J.Kirby, *Some pseudo-analytic functions on commutative algebraic groups*, arXiv:1512.04262

Part III

Abstract Elementary Classes

Chapter 4

Hanf Numbers and Presentation Theorems in AECs

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Abstract

We prove that a strongly compact cardinal is an upper bound for a Hanf number for amalgamation, etc. in AECs using both semantic and syntactic methods. To syntactically prove non-disjoint amalgamation, a different presentation theorem than Shelah's is needed. This relational presentation theorem has the added advantage of being *functorial*, which allows the transfer of amalgamation.

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4.1 Introduction

This paper addresses a number of fundamental problems in logic and the philosophy of mathematics by considering some more technical problems in model theory and set theory. The interplay between syntax and semantics is usually considered the hallmark of model theory. At first sight, Shelah's notion of abstract elementary class shatters that icon. As in the beginnings of the modern theory of structures ([Cor92]) Shelah studies certain classes of models and relations among them, providing an axiomatization in the Bourbaki ([Bou50]) as opposed to the Gödel or Tarski sense: mathematical requirements, not sentences in a formal language. This formalism-free approach ([Ken13]) was designed to circumvent confusion arising from the syntactical schemes of infinitary logic; if a logic is closed under infinite conjunctions, what is the sense of studying types? However, Shelah's presentation theorem and more strongly Boney's use [Bon] of AEC's as theories of $\mathbb{L}_{\kappa,\omega}$ (for κ strongly compact) reintroduce syntactical arguments. The issues addressed in this paper trace to the failure of infinitary logics to satisfy the *upward* Löwenheim-Skolem theorem or more specifically the compactness theorem. The compactness theorem allows such basic algebraic notions as amalgamation and joint embedding to be easily encoded in first order logic. Thus, all complete first order theories have amalgamation and joint embedding in all cardinalities. In contrast these and other familiar concepts from algebra and model theory turn out to be heavily cardinal-dependent for infinitary logic and specifically for abstract elementary classes. This is especially striking as one of the most important contributions of modern model theory is the freeing of first order model theory from its entanglement with axiomatic set theory ([Bal15a], Chapter 7 of [Bal15b]).

Two main issues are addressed here. We consider not the interaction of syntax and semantics in the usual formal language/structure dichotomy but methodologically. What are reasons for adopting syntactic and/or semantic approaches to a particular topic? We compare methods from the very beginnings of model theory with semantic methods powered by large cardinal hypotheses. Secondly, what then are the connections of large cardinal axioms with the cardinal dependence of algebraic properties in model theory. Here we describe the opening of the gates for potentially large interactions between set theorists (and incidentally graph theorists) and model theorists. More precisely, can the combinatorial properties of small large cardinals be coded as structural properties of abstract elementary classes so as to produce Hanf numbers intermediate in cardinality between 'well below the first inaccessible' and 'strongly compact'?

Most theorems in mathematics are either true in a specific small cardinality (at most the continuum) or in all cardinals. For example all, *finite* division rings are commutative, thus all finite Desarguesian planes are Pappian. But *all* Pappian planes are Desarguean and not conversely. Of course this stricture does not apply to set theory, but the distinctions arising in set theory are combinatorial. First order model theory, to some extent, and Abstract Elementary Classes (AEC) are beginning to provide a deeper exploration of Cantor's paradise: algebraic properties that are cardinality dependent. In this article, we explore whether certain key properties (amalgamation, joint embedding, and their relatives) follow this line. These algebraic properties are structural in the sense of [Cor04].

Much of this issue arises from an interesting decision of Shelah. Generalizing Fraïssé [Fra54] who considered only finite and countable stuctures, Jónsson laid the foundations for AEC by his study of universal and homogeneous relation systems [Jón56, Jón60]. Both of these authors assumed the amalgamation property (AP) and the joint embedding property (JEP), which in their context is cardinal independent. Variants such as disjoint or free amalgamation (DAP) are a well-studied notion in model theory and universal algebra. But Shelah omitted the requirement of amalgamation in defining AEC. Two reasons are evident for this: it is cardinal dependent in this context; Shelah's theorem (under weak diamond) that categoricity in κ and few models in κ^+ implies amalgamation in κ suggests that amalgamation might be a dividing line.

Grossberg [Gro02, Conjecture 9.3] first raised the question of the existence of Hanf numbers for joint embedding and amalgamation in Abstract Elementary Classes (AEC). We define four kinds of amalgamation properties (with various cardinal parameters) in Subsection 4.1.1 and a fifth at the end of Section 4.3.1. The first three notions are staples of the model theory and universal algebra since the fifties and treated for first order logic in a fairly uniform manner by the methods of Abraham Robinson. It is a rather striking feature of Shelah's presentation theorem that issues of disjointness require careful study for AEC, while disjoint amalgamation is trivial for complete first order theories.

Our main result is the following:

Theorem 4.1.1. *Let κ be strongly compact and $\mathbf{K}$ be an AEC with Löwenheim-Skolem number less than κ.*

If $\mathbf{K}$ satisfies[1] *AP/JEP/DAP/DJEP/ NDJEP for models of size $[\mu, < \kappa)$, then $\mathbf{K}$ satisfies AP/JEP/DAP/DJEP/ NDJEP for all models of size $\geq \mu$.*

We conclude with a survey of results showing the large gap for many properties between the largest cardinal where an 'exotic' structure exists and the smallest where eventual behavior is determined. Then we provide specific question to investigate this distinction.

Our starting place for this investigation was second author's work [Bon] that emphasized the role of large cardinals in the study of AEC. A key aspect of the definition of AEC is as a mathematical definition with no formal syntax - class of structures satisfying certain closure properties. However, Shelah's Presentation Theorem says that AECs are expressible in infinitary languages, $\mathbb{L}_{\kappa,\omega}$, which allowed a proof via sufficiently complete ultraproducts that, assuming enough strongly compact cardinals, all AEC's were eventually tame in the sense of [GV06].

Thus we approached the problem of finding a Hanf number for amalgamation, etc. from two directions: using ultraproducts to give purely semantic arguments and using Shelah's Presentation Theorem to give purely syntactic arguments. However, there was a gap: although syntactic arguments gave characterizations similar to those found in first order, they required looking

[1] This alphabet soup is decoded in Definition 4.1.3.

at the *disjoint* versions of properties, while the semantic arguments did not see this difference.

The requirement of disjointness in the syntactic arguments stems from a lack of canonicity in Shelah's Presentation Theorem: a single model has many expansions which means that the transfer of structural properties between an AEC $\mathbf{K}$ and it's expansion can break down. To fix this problem, we developed a new presentation theorem, called the *relational presentation theorem* because the expansion consists of relations rather than the Skolem-like functions from Shelah's Presentation Theorem.

Theorem 4.1.2 (The relational presentation theorem, Theorem 4.3.8)**.** *To each AEC $\mathbf{K}$ with $LS(K) = \kappa$ in vocabulary τ, there is an expansion of τ by predicates of arity κ and a theory T^* in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ such that $\mathbf{K}$ is exactly the class of τ reducts of models of T^*.*

Note that this presentation theorem works in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ and has symbols of arity κ, a far cry from the $\mathbb{L}_{(2^\kappa)^+,\omega}$ and finitary language of Shelah's Presentation Theorem. The benefit of this is that the expansion is canonical or functorial (see Definition 4.3.1). This functoriality makes the transfer of properties between $\mathbf{K}$ and $(\operatorname{Mod} T^*, \subset_{\tau^*})$ trivial (see Proposition 4.3.2). This allows us to formulate natural syntactic conditions for our structural properties.

Comparing the relational presentation theorem to Shelah's, another well-known advantage of Shelah's is that it allows for the computation of Hanf numbers for existence (see Section 4.4) because these exist in $\mathbb{L}_{\kappa,\omega}$. However, there is an advantage of the relational presentation theorem: Shelah's Presentation Theorem works with a sentence in the logic $\mathbb{L}_{(2^{LS(\mathbf{K})})^+,\omega}$ and there is little hope of bringing that cardinal down[2]. On the other hand, the logic and size of theory in the relational presentation theorem can be brought down by putting structure assumptions on the class $\mathbf{K}$, primarily on the number of non-isomorphic extensions of size $LS(\mathbf{K})$, $|\{(M,N)/\cong : M \prec_{\mathbf{K}} N \text{ from } \mathbf{K}_{LS(\mathbf{K})}\}|$.

We would like to thank Spencer Unger and Sebastien Vasey for helpful discussions regarding these results.

4.1.1 Preliminaries

We discuss the relevant background of AECs, especially for the case of disjoint amalgamation.

Definition 4.1.3. We consider several variations on the joint embedding property, written JEP or JEP$[\mu,\kappa)$.

1. Given a class of cardinals $\mathcal{F}$ and an AEC $\mathbf{K}$, $\mathbf{K}_{\mathcal{F}}$ denotes the collection of $M \in \mathbf{K}$ such that $|M| \in \mathcal{F}$. When $\mathcal{F}$ is a singleton, we write $\mathbf{K}_\kappa$

[2]Indeed an AEC $\mathbf{K}$ where the sentence is in a smaller logic would likely have to have satisfy the very strong property that there are $< 2^{LS(\mathbf{K})}$ many $\tau(\mathbf{K})$ structures that **are not** in $\mathbf{K}$

instead of $\mathbf{K}_{\{\kappa\}}$. Similarly, when $\mathcal{F}$ is an interval, we write $< \kappa$ in place of $[\mathrm{LS}(\mathbf{K}), \kappa)$; $\leq \kappa$ in place of $[\mathrm{LS}(\mathbf{K}), \kappa]$; $> \kappa$ in place of $\{\lambda \mid \lambda > \kappa\}$; and $\geq \kappa$ in place of $\{\lambda \mid \lambda \geq \kappa\}$.

2. An AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ has the joint embedding property, JEP, (on the interval $[\mu, \kappa)$) if any two models (from $\mathbf{K}_{[\mu,\kappa)}$) can be $\mathbf{K}$-embedded into a larger model.
3. If the embeddings witnessing the joint embedding property can be chosen to have disjoint ranges, then we call this the *disjoint embedding property* and write $DJEP$.
4. An AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ has the amalgamation property, AP, (on the interval $[\mu, \kappa)$) if, given any triple of models $M_0 \prec M_1, M_2$ (from $\mathbf{K}_{[\mu,\kappa)}$), M_1 and M_2 can be $\mathbf{K}$-embedded into a larger model by embeddings that agree on M_0.
5. If the embeddings witnessing the amalgamation property can be chosen to have disjoint ranges except for M_0, then we call this the *disjoint amalgamation property* and write DAP.

Definition 4.1.4. 1. A *finite diagram* or $EC(T, \Gamma)$*-class* is the class of models of a first order theory T which omit all types from a specified collection Γ of complete types in finitely many variables over the empty set.

2. Let Γ be a collection of first order types in finitely many variables over the empty set for a first order theory T in a vocabulary τ_1. A $PC(T, \Gamma, \tau)$ *class* is the class of reducts to $\tau \subset \tau_1$ of models of a first order τ_1-theory T which omit all members of the specified collection Γ of partial types.

4.2 Semantic arguments

It turns out that the Hanf number computation for the amalgamation properties is immediate from Boney's "Łoś' Theorem for AECs" [Bon, Theorem 4.3]. We will sketch the argument for completeness. For convenience here, we take the following of the many equivalent definitions of strongly compact; it is the most useful for ultraproduct constructions.

Definition 4.2.1 ([Jec06].20)**.** The cardinal κ is strongly compact iff for every S and every κ-complete filter on S can be extended to a κ-complete ultrafilter. Equivalently, for every $\lambda \geq \kappa$, there is a fine[3], κ-complete ultrafilter on $P_\kappa\lambda = \{\sigma \subset \lambda : |\sigma| < \kappa\}$.

[3] U is fine iff $G(\alpha) := \{z \in P_\kappa(\lambda) | \alpha \in z\}$ is an element of U for each $\alpha < \lambda$.

For this paper, "essentially below κ" means "$LS(K) < \kappa$."

Fact 4.2.1 (Łoś' Theorem for AECs). Suppose K is an AEC essentially below κ and U is a κ-complete ultrafilter on I. Then K and the class of K-embeddings are closed under κ-complete ultraproducts and the ultrapower embedding is a K-embedding.

The argument for Theorem 4.2.2 has two main steps. First, use Shelah's presentation theorem to interpret the AEC into $\mathbb{L}_{\kappa,\omega}$ and then use the fact that $\mathbb{L}_{\kappa,\omega}$ classes are closed under ultraproduct by *κ-complete* ultraproducts.

Theorem 4.2.2. *Let κ be strongly compact and $\mathbf{K}$ be an AEC with Löwenheim-Skolem number less than κ.*

- *If $\mathbf{K}$ satisfies $AP(< \kappa)$ then $\mathbf{K}$ satisfies AP.*
- *If $\mathbf{K}$ satisfies $JEP(< \kappa)$ then $\mathbf{K}$ satisfies JEP.*
- *If $\mathbf{K}$ satisfies $DAP(< \kappa)$ then $\mathbf{K}$ satisfies DAP.*

Proof: We first sketch the proof for the first item, AP, and then note the modifications for the other two.

Suppose that $\mathbf{K}$ satisfies $AP(< \kappa)$ and consider a triple of models (M, M_1, M_2) with $M \prec_{\mathbf{K}} M_1, M_2$ and $|M| \leq |M_1| \leq |M_2| = \lambda \geq \kappa$. Now we will use our strongly compact cardinal. An *approximation* of (M, M_1, M_2) is a triple $\mathbf{N} = (N^{\mathbf{N}}, N_1^{\mathbf{N}}, N_2^{\mathbf{N}}) \in (K_{<\kappa})^3$ such that $N^{\mathbf{N}} \prec M, N_\ell^{\mathbf{N}} \prec M_\ell, N^{\mathbf{N}} \prec N_\ell^{\mathbf{N}}$ for $\ell = 1, 2$. We will take an ultraproduct indexed by the set X below of approximations to the triple (M, M_1, M_2). Set

$$X := \{\mathbf{N} \in (\mathbf{K}_{<\kappa})^3 : \mathbf{N} \text{ is an approximation of } (M, M_1, M_2)\}$$

For each $\mathbf{N} \in X$, $AP(< \kappa)$ implies there is an amalgam of this triple. Fix $f_\ell^{\mathbf{N}} : N_\ell^{\mathbf{N}} \to N_*^{\mathbf{N}}$ to witness this fact. For each $(A, B, C) \in [M]^{<\kappa} \times [M_1]^{<\kappa} \times [M_2]^{<\kappa}$, define

$$G(A, B, C) := \{\mathbf{N} \in X : A \subset N^{\mathbf{N}}, B \subset N_1^{\mathbf{N}}, C \subset N_2^{\mathbf{N}}\}$$

These sets generate a κ-complete filter on X, so it can be extended to a κ-complete ultrafilter U on X; note that this ultrafilter will satisfy the appropriate generalization of fineness, namely that $G(A, B, C)$ is always a U-large set.

We will now take the ultraproduct of the approximations and their amalgam. In the end, we will end up with the following commuting diagram, which provides the amalgam of the original triple.

$$\begin{tikzcd}
 & M_1 \arrow[rr, "h_1"] & & \Pi N_1^{\mathbf{N}}/U \arrow[dr, "\Pi f_1^{\mathbf{N}}"] & \\
M \arrow[ur] \arrow[rr, "h"'] \arrow[dr] & & \Pi N^{\mathbf{N}}/U \arrow[ur] \arrow[dr] & & \Pi N_*^{\mathbf{N}}/U \\
 & M_2 \arrow[rr, "h_2"'] & & \Pi N_2^{\mathbf{N}}/U \arrow[ur, "\Pi f_2^{\mathbf{N}}"'] &
\end{tikzcd}$$

First, we use Łoś' Theorem for AECs to get the following maps:

$$\begin{aligned} h &: M \to \Pi N^{\mathbf{N}}/U \\ h_\ell &: M_\ell \to \Pi N_\ell^{\mathbf{N}}/U \qquad \text{for } \ell = 1, 2 \end{aligned}$$

h is defined by taking $m \in M$ to the equivalence class of constant function $\mathbf{N} \mapsto x$; this constant function is not always defined, but the fineness-like condition guarantees that it is defined on a U-large set (and h_1, h_2 are defined similarly). The uniform definition of these maps imply that $h_1 \restriction M = h \restriction M = h_2 \restriction M$.

Second, we can average the $f_\ell^{\mathbf{N}}$ maps to get ultraproduct maps

$$\Pi f_\ell^{\mathbf{N}} : \Pi N_\ell^{\mathbf{N}}/U \to \Pi N_*^{\mathbf{N}}/U$$

These maps agree on $\Pi N^{\mathbf{N}}/U$ since each of the individual functions do. As each M_ℓ embeds in $\Pi N_\ell^{\mathbf{N}}/U$ the composition of the f and h maps gives the amalgam.

There is no difficulty if one of M_0 or M_1 has cardinality $< \kappa$; many of the approximating triples will have the same first or second coordinates but this causes no harm. Similary, we get the JEP transfer if $M_0 = \emptyset$. And we can transfer disjoint amalgamation since in that case each $N_1^{\mathbf{N}} \cap N_2^{\mathbf{N}} = N^{\mathbf{N}}$ and this is preserved by the ultraproduct. $\dagger_{4.2.2}$

4.3 Syntactic Approaches

The two methods discussed in this section both depend on expanding the models of $\mathbf{K}$ to models in a larger vocabulary. We begin with a concept introduced in Vasey [Vasa, Definition 3.1].

Definition 4.3.1. A *functorial expansion* of an AEC $\mathbf{K}$ in a vocabulary τ is an AEC $\hat{\mathbf{K}}$ in a vocabulary $\hat{\tau}$ extending τ such that

1. each $M \in \mathbf{K}$ has a *unique* expansion to a $\hat{M} \in \hat{\mathbf{K}}$,

2. if $f : M \cong M'$ then $f : \hat{M} \cong \hat{M}'$, and
3. if M is a strong substructure of M' for $\mathbf{K}$, then $\hat{M}$ is strong substructure of $\hat{M}'$ for $\hat{\mathbf{K}}$.

This concept unifies a number of previous expansions: Morley's adding a predicate for each first order definable set, Chang adding a predicate for each $\mathbb{L}_{\omega_1,\omega}$ definable set, T^{eq}, [CHL85] adding predicates $R_n(\mathbf{x}, y)$ for closure (in an ambient geometry) of $\mathbf{x}$, and the expansion by naming the orbits in Fraîssè model[4].

An important point in both [Vasa] and our relational presentation is that the process does not just reduce the complexity of already definable sets (as Morley, Chang) but adds new definable sets. But the crucial distinction here is that the expansion in Shelah's presentation theorem is not 'functorial' in the sense here: each model has several expansions, rather than a single expansion. That is why there is an extended proof for amalgamation transfer in Section 4.3.1, while the transfer in Section 4.3.2 follows from the following result which is easily proved by chasing arrows.

Proposition 4.3.2. *Let* $\mathbf{K}$ *to* $\hat{\mathbf{K}}$ *be a functorial expansion.* $(K, \prec)$ *has* λ*-amalgamation [joint embedding, etc.] iff* $\hat{\mathbf{K}}$ *has* λ*-amalgamation [joint embedding, etc.].*

4.3.1 Shelah's Presentation Theorem

In this section, we provide syntactic characterizations of the various amalgamation properties in a finitary language. Our first approach to these results stemmed from the realization that the amalgamation property has the same syntactic characterization for $\mathbb{L}_{\kappa,\kappa}$ as for first order logic if κ is strongly compact, i.e., the compactness theorem hold for $\mathbb{L}_{\kappa,\kappa}$. Combined with Boney's recognition that one could code each AEC with Löwenheim-Skolem number less than κ in $\mathbb{L}_{\kappa,\kappa}$ this seemed a path to showing amalgamation. Unfortunately, this path leads through the trichotomy in Fact 4.3.1. The results depend directly (or with minor variations) on Shelah's Presentation Theorem and illustrate its advantages (finitary language) and disadvantage (lack of canonicity).

Fact 4.3.1 (Shelah's presentation theorem). If $\mathbf{K}$ is an AEC (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbf{K})$) with Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$, there is a vocabulary $\tau_1 \supseteq \tau$ with cardinality $|\mathrm{LS}(\mathbf{K})|$, a first order τ_1-theory T_1 and a set Γ of at most $2^{\mathrm{LS}(\mathbf{K})}$ partial types such that

1. $\mathbf{K} = \{M' \restriction \tau : M' \models T_1 \text{ and } M' \text{ omits } \Gamma\}$;
2. if M' is a τ_1-substructure of N' where M', N' satisfy T_1 and omit Γ then $M' \restriction \tau \prec_{\mathbf{K}} N' \restriction \tau$; and

[4] This has been done for years but there is a slight wrinkle in e.g. [BKL15] where the orbits are not first order definable.

3. if $M \prec N \in \mathbf{K}$ and $M' \in EC(T_1, \Gamma)$ such that $M' \restriction \tau = M$, then there is $N' \in EC(T_1, \Gamma)$ such that $M' \subset N'$ and $N' \restriction \tau = N$.

The exact assertion for part 3 is new in this paper; we don't include the slight modification in the standard proofs (e.g. [Bal09, Theorem 4.15]). Note that we have a weakening of Definition 4.3.1 caused by the possibility of multiple 'good' expansion of a model M.

Here are the syntactic conditions equivalent to DAP and DJEP.

Definition 4.3.3. • Ψ has $< \lambda$-*DAP satisfiability* iff for any expansion by constants $\mathbf{c}$ and all sets of atomic and negated atomic formulas (in $\tau(\Psi) \cup \{\mathbf{c}\}$) $\delta_1(\mathbf{x}, \mathbf{c})$ and $\delta_2(\mathbf{y}, \mathbf{c})$ of size $< \lambda$, if $\Psi \wedge \exists \mathbf{x} \left(\bigwedge \delta_1(\mathbf{x}, \mathbf{c}) \wedge \bigwedge x_i \neq c_j\right)$ and $\Psi \wedge \exists \mathbf{y} \left(\bigwedge \delta_2(\mathbf{y}, \mathbf{c}) \wedge \bigwedge y_i \neq c_j\right)$ are separately satisfiable, then so is

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}, \mathbf{c}) \wedge \bigwedge \delta_2(\mathbf{y}, \mathbf{c}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

• Ψ has $< \lambda$-*DJEP satisfiability* iff for all sets of atomic and negated atomic formulas (in $\tau(\Psi)$) $\delta_1(\mathbf{x})$ and $\delta_2(\mathbf{y})$ of size $< \lambda$, if $\Psi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Psi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ are separately satisfiable, then so is

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

We now outline the argument for $DJEP$; the others are similar. Note that $(2) \to (1)$ for the analogous result with DAP replacing DJEP has been shown by Hyttinen and Kesälä [HK06, 2.16].

Lemma 4.3.4. *Suppose that $\mathbf{K}$ is an AEC, $\lambda > LS(\mathbf{K})$, and T_1 and Γ are from Shelah's Presentation Theorem. Let Φ be the $\mathbb{L}_{LS(\mathbf{K})^+,\omega}$ theory that asserts the satisfaction of T_1 and omission of each type in Γ. Then the following are equivalent:*

1. *$\mathbf{K}_{<\lambda}$ has $DJEP$.*
2. *$(EC(T_1, \Gamma), \subset)_{<\lambda}$ has DJEP.*
3. *Φ has $< \lambda$-$DJEP$-satisfiability.*

Proof:

$(1) \leftrightarrow (2)$: First suppose that $\mathbf{K}_{<\lambda}$ has DJEP. Let $M_0^*, M_1^* \in EC(T_1, \Gamma)_{<\lambda}$ and set $M_\ell := M_\ell^* \restriction \tau$. By disjoint embedding for $\ell = 0, 1$, there is $N \in \mathbf{K}$ such that each $M_\ell \prec N$. Our goal is to expand N to be a member of $EC(T_1, \Gamma)$ in a way that respects the already existing expansions.

Recall from the proof of Fact 4.3.1 that expansions of $M \in \mathbf{K}$ to models $M^* \in EC(T_1, \Gamma)$ exactly come from writing M as a directed union of $LS(\mathbf{K})$-sized models indexed by $P_\omega|M|$, and then enumerating the models in the union. Thus, the expansion of M_ℓ to M_ℓ^* come from $\{M_{\ell,\mathbf{a}} \in \mathbf{K}_{LS(\mathbf{K})} \mid \mathbf{a} \in M_\ell\}$, where $|M_{\ell,\mathbf{a}}| = \{\left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a}) \mid i < LS(\mathbf{K})\}$ and the functions F^i_n are from the expansion. Because M_1 and M_2 are disjoint strong submodels of N, we can write N as a directed union of $\{N_\mathbf{a} \in \mathbf{K}_{LS(\mathbf{K})} \mid \mathbf{a} \in N\}$ such that $\mathbf{a} \in M_\ell$ implies that $M_{\ell,\mathbf{a}} = N_\mathbf{a}$. Now, any enumeration of the universes of these models of order type $LS(\mathbf{K})$ will give rise to an expansion of N to $N^* \in EC(T_1, \Gamma)$ by setting $\left(F^i_{|\mathbf{a}|}\right)^{N^*}(\mathbf{a})$ to be the ith element of $|N_\mathbf{a}|$.

Thus, choose an enumeration of them that agrees with the original enumerations from M_ℓ^*; that is, if $\mathbf{a} \in M_\ell$, then the ith element of $|N_\mathbf{a}| = |M_{\ell,\mathbf{a}}|$ is $\left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a})$ (note that, as used before, the disjointness guarantees that there is at most one ℓ satisfying this). In other words, our expansion N^* will have

$$\mathbf{a} \in M_\ell \implies \left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a}) = \left(F^i_{|\mathbf{a}|}\right)^{N^*}(\mathbf{a}) \text{ for all } i < LS(\mathbf{K})$$

This precisely means that $M_\ell^* \subset N^*$, as desired. Furthermore, we have constructed the expansion so $N^* \in EC(T_1, \Gamma)$. Thus, $(EC(T_1, \Gamma), \subset)_{<\lambda}$ has DJEP.

Second, suppose that $EC(T_1, \Gamma)$ has λ-DJEP. Let $M_0, M_1 \in \mathbf{K}$; WLOG, $M_0 \cap M_1 = \emptyset$. Using Shelah's Presentation Theorem, we can expand to $M_0^*, M_1^* \in EC(T_1, \Gamma)$. Then we can use disjoint embedding to find $N^* \in EC(T_1, \Gamma)$ such that $M_1^*, M_2^* \subset N^*$. By Shelah's Presentation Theorem 4.3.1.(1), $N := N^* \restriction \tau$ is the desired model.

(2) $\leftrightarrow$ (3): First, suppose that Φ has $<\lambda$-DJEP satisfiability. Let $M_0^*, M_1^* \in EC(T_1, \Gamma)$ be of size $<\lambda$. Let $\delta_0(\mathbf{x})$ be the quantifier-free diagram of $M_0^*$and $\delta_1(\mathbf{y})$be the quantifier-free diagram of M_1^*. Then $M_0^* \vDash \Phi \wedge \exists \mathbf{x} \bigwedge \delta_0(\mathbf{x})$; similarly, $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_1(\mathbf{y})$ is satisfiable. By the satisfiability property, there is N^* such that

$$N^* \vDash \Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_0(\mathbf{x}) \wedge \bigwedge \delta_1(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Then $N^* \in EC(T_1, \Gamma)$ and contains disjoint copies of M_0^* and M_1^*, represented by the witnesses of $\mathbf{x}$ and $\mathbf{y}$, respectively.

Second, suppose that $(EC(T_1, \Gamma), \subset)_{<\lambda}$ has DJEP. Let $\Phi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ be as in the hypothesis of $<\lambda$-DJEP satisfiability. Let M_0^* witness the satisfiability of the first and M_1^* witness the satisfiability of the second; note both of these are in $EC(T_1, \Gamma)$. By DJEP, there is

$N \in EC(T_1, \Gamma)$ that contains both as substructures. This witnesses

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Note that the formulas in δ_1 and δ_2 transfer up because they are atomic or negated atomic.

†

The following is a simple use of the syntactic characterization of strongly compact cardinals.

Lemma 4.3.5. *Assume κ is strongly compact and let $\Psi \in \mathbb{L}_{\kappa,\omega}(\tau_1)$ and $\lambda > \kappa$. If Ψ has $< \kappa$-DJEP-satisfiability, then Ψ has $< \lambda$-DJEP-satisfiability.*

Proof: $< \lambda$-DJEP satisfiability hinges on the consistency of a particular $\mathbb{L}_{\kappa,\omega}$ theory. If Ψ has $< \kappa$-DJEP-satisfiability, then every $< \kappa$ sized subtheory is consistent, which implies the entire theory is by the syntactic version of strong compactness we introduced at the beginning of this section.

†

Obviously the converse (for $\Psi \in \mathbb{L}_{\infty,\omega}$) holds without any large cardinals.

Proof of Theorem 4.1.1 for *DAP* and *DJEP*: We first complete the proof for DJEP. By Lemma 4.3.4, $< \kappa$-DJEP implies that Φ has $< \kappa$-DJEP satisfiability. By Lemma 4.3.5, Φ has $< \lambda$-DJEP satisfiability for every $\lambda \geq \kappa$. Thus, by Lemma 4.3.4 again, $\mathbf{K}$ has DJEP. The proof for DAP is exactly analogous. †

4.3.2 The relational presentation theorem

We modify Shelah's Presentation Theorem by eliminating the two instances where an arbitrary choice must be made: the choice of models in the cover and the choice of an enumeration of each covering model. Thus the new expansion is functorial (Definition 4.3.1). However, there is a price to pay for this canonicity. In order to remove the choices, we must add predicates of arity $LS(K)$ and the relevant theory must allow $LS(K)$-ary quantification, potentially putting it in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, where $\kappa = LS(K)$; contrast this with a theory of size $\leq 2^\kappa$ in $\mathbb{L}_{\kappa^+,\omega}$ for Shelah's version. As a possible silver lining, these arities can actually be brought down to $\mathbb{L}_{(I(\mathbf{K},\kappa)+\kappa)^+,\kappa^+}$. Thus, properties of the AEC, such as the number of models in the Löwenheim-Skolem cardinal are reflected in the presentation, while this has no effect on the Shelah version.

We fix some notation. Let $\mathbf{K}$ be an AEC in a vocabulary τ and let $\kappa = LS(\mathbf{K})$. We assume that $\mathbf{K}$ contains no models of size $< \kappa$. The same arguments could be done with $\kappa > LS(\mathbf{K})$, but this case reduces to applying our result to $\mathbf{K}_{\geq \kappa}$.

We fix a collection of compatible enumerations for models $M \in K_\kappa$. *Compatible enumerations* means that each M has an enumeration of its universe, denoted $\mathbf{m}^M = \langle m_i^M : i < \kappa \rangle$, and, if $M \cong M'$, there is some fixed isomorphism $f_{M,M'} : M \cong M'$ such that $f_{M,M'}(m_i^M) = m_i^{M'}$ and if $M \cong M' \cong M''$, then $f_{M,M''} = f_{M',M''} \circ f_{M,M'}$.

For each isomorphism type $[M]_\cong$ and $[M \prec N]_\cong$ with $M, N \in K_\kappa$, we add to τ

$$R_{[M]}(\mathbf{x}) \text{ and } R_{[M \prec N]}(\mathbf{x}; \mathbf{y})$$

as κ-ary and $\kappa 2$-ary predicates to form τ^*.

A skeptical reader might protest that we have made many arbitrary choices so soon after singing the praises of our choiceless method. The difference is that all choices are made prior to defining the presentation theory, T^*.

Once T^* is defined, no other choices are made.

The goal of the theory T^* is to recognize every strong submodel of size κ and every strong submodel relation between them via our predicates. This is done by expressing in the axioms below concerning sequences $\mathbf{x}$ of length at most κ the following properties connecting the canonical enumerations with structures in $\mathbf{K}$.

$$\begin{gathered} R_{[M]}(\mathbf{x}) \text{ holds iff } x_i \mapsto m_i^M \text{ is an isomorphism} \\ R_{[M \prec N]}(\mathbf{x}, \mathbf{y}) \text{ holds iff } x_i \mapsto m_i^M \text{ and } y_i \mapsto m_i^N \text{ are isomorphisms and} \\ x_i = y_j \text{ iff } m_i^M = m_j^N \end{gathered}$$

Note that, by the coherence of the isomorphisms, the choice of representative from $[M]_\cong$ doesn't matter. Also, we might have $M \cong M'$; $N \cong N'$; $M \prec N$ and $M' \prec N'$; but not $(M, N) \cong (M', N')$. In this case $R_{[M \prec N]}$ and $R_{[M' \prec N']}$ are different predicates.

We now write the axioms for T^*. *A priori* they are in the logic $\mathbb{L}_{(2^\kappa)^+,\kappa^+}(\tau^*)$ but the theorem states a slightly finer result. To aid in understanding, we include a description prior to the formal statement of each property.

Definition 4.3.6. The theory T^* in $\mathbb{L}_{(I(\mathbf{K},\kappa)+\kappa)^+,\kappa^+}(\tau^*)$ is the collection of the following schema:

1. *If $R_{[M]}(\mathbf{x})$ holds, then $x_i \mapsto m_i^M$ should be an isomorphism.*
   If $\phi(z_1, \ldots, z_n)$ is an atomic or negated atomic τ-formula that holds of $m_{i_1}^M, \ldots, m_{i_n}^M$, then include

$$\forall \mathbf{x} \left(R_{[M]}(\mathbf{x}) \to \phi(x_{i_1}, \ldots, x_{i_n}) \right)$$

2. *If $R_{[M \prec N]}(\mathbf{x}, \mathbf{y})$ holds, then $x_i \mapsto m_i^M$ and $y_i \mapsto m_i^N$ should be isomorphisms and the correct overlap should occur.*
   If $M \prec N$ and $i \mapsto j_i$ is the function such that $m_i^M = m_{j_i}^N$, then include

$$\forall \mathbf{x}, \mathbf{y} \left(R_{[M \prec N]}(\mathbf{x}, \mathbf{y}) \to \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i} \right) \right)$$

3. *Every κ-tuple is covered by a model.*
Include the following where $\lg(\mathbf{x}) = \lg(\mathbf{y}) = \kappa$

$$\forall \mathbf{x} \exists \mathbf{y} \left(\bigvee_{[M]_{\cong} \in K_\kappa / \cong} R_{[M]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} \bigvee_{j<\kappa} x_i = y_{j_i} \right)$$

4. *If $R_{[N]}(\mathbf{x})$ holds and $M \prec N$, then $R_{[M \prec N]}(\mathbf{x}^\circ, \mathbf{x})$ should hold for the appropriate subtuple $\mathbf{x}^\circ$ of $\mathbf{x}$.*
If $M \prec N$ and $\pi : \kappa \to \kappa$ is the unique map so $m_i^M = m_{\pi(i)}^N$, then denote $\mathbf{x}^\pi$ to be the subtuple of $\mathbf{x}$ such that $x_i^\pi = x_{\pi(i)}$ and include

$$\forall \mathbf{x} \left(R_{[N]}(\mathbf{x}) \to R_{[M \prec N]}(\mathbf{x}^\pi, \mathbf{x}) \right)$$

5. *Coherence: If $M \subset N$ are both strong substructures of the whole model, then $M \prec N$.*
If $M \prec N$ and $m_i^M = m_{j_i}^N$, then include

$$\forall \mathbf{x}, \mathbf{y} \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i} \to R_{[M \prec N]}(\mathbf{x}, \mathbf{y}) \right)$$

Remark 4.3.7. We have intentionally omitted the converse to Definition 4.3.6.(1), namely

$$\forall \mathbf{x} \left(\bigwedge_{\phi(z_{i_1}, \ldots, z_{i_n}) \in tp_{qf}(M/\emptyset)} \phi(x_{i_1}, \ldots, x_{i_n}) \to R_{[M]}(\mathbf{x}) \right)$$

because it is not true. The "toy example" of a nonfinitary AEC–the $L(Q)$-theory of an equivalence relation where each equivalence class is countable–gives a counter-example.

For any $M^* \vDash T^*$, denote $M^* \restriction \tau$ by M.

Theorem 4.3.8 (Relational Presentation Theorem). *1. If $M^* \vDash T^*$ then $M^* \restriction \tau \in K$. Further, for all $M_0 \in K_\kappa$, we have $M^* \vDash R_{[M_0]}(\mathbf{m})$ implies that $\mathbf{m}$ enumerates a strong substructure of M.*

2. Every $M \in K$ has a unique expansion M^* that models T^*.

3. If $M \prec N$, then $M^* \subset N^*$.

4. If $M^* \subset N^*$ both model T^*, then $M \prec N$.

5. If $M \prec N$ and $M^* \vDash T$ such that $M^* \restriction \tau = M$, then there is $N^* \vDash T$ such that $M^* \subset N^*$ and $N^* \restriction \tau = N$.

Moreover, this is a functorial expansion in the sense of Vasey [Vasa, Definition 3.1] and $(\mathrm{Mod}\, T^*, \subset)$ is an AEC except that it allows κ-ary relations.

Note that although the vocabulary τ^* is κ-ary, the structure of objects and embeddings from $(\operatorname{Mod} T^*, \subset)$ still satisfies all of the category theoretic conditions on AECs, as developed by Lieberman and Rosicky [LR]. This is because $(\operatorname{Mod} T^*, \subset)$ is equivalent to an AEC, namely $\mathbf{K}$, via the forgetful functor.

Proof: (1): We will build a $\prec$-directed system $\{M_{\mathbf{a}} \subset M : \mathbf{a} \in {}^{<\omega}M\}$ that are members of K_κ. We don't (and can't) require in advance that $M_{\mathbf{a}} \prec M$, but this will follow from our argument.

For singletons $a \in M$, taking $\mathbf{x}$ to be $\langle a : i < \kappa \rangle$ in (4.3.6.3), implies that there is $M'_a \in K_\kappa$ and $\mathbf{m}^a \in {}^\kappa M$ with $a \in \mathbf{m}^a$ such that $M \vDash R_{[M'_a]}(\mathbf{m}^a)$. By (1), this means that $m_i^a \mapsto m_i^{M'_a}$ is an isomorphism. Set $M_a := \mathbf{m}^a$.[5]

Suppose $\mathbf{a}$ is a finite sequence in M and $M_{\mathbf{a}'}$ is defined for every $\mathbf{a}' \subsetneq \mathbf{a}$. Using the union of the universes as the $\mathbf{x}$ in (4.3.6.3), there is some $N \in K_\kappa$ and $\mathbf{m}^{\mathbf{a}} \in {}^\kappa M$ such that

- $|M_{\mathbf{a}'}| \subset \mathbf{m}^{\mathbf{a}}$ for each $\mathbf{a}' \subsetneq \mathbf{a}$.
- $M \vDash R_{[N]}(\mathbf{m}^{\mathbf{a}})$.

By (4.3.6.4), this means that $M \vDash R_{\bar{M}_{\mathbf{a}'} \prec \bar{N}}(\mathbf{m}^{\mathbf{a}'}, \mathbf{m}^{\mathbf{a}})$, after some permutation of the parameters. By (2) and (1), this means that $M_{\mathbf{a}'} \prec N$; set $M_{\mathbf{a}} := \mathbf{m}^{\mathbf{a}}$.

Now that we have finished the construction, we are done. AECs are closed under directed unions, so $\cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$. But this model has the same universe as M and is a substructure of M; thus $M = \cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$.

For the further claim, suppose $M^* \vDash R_{[M_0]}(\mathbf{m})$. We can redo the same proof as above with the following change: whenever $\mathbf{a} \in M$ is a finite sequence such that $\mathbf{a} \subset \mathbf{m}$, then set $\mathbf{m}^{\mathbf{a}} = \mathbf{m}$ directly, rather than appealing to (4.3.6.3) abstractly. Note that $\mathbf{m}$ witnesses the existential in that axiom, so the rest of the construction can proceed without change. At the end, we have

$$\mathbf{m} = M_{\mathbf{a}} \prec \bigcup_{\mathbf{a}' \in {}^{<\omega}M} M_{\mathbf{a}'} = M$$

(2): First, it's clear that $M \in K$ has an expansion; for each $M_0 \prec M$ of size κ, make $R_{[M_0]}(\langle m_i^{M_0} : i < \kappa \rangle)$ hold and, for each $M_0 \prec N_0 \prec M$ of size κ, make $R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$ hold. Now we want to show this expansion is the unique one.

Suppose $M^+ \vDash T^*$ is an expansion of M. We want to show this is in fact the expansion described in the above paragraph. Let $M_0 \prec M$. By (4.3.6.3) and (1) of this theorem, there is $N_0 \prec M$ and $\mathbf{n} \in {}^\kappa M$ such that

[5] We mean that we set M_a to be τ-structure with universe the range of $\mathbf{m}^a$ and functions and relations inherited from M'_a via the map above.

- $M^+ \vDash R_{[N_0]}(\mathbf{n})$
- $|M_0| \subset \mathbf{n}$

By coherence, $M_0 \prec \mathbf{n}$. Since $n_i \mapsto m_i^{N_0}$ is an isomorphism, there is $M_0^* \cong M_0$ such that $M_0^* \prec N_0$. Note that $T^* \models \forall \mathbf{x} R_{[M_0^*]}(\mathbf{x}) \leftrightarrow R_{[M_0]}(\mathbf{x})$. By (4.3.6.4),

$$M^+ \vDash R_{[M_0^* \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \mathbf{n})$$

By (4.3.6.2), $M^+ \vDash R_{[M_0^*]}(\langle m_i^{M_0} : i < \kappa \rangle)$, which gives us the conclusion by the further part of (1) of this theorem.

Similarly, if $M_0 \prec N_0 \prec M$, it follows that

$$M^+ \vDash R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$$

Thus, this arbitrary expansion is actually the intended one.

(3): Apply the uniqueness of the expansion and the transitivity of $\prec$.

(4): As in the proof of (1), we can build $\prec$-directed systems $\{M_{\mathbf{a}} : \mathbf{a} \in {}^{<\omega}M\}$ and $\{N_{\mathbf{b}} : \mathbf{b} \in {}^{<\omega}N\}$ of submodels of M and N, so that $M_{\mathbf{a}} = N_{\mathbf{a}}$ when $\mathbf{a} \in {}^{<\omega}M$. From the union axioms of AECs, we see that $M \prec N$.

(5): This follows from (3), (4) of this theorem and the uniqueness of the expansion.

Recall that the map $M^* \in \operatorname{Mod} T^*$ to $M^* \restriction \tau \in \mathbf{K}$ is a an abstract Morleyization if it is a bijection such that every isomorphism $f : M \cong N$ in $\mathbf{K}$ lifts to $f : M^* \cong N^*$ and $M \prec N$ implies $M^* \subset N^*$. We have shown that this is true of our expansion. †

Remark 4.3.9. *The use of infinitary quantification might remind the reader of the work on the interaction between AECs and $\mathbb{L}_{\infty,\kappa^+}$ by Shelah [She09, Chapter IV] and Kueker [Kue08] (see also Boney and Vasey [BV] for more in this area). The main difference is that, in working with $\mathbb{L}_{\infty,\kappa^+}$, those authors make use of the semantic properties of equivalence (back and forth systems and games). In contrast, particularly in the following transfer result we look at the syntax of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$.*

The functoriality of this presentation theorem allows us to give a syntactic proof of the amalgamation, etc. transfer results without assuming disjointness (although the results about disjointness follow similarly). We focus on amalgamation and give the details only in this case, but indicate how things are changed for other properties.

Proposition 4.3.2 applied to this context yields the following result.

Proposition 4.3.10. $(K, \prec)$ *has λ-amalgamation [joint embedding, etc.] iff* $(\operatorname{Mod} T^*, \subset)$ *has λ-amalgamation [joint embedding, etc.].*

Now we show the transfer of amalgamation between different cardinalities using the technology of this section.

Notation 4.3.11. *Fix an AEC* $\mathbf{K}$ *and the language* τ^* *from Theorem 4.3.8.*

1. *Given* τ^**-structures* $M_0^* \subset M_1^*, M_2^*$*, we define the* amalgamation diagram $AD(M_1^*, M_2^*/M_0^*)$ *to be*

$$\{\phi(\mathbf{c}_{\mathbf{m}_0}, \mathbf{c}_{\mathbf{m}_1})) : \phi \textit{ is quantifier-free from } \tau^* \textit{ and for } \ell = 0 \textit{ or } 1, \\ M_\ell^* \vDash \phi(\mathbf{c}_{\mathbf{m}_0}, \mathbf{c}_{\mathbf{m}_1}), \textit{ with } \mathbf{m}_0 \in M_0^* \textit{ and } \mathbf{m}_1 \in M_\ell^*\}$$

in the vocabulary $\tau^* \cup \{c_m : m \in M_1^* \cup M_2^*\}$ *where each constant is distinct except for the common submodel* M_0 *and* $\mathbf{c}_\mathbf{m}$ *denotes the finite sequence of constants* $c_{m_1}, \ldots, c_{m_n}$*.*
The disjoint amalgamation diagram $DAD(M_1^*, M_2^*/M_0^*)$ *is*

$$AD(M_1^*, M_2^*/M_0^*) \cup \{c_{m_1} \neq c_{m_2} : m_\ell \in M_\ell^* - M_0^*\}$$

2. *Given* τ^**-structures* M_0^*, M_1^**, we define the* joint embedding diagram $JD(M_0^*, M_1^*)$ *to be*

$$\{\phi(\mathbf{c}_\mathbf{m}) \quad : \quad \phi \textit{ is quantifier-free from } \tau^* \textit{ and} \\ \textit{for } \ell = 0 \textit{ or } 1, M_\ell^* \vDash \phi(\mathbf{c}_\mathbf{m}) \textit{ with } \mathbf{m} \in M_\ell^*\}$$

in the vocabulary $\tau^* \cup \{c_m : m \in M_1^* \cup M_2^*\}$ *where each constant is distinct.*
The disjoint amalgamation diagram $DJD(M_0^*, M_1^*)$ *is*

$$AD(M_1^*, M_2^*/M_0^*) \cup \{c_{m_1} \neq c_{m_2} : m_\ell \in M_\ell^* - M_0^*\}$$

The use of this notation is obvious.

Claim 4.3.1.1. *Any amalgam of* M_1 *and* M_2 *over* M_0 *is a reduct of a model of*

$$T^* \cup AD(M_1^*, M_2^*/M_0^*)$$

Proof: An amalgam of $M_0 \prec M_1, M_2$ is canonically expandable to an amalgam of $M_0^* \subset M_1^*, M_2^*$, which is precisely a model of $T^* \cup AD(M_1^*, M_2^*/M_0^*)$. Conversely, a model of that theory will reduct to a member of $\mathbf{K}$ with embeddings of M_1 and M_2 that fix M_0. †

There are similar claims for other properties. Thus, we have connected amalgamation in $\mathbf{K}$ to amalgamation in $(\mathrm{Mod}\, T^*, \subset)$ to a syntactic condition, similar to Lemma 4.3.4. Now we can use the compactness of logics in various large cardinals to transfer amalgamation between cardinals. To do this, recall the notion of an amalgamation base.

Definition 4.3.12. For a class of cardinals $\mathcal{F}$, we say $M \in K_\mathcal{F}$ is a $\mathcal{F}$-*amalgamation base* ($\mathcal{F}$-a.b.) if any pair of models from $K_\mathcal{F}$ extending M can be amalgamated over M. We use the same rewriting conventions as in Definition 4.1.3.(1), e. g., writing $\leq \lambda$-a.b. for $[LS(K), \lambda]$-amalgamation base.

We need to specify two more large cardinal properties.

Definition 4.3.13. 1. A cardinal κ is weakly compact if it is strongly inaccessible and every set of κ sentence in $\mathbb{L}_{\kappa,\kappa}$ that is $<\kappa$-satisfiable is satisfiable is satisfiable[6].

2. A cardinal κ is measurable if there exists a κ-additive, non-trivial, $\{0,1\}$-valued measure on the power set of κ.

3. κ is (δ,λ)-strongly compact for $\delta \leq \kappa \leq \lambda$ if there is a δ-complete, fine ultrafilter on $P_\kappa(\lambda)$.
   κ is λ-strongly compact if it is (κ,λ)-strongly compact.

This gives us the following results syntactically.

Proposition 4.3.14. *Suppose $LS(K) < \kappa$.*

- *Let κ be weakly compact and $M \in K_\kappa$. If M can be written as an increasing union $\cup_{i<\kappa} M_i$ with each $M_i \in K_{<\kappa}$ being a $<\kappa$-a.b., then M is a κ-a.b.*
- *Let κ be measurable and $M \in K$. If M can be written as an increasing union $\cup_{i<\kappa} M_i$ with each M_i being a λ_i-a.b., then M is a $(\sup_{i<\kappa}\lambda_i)$-a.b.*
- *Let κ be λ-strongly compact and $M \in K$. If M can be written as a directed union $\cup_{x\in P_\kappa\lambda} M_x$ with each M_x being a $<\kappa$-a.b., then M is a $\leq\lambda$-a.b.*

Proof: The proof of the different parts are essentially the same: take a valid amalgamation problem over M and formulate it syntactically via Claim 4.3.1.1 in $\mathbb{L}_{\kappa,\kappa}(\tau^*)$. Then use the appropriate syntactic compactness for the large cardinal to conclude the satisfiability of the appropriate theory.

First, suppose κ is weakly compact and $M = \cup_{i<\kappa} M_i \in K_\kappa$ where $M_i \in K_{<\kappa}$ is a $<\kappa$-a.b. Let $M \prec M^1, M^2$ is an amalgamation problem from K_κ. Find resolutions $\langle M_i^\ell \in K_{<\kappa} : i < \kappa\rangle$ with $M_i \prec M_i^\ell$ for $\ell = 1, 2$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{i<\kappa} \left(T^* \cup AD(M_i^{1*}, M_i^{2*}/M_i^*)\right)$$

and is of size κ. Each member of the union is satisfiable (by Claim 4.3.1.1 because M_i is a $<\kappa$-a.b.) and of size $<\kappa$, so $T^* \cup AD(M^{1*}, M^{2*}/M^*)$ is satisfiable. Since $M^1, M^2 \in K_\kappa$ were arbitrary, M is a κ-a.b.

Second, suppose that κ is measurable and $M = \cup_{i<\kappa} M_i$ where M_i is a λ_i-a.b. Set $\lambda = \sup_{i<\kappa}\lambda_i$ and let $M \prec M^1, M^2$ is an amalgamation problem from K_λ. Find resolutions $\langle M_i^\ell \in K : i < \kappa\rangle$ with $M_i \prec M_i^\ell$ for $\ell = 1, 2$ and $\|M_i^\ell\| = \lambda_i$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{i<\kappa} \left(T^* \cup AD(M_i^{1*}, M_i^{2*}/M_i^*)\right)$$

[6] At one time strong inaccessiblity was not required, but this is the current definition

Each member of the union is satisfiable because M_i is a λ_i-a.b. By the syntactic characterization of measurable cardinals (see [CK73, Exercise 4.2.6]), the union is satisfiable. Thus, M is λ-a.b.

Third, suppose that κ is λ-strongly compact and $M = \cup_{x \in P_\kappa \lambda} M_x$ with each M_x being a $< \kappa$-a.b. Let $M \prec M^1, M^2$ be an amalgamation problem from K_λ. Find directed systems $\langle M_x^\ell \in K_{<\kappa} \mid x \in P_\kappa \lambda \rangle$ with $M_x \prec M_x^\ell$ for $\ell = 1, 2$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{x \in P_\kappa \lambda} \left(T^* \cup AD(M_x^{1*}, M_x^{2*}/M_x^*) \right)$$

Every subset of the left side of size $< \kappa$ is contained in a member of the right side because $P_\kappa \lambda$ is $< \kappa$-directed, and each member of the union is consistent because each M_x is an amalgamation base. Because κ is λ-strongly compact, this means that the entire theory is consistent. Thus, M is a λ-a.b. †

From this, we get the following corollaries computing upper bounds on the Hanf number for the $\leq \lambda$-AP.

Corollary 4.3.15. *Suppose $LS(K) < \kappa$.*

- *If κ is weakly compact and K has $< \kappa$-AP, then K has $\leq \kappa$-AP.*
- *If κ is measurable,* cf $\lambda = \kappa$*, and K has $< \lambda$-AP, then K has $\leq \lambda AP$.*
- *If κ is λ-strongly compact and K has $< \kappa$-AP, then K has $\leq \lambda$-AP.*

Moreover, when κ is strongly compact, we can imitate the proof of [MS90, Corollary 1.6] to show that being an amalgamation base follows from being a $< \kappa$-existentially closed model of T^*. This notion turns out to be the same as the notion of $< \kappa$-universally closed from [Bon], and so this is an alternate proof of [Bon, Lemma 7.2].

4.4 The Big Gap

This section concerns examples of 'exotic' behavior in small cardinalities as opposed to behavior that happens unboundedly often or even eventually. We discuss known work on the spectra of existence, amalgamation of various sorts, tameness, and categoricity.

Intuitively, Hanf's principle is that if a certain property can hold for only set-many objects then it is eventually false. He refines this twice. First, if $\mathcal{K}$ a *set* of collections of structures $\mathbf{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\mathbf{K}, \lambda)$ means some member of $\mathbf{K}$ with cardinality λ satisfies P then there is a cardinal κ_P such that for any $\mathbf{K} \in \mathcal{K}$, if $\phi(\mathbf{K}, \kappa')$ holds for some $\kappa' \geq \kappa_P$, then $\phi(\mathbf{K}, \lambda)$ holds for arbitrarily large λ. Secondly, he observed that if the property P is closed down for sufficiently large members of each $\mathbf{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Existence: Morley (plus the Shelah presentation theorem) gives a decisive concrete example of this principle to AEC's. Any AEC in a countable vocabulary with countable Löwenheim-Skolem number with models up to $\beth_{\omega_1}$ has arbitrarily large models. And Morley [Mor65] gave easy examples showing this bound was tight for arbitrary sentences of $\mathbb{L}_{\omega_1,\omega}$. But it was almost 40 years later that Hjorth [Hjo02, Hjo07] showed this bound is also tight for *complete*-sentences of $\mathbb{L}_{\omega_1,\omega}$. And a fine point in his result is interesting.

We say a ϕ *characterizes* κ, if there is a model of ϕ with cardinality κ but no larger. Further, ϕ *homogeneously* [Bau74] characterizes κ if ϕ is a complete sentence of $\mathbb{L}_{\omega_1,\omega}$ that characterizes κ, contains a unary predicate U such that if M is the countable model of ϕ, every permutation of $U(M)$ extends to an automorphism of M (i.e. $U(M)$ is a set of absolute indiscernibles.) and there is a model N of ϕ with $|U(N)| = \kappa$.

In [Hjo02], Hjorth found, by an inductive procedure, for each $\alpha < \omega_1$, a countable (finite for finite α) set S_α of complete $\mathbb{L}_{\omega_1,\omega}$-sentences such that some $\phi_\alpha \in S_\alpha$ characterizes $\aleph_\alpha$[7]. This procedure was nondeterministic in the sense that he showed one of (countably many if α is infinite) sentences worked at each $\aleph_\alpha$; it is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. In [BKL15], we show a modification of the Laskowski-Shelah example (see [LS93, BFKL16]) gives a family of $\mathbb{L}_{\omega_1,\omega}$-sentences ϕ_r, such that ϕ_r homogeneously characterizes $\aleph_r$ for $r < \omega$. Thus for the first time [BKL15] establishes in ZFC, the existence of specific sentences ϕ_r characterizing $\aleph_r$.

Amalgamation: In this paper, we have established a similar upper bound for a number of amalgamation-like properties. Moreover, although it is not known beforehand that the classes are eventually downward closed, that fact falls out of the proof. In all these cases, the known lower bounds (i. e., examples where AP holds initially and eventually fails) are far smaller. We state the results for countable Löwenheim-Skolem numbers, although the [BKS09, KLH14] results generalize to larger cardinalities.

The best lower bounds for the disjoint amalgamation property is $\beth_{\omega_1}$ as shown in [KLH14] and [BKS09]. In [BKS09], Baldwin, Kolesnikov, and Shelah gave examples of $\mathbb{L}_{\omega_1,\omega}$-definable classes that had disjoint embedding up to $\aleph_\alpha$ for every countable α (but did not have arbitrarily large models). Kolesnikov and Lambie-Hanson [KLH14] show that for the collection of all coloring classes (again $\mathbb{L}_{\omega_1,\omega}$-definable when α is countable) in a vocabulary of a fixed size κ, the Hanf number for amalgamation (equivalently in this example disjoint amalgamation) is precisely $\beth_{\kappa^+}$ (and many of the classes have arbitrarily large

[7] Malitz [Mal68] (under GCH) and Baumgartner [Bau74] had earlier characterized the $\beth_\alpha$ for countable α.

models). In [BKL15], Baldwin, Koerwein, and Laskowski construct, for each $r < \omega$, a *complete* $\mathbb{L}_{\omega_1,\omega}$-sentence ϕ^r that has disjoint 2-amalgamation up to and including $\aleph_{r-2}$; disjoint amalgamation and even amalgamation fail in $\aleph_{r-1}$ but hold (trivially) in $\aleph_r$; there is no model in $\aleph_{r+1}$.

The joint embedding property and the existence of maximal models are closely connected[8]. The main theorem of [BKS16] asserts: If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $\mathbb{L}_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Thus, a lower bound on the Hanf number for either maximal models of the joint embedding property is again $\beth_{\omega_1}$. Again, the result is considerably more complicated for complete sentences. But [BS15b] show that there is a sentence ϕ in a vocabulary with a predicate X such that if $M \models \phi$, $|M| \leq |X(M)|^+$ and for every κ there is a model with $|M| = \kappa^+$ and $|X(M)| = \kappa$. Further they note that if there is a sentence ϕ that homogenously characterizes κ, then there is a sentence ϕ' with a new predicate B such that ϕ' also characterizes κ, B defines a set of absolute indiscernibles in the countable model, and there are models M_λ for $\lambda \leq \kappa$ such that $(|M|, |B(M_\lambda)|) = (\kappa, \lambda)$. Combining these two with earlier results of Souldatos [Sou13] one obtains several different ways to show the lower bound on the Hanf number for a complete $\mathbb{L}_{\omega_1,\omega}$-sentence having maximal models is $\beth_{\omega_1}$. In contrast to [BKS16], all of these examples have no models beyond $\beth_{\omega_1}$.

No maximal models: Baldwin and Shelah [BS15a] have announced that the exact Hanf number for the non-existence of maximal models is the first measurable cardinal. Souldatos observed that this implies the lower bound on the Hanf number for $\mathbf{K}$ has joint embedding of models at least μ is the first measurable.

Tameness: Note that the definition of a Hanf number for tameness is more complicated as tameness is fundamentally a property of two variables: $\mathbf{K}$ is $(< \chi, \mu)$-tame if for any $N \in \mathbf{K}_\mu$, if the Galois types p and q over N are distinct, there is an $M \prec N$ with $|M| < \chi$ and $p \upharpoonright M \neq q \upharpoonright M$.

Thus, we define the *Hanf number for* $<\kappa$*-tameness* to be the minimal λ such that the following holds:

if $\mathbf{K}$ is an AEC with $LS(\mathbf{K}) < \kappa$ that is $(< \kappa, \mu)$-tame for *some* $\mu \geq \lambda$, then it is $(< \kappa, \mu)$-tame for arbitrarily large μ.

The results of [Bon] show that Hanf number for $< \kappa$-tameness is κ when κ is

[8]Note that, under joint embedding, the existence of a maximal model is equivalent to the non-existence of arbitrarily large models

strongly compact[9]. However, this is done by showing a much stronger "global tameness" result that ignores the hypothesis: *every* AEC $\mathbf{K}$ with $LS(\mathbf{K}) < \kappa$ is $(< \kappa, \mu)$-tame for all $\mu \geq \kappa$. Boney and Unger [BU], building on earlier work of Shelah [Shec], have shown that this global tameness result is actually an equivalence (in the almost strongly compact form). Also, due to monotonicity results for tameness, the Boney results show that the Hanf number for $< \lambda$-tameness is at most the first almost strongly compact above λ (if such a thing exists). The results [BU, Theorem 4.9] put a large restriction on the structure of the tameness spectrum for any ZFC Hanf number. In particular, the following

Fact 4.4.1. Let $\sigma = \sigma^\omega < \kappa \leq \lambda$. Every AEC $\mathbf{K}$ with $LS(\mathbf{K}) = \sigma$ is $\left(< \kappa, \sigma^{(\lambda^{<\kappa})}\right)$-tame iff κ is (σ^+, λ)-strongly compact.

This means that a ZFC (i. e., not a large cardinal) Hanf number for $< \kappa$-tameness would consistently have to avoid cardinals of the form $\sigma^{(\lambda^{<\kappa})}$ (under GCH, all cardinals are of this form except for singular cardinals and successors of singulars of cofinality less than κ).

One could also consider a variation of a Hanf number for $< \kappa$ that requires $(< \kappa, \mu)$-tameness *on a tail* of μ, rather than for arbitrarily large μ. The argument above shows that that is exactly the first strongly compact above κ.

Categoricity: Another significant instance of Hanf's observation is Shelah's proof in [She99a] that if $\mathcal{K}$ is taken as all AEC's $\mathbf{K}$ with $LS_{\mathbf{K}}$ bounded by a cardinal κ, then there is such an eventual Hanf number for categoricity in a successor. Boney [Bon] places an upper bound on this Hanf number as the first strongly compact above κ. This depended on the results on tameness discussed in the previous paragraphs.

Building on work of Shelah [She09, She10], Vasey [Vasb] proves that if a universal class (see [She87]) is categorical in a λ at least the Hanf number for existence, then it has amalgamation in all $\mu \geq \kappa$. The he shows that for universal class in a countable vocabulary, *that satisifies amalgamation*, the Hanf number for categoricity is at most $\beth_{\beth_{(2^\omega)^+}}$. Note that the lower bound for the Hanf number for categoricity is $\aleph_\omega$, ([HS90, BK09]).

Question 4.4.1. 1. Can one calculate in ZFC an upper bound on these Hanf numbers for 'amalgamation'? Can[10] the gaps in the upper and lower bounds of the Hanf numbers reported here be closed in ZFC? Will smaller large cardinal axioms suffice for some of the upper bounds? Does categoricity help?

2. (Vasey) Are there any techniques for downward transfer of amalgamation[11]?

[9]This can be weakened to almost strongly compact; see Brooke-Taylor and Rosický [BTR15] or Boney and Unger [BU].

[10]Grossberg initiated this general line of research.

[11]Note that there is an easy example in [BKS09] of a sentence in $\mathbb{L}_{\omega_1,\omega}$ that is categorical and has amalgamation in every uncountable cardinal but it fails both in $\aleph_0$.

3. Does every AEC have a functional expansion to a $PC\Gamma$ class. Is there a natural class of AEC's with this property — e.g. solvable groups?
4. Can[12] one define in ZFC a sequence of sentences ϕ_α for $\alpha < \omega_1$, such that ϕ_α characterizes $\aleph_\alpha$?
5. (Shelah) If $\aleph_{\omega_1} < 2^{\aleph_0}$ $\mathbb{L}_{\omega_1,\omega}$-sentence has models up to $\aleph_{\omega_1}$, must it have a model in $2^{\aleph_0}$? (He proves this statement is consistent in [She99b]).
6. (Souldatos) Is any cardinal except $\aleph_0$ characterized by a complete sentence of $\mathbb{L}_{\omega_1,\omega}$ but not homogeneously?

[12]This question seems to have originated from discussions of Baldwin, Souldatos, Laskowski, and Koerwien.

Bibliography

[Bal09] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, Providence, USA, 2009. www.math.uic.edu/˜ jbaldwin.

[Bal15a] J.T. Baldwin. The entanglement of model theory and set theory. lecture slides and audio `http://homepages.math.uic.edu/\~jbaldwin/pub/osuapr15.pdf`, 2015.

[Bal15b] J.T. Baldwin. Formalization without foundationalism; model theory and the philosophy of mathematics practice. book manuscript available on request, 2015.

[Bau74] J. Baumgartner. The Hanf number for complete $L_{\omega_1,\omega}$-sentences (without GCH). *Journal of Symbolic Logic*, 39:575–578, 1974.

[BFKL16] J.T. Baldwin, Sy Friedman, M. Koerwien, and C. Laskowski. Three red herrings around Vaught's conjecture. *Transactions of the American Math Society*, 368:22, 2016. Published electronically: November 6, 2015.

[BK09] John T. Baldwin and Alexei Kolesnikov. Categoricity, amalgamation, and tameness. *Israel Journal of Mathematics*, 170:411–443, 2009. also at `www.math.uic.edu/\~\jbaldwin`.

[BKL15] J.T. Baldwin, M. Koerwien, and C. Laskowski. Amalgamation, characterizing cardinals, and locally finite aec. *Journal of Symbolic Logic*, page 11, 2015. to appear, preprint:`http://homepages.math.uic.edu/~jbaldwin/pub/LS-2014-03-10jb.pdf`.

[BKS09] J.T. Baldwin, A. Kolesnikov, and S. Shelah. The amalgamation spectrum. *Journal of Symbolic Logic*, 74:914–928, 2009.

[BKS16] J.T. Baldwin, M. Koerwien, and I. Souldatos. The joint embedding property and maximal models. *Archive for Mathematical Logic*, 55:545–565, 2016.

[Bon] W. Boney. Tameness from large cardinal axioms. preprint, to appear: Journal of Symbolic Logic.

[Boo76] Boos. Infinitary compactness without strong inaccessibility. *The Journal of Symbolic Logic*, 41:33–38, 1976.

[Bou50] Nicholas Bourbaki. The architecture of mathematics. *The American Mathematical Monthly*, 57:221–232, 1950.

[BS15a] J.T. Baldwin and S. Shelah. Hanf numbers for maximality. preprint Shelah F1533, 2015.

[BS15b] J.T. Baldwin and I. Souldatos. Complete $L_{\omega_1,\omega}$-sentences with maximal models in multiple cardinalities. submitted, 2015.

[BTR15] Andrew Brooke-Taylor and Jiří Rosický. Accessible images revisited. Submitted, arXiv:1506.01986, 2015.

[BU] W. Boney and S. Unger. Large cardinal axioms from tameness in AECs. In preparation: arXiv:1509.01191.

[BV] W. Boney and S. Vasey. Categoricity and infinitary logics. preprint.

[CCHJ] B. Cody, S. Cox, J. Hamkins, and T. Johnston. The weakly compact embedding property. In preparation.

[CHL85] G.L. Cherlin, L. Harrington, and A.H. Lachlan. $\aleph_0$-categorical, $\aleph_0$-stable structures. *Annals of Pure and Applied Logic*, 28:103–135, 1985.

[CK73] C.C. Chang and H.J Keisler. *Model theory.* North-Holland, 1973. 3rd edition 1990.

[Cor92] Leo Corry. Nicolas bourbaki and the concept of mathematical structure. *Synthese*, 92:315–348, 1992.

[Cor04] Leo Corry. *Modern Algebra and the Rise of Mathematical Structures.* Birkhuser Verlag, 2004.

[Fra54] R. Fraïssé. Sur quelques classifications des systèmes de relations. *Publ. Sci. Univ. Algeria Sèr. A*, 1:35–182, 1954.

[Gro02] R. Grossberg. Classification theory for non-elementary classes. In Yi Zhang, editor, *Logic and Algebra*, pages 165–204. AMS, 2002. Contemporary Mathematics 302.

[GV06] R. Grossberg and M. VanDieren. Galois stability for tame abstract elementary classes. *Journal of Mathematical Logic*, 6:1–24, 2006.

[Hjo02] Greg Hjorth. Knight's model, its automorphism group, and characterizing the uncountable cardinals. *Journal of Mathematical Logic*, pages 113–144, 2002.

[Hjo07] Greg Hjorth. A note on counterexamples to Vaught's conjecture. *Notre Dame Journal of Formal Logic*, 2007.

[HK06] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Ann. Pure Appl. Logic*, 143(1-3):103–138, 2006.

[HS90] B. Hart and S. Shelah. Categoricity over P for first order T or categoricity for $\phi \in \mathrm{l}_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$. *Israel Journal of Mathematics*, 70:219–235, 1990.

[Jec06] T. Jech. *Set Theory: 3rd Edition.* Springer Monographs in Mathematics. Springer, 2006.

[Jón56] B. Jónsson. Universal relational systems. *Mathematica Scandinavica*, 4:193–208, 1956.

[Jón60] B. Jónsson. Homogeneous universal relational systems. *Mathematica Scandinavica*, 8:137–142, 1960.

[Ken13] Juliette Kennedy. On formalism freeness: Implementing gödel's 1946 princeton bicentennial lecture. *Bulletin of Symbolic Logic*, 19:351–393, 2013.

[KLH14] Alexei Kolesnikov and Christopher Lambie-Hanson. Hanf numbers for amalgamation of coloring classes. preprint, 2014.

[Kue08] D. W. Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156:274–286, 2008.

[LR] M. Lieberman and Jirí Rosický. Classification theory of accessible categories. to appear JSL.

[LS93] Michael C. Laskowski and Saharon Shelah. On the existence of atomic models. *J. Symbolic Logic*, 58:1189–1194, 1993.

[Mal68] J. Malitz. The Hanf number for complete $L_{\omega_1,\omega}$ sentences. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, LNM 72, pages 166–181. Springer-Verlag, 1968.

[Mor65] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.

[MS90] Michael Makkai and Saharon Shelah. Categoricity of theories in $l_{\kappa,\omega}$, with κ a compact cardinal. *Annals of Pure and Applied Logic*, 47:41–97, 1990.

[She] S. Shelah. Maximal failures of sequence locality in a.e.c. preprint on archive.

[She87] Saharon Shelah. Universal classes, part I. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 264–419. Springer, Berlin, 1987. paper 300:Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.

[She99a] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394. Consult Shelah for post-publication revisions.

[She99b] Saharon Shelah. Borel sets with large squares. *Fundamenta Mathematica*, 159:1–50, 1999.

[She09] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications `www.collegepublications.co.uk`, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

[She10] S. Shelah. *Classification Theory for Abstract Elementary Classes: II*. Studies in Logic. College Publications <www.collegepublications.co.uk>, 2010. Binds together papers 300 A-G, E46, 838.

[Sou13] Ioannis Souldatos. Characterizing the powerset by a complete (Scott) sentence. *Fundamenta Mathematica*, 222:131–154, 2013.

[Vasa] Sebastien Vasey. Infinitary stability theory. xarchive: http://arxiv.org/abs/1412.3313.

[Vasb] Sebastien Vasey. Shelah's eventual categoricity conjecture in universal classes. preprint.

theorem]Theorem

Chapter 5

A survey on tame abstract elementary classes

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Abstract

Tame abstract elementary classes are a broad nonelementary framework for model theory that encompasses several examples of interest. In recent years, progress toward developing a classification theory for them has been made. Abstract independence relations such as Shelah's good frames have been found to be key objects. Several new categoricity transfers have been obtained. We survey these developments using the following result (due to the second author) as our guiding thread:

[. *If a universal class is categorical in cardinals of arbitrarily high cofinality, then it is categorical on a tail of cardinals.*

5.1 Introduction

Abstract elementary classes (AECs) are a general framework for nonelementary model theory. They encompass many examples of interest while still allowing some classification theory, as exemplified by Shelah's recent two-volume book [She09b, She09c] titled *Classification Theory for Abstract Elementary Classes*.

So why study the classification theory of *tame* AECs in particular? Before going into a technical discussion, let us simply say that several key results can be obtained assuming tameness that provably cannot be obtained (or are very hard to prove) without assuming it. Among these results are the construction, assuming a stability or superstability hypothesis, of certain global independence notions akin to first-order forking. Similarly to the first-order case, the existence of such notions allows us to prove several more structural properties of the class (such as a bound on the number of types or the fact that the union of a chain of saturated models is saturated). After enough of the theory is developed, categoricity transfers (in the spirit of Morley's celebrated categoricity theorem [Mor65]) can be established.

A survey of such results (with an emphasis on forking-like independence) is in Section 5.5. However, we did not want to overwhelm the reader with a long list of sometimes technical theorems, so we thought we would first present an application: the categoricity transfer for universal classes from the abstract (Section 5.4). We chose this result for several reasons. First, its statement is simple and does not mention tameness or even abstract elementary classes. Second, the proof revolves around several notions (such as good frames) that might seem overly technical and ill-motivated if one does not see them in action first. Third, the result improves on earlier categoricity transfers in several ways, for example not requiring that the categoricity cardinal be a successor and not assuming the existence of large cardinals. Finally, the method of proof leads to Theorem 5.5.47, the currently best known ZFC approximation to Shelah's eventual categoricity conjecture (which is the main test question for AECs, see below).

Let us go back to what tameness is. Briefly, tameness is a property of AECs saying that Galois (or orbital) types are determined locally: two distinct Galois types must already be distinct when restricted to some small piece of their domain. This holds in elementary classes: types as sets of formulas can be characterized in terms of automorphisms of the monster model and distinct types can be distinguished by a finite set of parameters. However, Galois types in general AECs are not syntactic and their behavior can be wild, with "new" types springing into being at various cardinalities and increasing chains of Galois types having no unique upper bound (or even no upper bound at all). This wild behavior makes it very hard to transfer results between cardinalities.

For a concrete instance, consider the problem of developing a forking-like independence notion for AECs. In particular, we want to be able to extend each (Galois) type p over M to a unique nonforking[1] extension over every larger set N. If the AEC, $\mathbf{K}$, is nice enough, one might be able to develop an independence notion allows one to obtain a nonforking extension q of p over N. But suppose that $\mathbf{K}$ is not tame and that this non-tameness is witnessed by q. Then there is another type q' over N that has all the same small restrictions as q. In particular it extends p and (assuming our independence notion has a reasonable continuity property) is a nonforking extension. In this case the quest to have a unique nonforking extension is (provably, see Example 5.3.2.1.(4)) doomed.

This failure has led, in part, to Shelah's work on a local approach where the goal is to build a structure theory cardinal by cardinal without any "traces of compactness" (see [She01a, p. 5]). The central concept there is that of a good λ-frame (the idea is, roughly, that if an AEC $\mathbf{K}$ has a good λ-frame, then $\mathbf{K}$ is "well-behaved in λ"). Multiple instances of categoricity together with non-ZFC hypotheses (such as the weak generalized continuum hypothesis: $2^\mu < 2^{\mu^+}$ for all μ) are used to build a good λ-frame [She01a], to push it up to models of size λ^+ (changing the class in the process) [She09b, Chapter II], and finally to push it to models of sizes $\lambda^{+\omega}$ and beyond in [She09b, Chapter III] (see Section 5.2.5).

In contrast, the amount of compactness offered by tameness and other locality properties has been used to prove similar results in simpler ways and with fewer assumptions (after tameness is accounted for). In particular, the work can often be done in ZFC.

In tame AECs, Galois types are determined by their small restrictions and the behavior of these small restrictions can now influence the behavior of the full type. An example can be seen in uniqueness results for nonsplitting extensions: in general AECs, uniqueness results hold for non-μ-splitting extensions to models of size μ, but no further (Theorem 5.2.24). However, in μ-tame AECs, uniqueness results hold for non-μ-splitting extensions to models *of all sizes* (Theorem 5.5.15). Indeed, the parameter μ in non-μ-splitting becomes irrelevant. Thus, tameness can replace several extra assumptions. Compared to the good frame results above, categoricity in a single cardinal, tameness, and amalgamation are enough to show the existence of a good frame (Theorem 5.5.44) and tameness and amalgamation are enough to transfer the frame upwards without any change of the underlying class (Theorem 5.5.26).

Although tameness seems to only be a weak approximation to the nice properties enjoyed by first-order logic, it is still *strong enough* to import more of the model-theoretic intuition and technology from first-order. When dealing with tame AECs, a type can be identified with the set of all of its restrictions to small domains, and these small types play a role similar to formulas. This can be made precise: one can even give a sense in which types are sets of (infini-

[1] In the sense of the independence notion mentioned above. This will often be different from the first-order definition.

tary) formulas (see Theorem 5.3.5). This allows several standard arguments to be painlessly repeated in this context. For instance, the proof of the properties of $< \kappa$-satisfiability and the equivalence between Galois-stability and no order property in this context are similar to their first-order counterparts (see Section 5.5.2). On the other hand, several arguments from the theory of tame AECs have no first-order analog (see for example the discussion around amalgamation in Section 5.4).

On the other side, while tameness is in a sense a form of the first-order compactness theorem, it is *sufficiently weak* that several examples of nonelementary classes are tame. Section 5.3.2.1 goes into greater depth, but diverse classes like locally finite groups, rank one valued fields, and Zilber's pseudoexponentiation all turn out to be tame. Tameness can also be obtained for free from a large cardinal axiom, and a weak form of it follows from model-theoretic hypotheses such as the combination of amalgamation and categoricity in a high-enough cardinal.

Indeed, examples of *non*-tame AECs are in short supply (Section 5.3.2.2). All known examples are set-theoretic in nature, and it is open whether there are non-tame "natural" mathematical classes (see (5) in Section 5.6). The focus on ZFC results for tame AECs allows us to avoid situations where, for example, conclusions about rank one valued fields depend on whether $2^{\aleph_0} < 2^{\aleph_1}$. This replacing of set-theoretic hypotheses with model-theoretic ones suggests that developing a classification theory for tame AECs is possible *within ZFC*.

Thus, tame AECs seem to strike an important balance: they are *general* enough to encompass several nonelementary classes and yet *well-behaved/specific* enough to admit a classification theory. Even if one does not believe that tameness is a justified assumption, it can be used as a first approximation and then one can attempt to remove (or weaken) it from the statement of existing theorems. Indeed, there are several results in the literature (see the end of Section 5.2.4) which do not directly assume tameness, but whose proof starts by deducing some weak amount of tameness from the other assumptions, and then use this tameness in crucial ways.

We now highlight some results about tame AECs that will be discussed further in the rest of this survey. We first state two motivating test questions. The first is the well-known categoricity conjecture which can be traced back to an open problem in [She78]. The following version appears as [She09b, Conjecture N.4.2]:

Conjecture 5.1.1 (Shelah's eventual categoricity conjecture)**.** There exists a function $\mu \mapsto \lambda(\mu)$ such that if $\mathbf{K}$ is an AEC categorical in *some* $\lambda \geq \lambda(\mathrm{LS}(\mathbf{K}))$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \lambda(\mathrm{LS}(\mathbf{K}))$.

Shelah's categoricity conjecture is the main test question for AECs and remains the yardstick by which progress is measured. Using this yardstick, tame AECs are well-developed. Grossberg and VanDieren [GV06b] isolated tameness from Shelah's proof of a downward categoricity transfer in AECs with amalgamation [She99]. Tameness was one of the key (implicit) properties there (in the proof of [She99, Main claim 2.3], where Shelah proves that categoricity in a high-enough successor implies that types over Galois-saturated models are determined by their small restrictions[2], this property has later been called weak tameness). Grossberg and VanDieren defined tameness without the assumption of saturation and developed the theory in a series of papers [GV06b, GV06c, GV06a], culminating in the proof of Shelah's eventual categoricity conjecture from a successor in tame AECs with amalgamation[3].

Progress towards other categoricity transfers often proceed by first proving tameness and then using it to transfer categoricity. One of the achievements of developing the classification theory of tame AECs is the following result, due to the second author [Vasf]:

Theorem 5.1.2. *Shelah's eventual categoricity conjecture is true when* $\mathbf{K}$ *is a universal class with amalgamation. In this case, one can take* $\lambda(\mu) := \beth_{(2^\mu)^+}$*. Moreover amalgamation can be derived from categoricity in cardinals of arbitrarily high cofinality.*

The proof starts by observing that every universal class is tame (a result of the first author [Bonc], see Theorem 5.3.11).

The second test question is more vague and grew out of the need to generalize some of the tools of first-order stability theory to AECs.

Question 5.1.3. Let $\mathbf{K}$ be an AEC categorical in a high-enough cardinal. Does there exists a cardinal χ such that $\mathbf{K}_{\geq \chi}$ admits a notion of independence akin to first-order forking?

The answer is positive for universal classes with amalgamation (see Theorem 5.5.41), and more generally for classes with amalgamation satisfying a certain strengthening of tameness:

Theorem 5.5.53. *Let* $\mathbf{K}$ *be a fully* $< \aleph_0$*-tame and -type short AEC with amalgamation. If* $\mathbf{K}$ *is categorical in a* $\lambda > \mathrm{LS}(\mathbf{K})$*, then* $\mathbf{K}_{\geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}}$ *has (in a precise sense) a superstable forking-like independence notion.*

Varying the locality assumption, one can obtain weaker, but still powerful, conclusions that are key in the proof of Theorem 5.1.2.

One of the big questions in developing classification theory outside of first-order is which of the characterizations of dividing lines to take as the definition (see Section 5.2.1). This is especially true when dealing with superstability. In the first-order context, this is characterizable by forking having certain properties or the union of saturated models being saturated or one of several other properties. In tame AECs, these characterizations have recently been proven to also be equivalent! See Theorem 5.5.23.

[2]In [She01a, Definition 0.24], Shelah defines a type to be *local* if it is defined by all its $\mathrm{LS}(\mathbf{K})$-sized restrictions.

[3]The work on [GV06b] was done in 2000-2001 and preprints of [GV06c, GV06a] were circulated in 2004.

This survey is organized as follows: Section 5.2 reviews concepts from the study of general AEC. This begins with definitions and basic notions (Galois type, etc.) that are necessary for work in AECs. Subsection 5.2.1 is a review of classification theory without tameness. The goal here is to review the known results that do not involve tameness in order to emphasize the strides that assuming tameness makes. Of course, we also setup notation and terminology. Previous familiarity with the basics of AECs as laid out in e.g. [Bal09, Chapter 4] would be helpful. We also assume that the reader knows the basics of first-order model theory.

Section 5.3 formally introduces tameness and related principles. Subsection 5.3.2.1 reviews the known examples of tameness and non-tameness.

Section 5.4 outlines the proof of Shelah's Categoricity Conjecture for universal classes. The goal of this outline is to highlight several of the tools that exist in the classification theory of tame AECs and tie them together in a single result. After whetting the reader's appetite, Section 5.5 goes into greater detail about the classification-theoretic tools available in tame AECs.

This introduction has been short on history and attribution and the historical remarks in Section 5.7 fill this gap. We have written this survey in a somewhat informal style where theorems are not attributed when they are stated: the reader should also look at Section 5.7, where proper credits are given. *It should not be assumed that an unattributed result is the work of the authors.*

Let us say a word about what is *not* discussed: We have chosen to focus on recent material which is not already covered in Baldwin's book [Bal09], so while its interest cannot be denied, we do not for example discuss the proof of the Grossberg-VanDieren upward categoricity transfer [GV06c, GV06a]. Also, we focus on tame AECs, so tameness-free results (such as Shelah's study of Ehrenfeucht-Mostowski models in [She09b, Chapter IV], [Shea], or the work of VanDieren and the second author on the symmetry property [Vanb, Van16, VV]) are not emphasized. Related frameworks which we do not discuss much are homogeneous model theory (see Example 5.3.2.1.(7)), tame finitary AECs (Example 5.3.2.1.(6)), and tame metric AECs (see Example 5.3.2.1.(9)).

Finally, let us note that the field is not a finished body of work but is still very much active. Some results may become obsolete soon after, or even before, this survey is published[4] . Still, we felt there was a need to write this paper, as the body of work on tame AECs has grown significantly in recent years and there is, in our opinion, a need to summarize the essential points.

5.1.1 Acknowledgments

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[4]Indeed, since this paper was first circulated (in December 2015) the amalgamation assumption has been removed from Theorem 5.1.2 [Vasg] and Question 5.2.37 has been answered positively [Vasd].

and he would like to thank Professor Grossberg for his guidance and assistance in his research in general and in this work specifically.

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5.2 A primer in abstract elementary classes without tameness

In this section, we give an overview of some of the main concepts of the study of abstract elementary classes. This is meant both as a presentation of the basics and as a review of the "pre-tameness" literature, with an emphasis of the difficulties that were faced. By the end of this section, we give several state-of-the-art results on Shelah's categoricity conjecture. While tameness is not assumed, deriving a weak version from categoricity is key in the proofs.

We only sketch the basics here and omit most of the proofs. The reader who wants a more thorough introduction should consult [Gro02], [Bal09], or the upcoming [Gro]. We are light on history and motivation for this part; interested readers should consult one of the references or Section 5.7.

Abstract elementary classes (AECs) were introduced by Shelah in the mid-seventies. The original motivation was axiomatizing classes of models of certain infinitary logics ($\mathbb{L}_{\omega_1,\omega}$ and $\mathbb{L}(Q)$), but the definition can also be seen as extracting the category-theoretic essence of first-order model theory (see [Lie11a]).

Definition 5.2.1. An abstract elementary class (AEC) is a pair $(\mathbf{K}, \leq)$ satisfying the following conditions:

1. $\mathbf{K}$ is a class of L-structures for a fixed language $L := L(\mathbf{K})$.
2. $\leq$ is a reflexive and transitive relation on $\mathbf{K}$.
3. Both $\mathbf{K}$ and $\leq$ are closed under isomorphisms: If $M, N \in \mathbf{K}$, $M \leq N$, and $f : N \cong N'$, then $f[M], N' \in \mathbf{K}$ and $f[M] \leq N'$.
4. If $M \leq N$, then M is an L-substructure of N (written[5] $M \subseteq N$).
5. Coherence axiom: If $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq M_2$, and $M_0 \leq M_2$, then $M_0 \leq M_1$.
6. Tarski-Vaught chain axioms: If δ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is an increasing chain (that is, for all $i < j < \delta$, $M_i \in \mathbf{K}$ and $M_i \leq M_j$), then:
    (a) $M_\delta := \bigcup_{i<\delta} M_i \in \mathbf{K}$.

[5] We write $|M|$ for the universe of an L-structure M and $\|M\|$ for the cardinality of the universe. Thus $M \subseteq N$ means M is a substructure of N while $|M| \subseteq |N|$ means that the universe of M is a subset of the universe of N.

(b) $M_i \leq M_\delta$ for all $i < \delta$.

(c) If $N \in \mathbf{K}$ and $M_i \leq N$ for all $i < \delta$, then $M_\delta \leq N$.

7. Löwenheim-Skolem-Tarski axiom[6]: There exists a cardinal $\mu \geq |L(\mathbf{K})| + \aleph_0$ such that for every $M \in \mathbf{K}$ and every $A \subseteq |M|$, there exists $M_0 \leq M$ so that $A \subseteq |M_0|$ and $\|M_0\| \leq \mu + |A|$. We define the Löwenheim-Skolem-Tarski number of $\mathbf{K}$ (written $\mathrm{LS}(\mathbf{K})$) to be the least such cardinal.

We often will not distinguish between $\mathbf{K}$ and the pair $(\mathbf{K}, \leq)$. We write $M < N$ when $M \leq N$ and $M \neq N$.

Example 5.2.2. $(\mathrm{Mod}(T), \preceq)$ for T a first-order theory, and more generally $(\mathrm{Mod}(\psi), \preceq_\Phi)$ for ψ an $\mathbb{L}_{\lambda,\omega}$ sentence and Φ a fragment containing ψ are among the motivating examples. The Löwenheim-Skolem-Tarski numbers in those cases are respectively $|L(T)| + \aleph_0$ and $|\Phi| + |L(\Phi)| + \aleph_0$. In the former case, we say that the class is *elementary*. See the aforementioned references for more examples.

Notation 5.2.3. *For $\mathbf{K}$ an AEC, we write $\mathbf{K}_\lambda$ for the class of $M \in \mathbf{K}$ with $\|M\| = \lambda$, and similarly for variations such as $\mathbf{K}_{\geq\lambda}$, $\mathbf{K}_{<\lambda}$, $\mathbf{K}_{[\lambda,\theta)}$, etc.*

Remark 5.2.4 (Existence of resolutions)**.** Let $\mathbf{K}$ be an AEC and let $\lambda > \mathrm{LS}(\mathbf{K})$. If $M \in \mathbf{K}_\lambda$, it follows directly from the axioms that there exists an increasing chain $\langle M_i : i \leq \lambda \rangle$ which is continuous[7] and so that $M_\lambda = M$ and $M_i \in \mathbf{K}_{<\lambda}$ for all $i < \lambda$; such a chain is called a *resolution* of M. We also use this name to refer to the initial segment $\langle M_i : i < \lambda \rangle$ with $M_\lambda = M = \bigcup_{i<\lambda} M_i$ left implicit.

Remark 5.2.5. Let $\mathbf{K}$ be an AEC. A few quirks are not ruled out by the definition:

- $\mathbf{K}$ could be empty.
- It could be that $\mathbf{K}_{<\mathrm{LS}(\mathbf{K})}$ is nonempty. This can be remedied by replacing $\mathbf{K}$ with $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ (also an AEC with the same Löwenheim-Skolem-Tarski number as $\mathbf{K}$). Note however that in some examples, the models below $\mathrm{LS}(\mathbf{K})$ give a lot of information on the models of size $\mathrm{LS}(\mathbf{K})$, see Baldwin, Koerwein, and Laskowski [BKL].

Most authors implicitly assume that $\mathbf{K}_{<\mathrm{LS}(\mathbf{K})} = \emptyset$ and $\mathbf{K}_{\mathrm{LS}(\mathbf{K})} \neq \emptyset$, and the reader can safely make these assumptions throughout. However, we will try to be careful about these details when stating results.

An AEC $\mathbf{K}$ may not have certain structural properties that always hold in the elementary case:

[6] This axiom was initially called the Löwenheim-Skolem axiom, which explains why it is written $\mathrm{LS}(\mathbf{K})$. However, later works have referred to it this way (and sometimes written $\mathrm{LST}(\mathbf{K})$) as an acknowledgment of Tarski's role in the corresponding first-order result.

[7] That is, for every limit i, $M_i = \bigcup_{j<i} M_j$.

Definition 5.2.6. Let $\mathbf{K}$ be an AEC.

1. $\mathbf{K}$ has *amalgamation* if for any $M_0, M_1, M_2 \in \mathbf{K}$ with $M_0 \leq M_\ell$, $\ell = 1, 2$, there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$, $\ell = 1, 2$.

$$\begin{tikzcd} M_1 \arrow[r, dotted, "f_1"] & N \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u, dotted, "f_2"'] \end{tikzcd}$$

2. $\mathbf{K}$ has *joint embedding* if for any $M_1, M_2 \in \mathbf{K}$, there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \to N$, $\ell = 1, 2$.
3. $\mathbf{K}$ has *no maximal models* if for any $M \in \mathbf{K}$ there exists $N \in \mathbf{K}$ with $M < N$.
4. $\mathbf{K}$ has *arbitrarily large models* if for any cardinal λ, $\mathbf{K}_{\geq \lambda} \neq \emptyset$.

We define localizations of these properties in the expected way. For example, we say that $\mathbf{K}_\lambda$ *has amalgamation* or $\mathbf{K}$ has *amalgamation in* λ (or λ-*amalgamation*) if the definition of amalgamation holds when all the models are required to be of size λ.

There are several easy relationships between these properties. We list here a few:

Proposition 5.2.7. *Let $\mathbf{K}$ be an AEC, $\lambda \geq \mathrm{LS}(\mathbf{K})$.*

1. *If $\mathbf{K}$ has joint embedding and arbitrarily large models, then $\mathbf{K}$ has no maximal models.*
2. *If $\mathbf{K}$ has joint embedding in λ, $\mathbf{K}_{<\lambda}$ has no maximal models, and $\mathbf{K}_{\geq \lambda}$ has amalgamation, then $\mathbf{K}$ has joint embedding.*
3. *If $\mathbf{K}$ has amalgamation in every $\mu \geq \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ has amalgamation.*

In a sense, joint embedding says that the AEC is "complete". Assuming amalgamation, it is possible to partition the AEC into disjoint classes each of which has amalgamation and joint embedding.

Proposition 5.2.8. *Let $\mathbf{K}$ be an AEC with amalgamation. For $M_1, M_2 \in \mathbf{K}$, say $M_1 \sim M_2$ if and only if M_1 and M_2 embed inside a common model (i.e. there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \to N$). Then $\sim$ is an equivalence relation, and its equivalence classes partition $\mathbf{K}$ into at most $2^{\mathrm{LS}(\mathbf{K})}$-many AECs with joint embedding and amalgamation.*

Thus if $\mathbf{K}$ is an AEC with amalgamation and arbitrarily large models, we can find a sub-AEC of it which has amalgamation, joint embedding, and no maximal models. In that sense, global amalgamation implies all the other properties (see also Corollary 5.2.12).

Using the existence of resolutions, it is not difficult to see that an AEC $(\mathbf{K}, \leq)$ is determined by its restriction to size λ $(\mathbf{K}_\lambda, \leq \cap (\mathbf{K}_\lambda \times \mathbf{K}_\lambda))$. Thus, there is only a set of AECs with a fixed Löwenheim-Skolem-Tarski number and hence there is a Hanf number for the property that the AEC has arbitrarily large models.

While this analysis only gives an existence proof for the Hanf number, Shelah's presentation theorem actually allows a computation of the Hanf number by establishing a connection between $\mathbf{K}$ and $\mathbb{L}_{\infty,\omega}$.

Theorem 5.2.9 (Shelah's presentation theorem)**.** *If $\mathbf{K}$ is an AEC with $L(\mathbf{K}) = L$, there exists a language $L' \supseteq L$ with $|L'| + \mathrm{LS}(\mathbf{K})$, a first-order L'-theory T', and a set of T'-types Γ such that*

$$\mathbf{K} = PC(T', \Gamma, L) := \{M' \upharpoonright L \mid M' \models T' \textit{ and } M' \textit{ omits all the types in } \Gamma\}$$

The proof proceeds by adding $\mathrm{LS}(\mathbf{K})$-many functions of each arity. For each M, we can write it as the union of a directed system $\{N_{\bar{a}} \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})} : \bar{a} \in {}^{<\omega}|M|\}$ with $\bar{a} \in N_{\bar{a}}$. Then, the intended expansion M' of M is where the universe of $N_{\bar{a}}$ is enumerated by new functions of arity $\ell(\bar{a})$ applied to $\bar{a}$. The types of Γ are chosen such that M' omits them if and only if the reducts of the substructures derived in this way actually form a directed system[8].

In particular, $\mathbf{K}$ is the reduct of a class of models of an $\mathbb{L}_{\mathrm{LS}(\mathbf{K})^+,\omega}$-theory. An important caveat is that if $\mathbf{K}$ was given by the models of some $\mathbb{L}_{\mathrm{LS}(\mathbf{K})^+,\omega}$-theory, the axiomatization given by Shelah's Presentation Theorem is different and uninformative. However, it is enough to allow the computation of the Hanf number for existence.

Corollary 5.2.10. *If $\mathbf{K}$ is an AEC such that $\mathbf{K}_{\geq \chi} \neq \emptyset$ for all $\chi < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$, then $\mathbf{K}$ has arbitrarily large models.*

The cardinal $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ appears frequently in studying AECs, so has been given a name:

Notation 5.2.11. *For λ an infinite cardinal, write $h(\lambda) := \beth_{(2^\lambda)^+}$. When $\mathbf{K}$ is a fixed AEC, we write $H_1 := h(\mathrm{LS}(\mathbf{K}))$ and $H_2 := h(h(\mathrm{LS}(\mathbf{K})))$.*

We obtain for example that any AEC with amalgamation and joint embedding in a single cardinal eventually has all the structural properties of Definition 5.2.6.

Corollary 5.2.12. *Let $\mathbf{K}$ be an AEC with amalgamation. If $\mathbf{K}$ has joint embedding in some $\lambda \geq \mathrm{LS}(\mathbf{K})$, then there exists $\chi < H_1$ so that $\mathbf{K}_{\geq \chi}$ has amalgamation, joint embedding, and no maximal models. More precisely, there exists an AEC $\mathbf{K}^*$ such that:*

1. $\mathbf{K}^* \subseteq \mathbf{K}$.

[8]Note that there are almost always the maximal number of types in Γ.

2. $\mathrm{LS}(\mathbf{K}^*) = \mathrm{LS}(\mathbf{K})$.
3. $\mathbf{K}^*$ *has amalgamation, joint embedding, and no maximal models.*
4. $\mathbf{K}_{\geq \chi} = (\mathbf{K}^*)_{\geq \chi}$.

Proof sketch. First use Proposition 5.2.8 to decompose the AEC into at most $2^{\mathrm{LS}(\mathbf{K})}$ many subclasses, each of which has amalgamation and joint embedding. Now if one of these partitions does not have arbitrarily large models, then there must exists a $\chi_0 < H_1$ in which it has no models. Take the sup of all such χ_0s and observe that $\mathrm{cf}(H_1) = (2^{\mathrm{LS}(\mathbf{K})})^+ > 2^{\mathrm{LS}(\mathbf{K})}$. □

If $\mathbf{K}$ is an AEC with joint embedding, amalgamation, and no maximal models, we may build a proper-class[9] sized model-homogeneous universal model $\mathfrak{C}$, where:

Definition 5.2.13. Let $\mathbf{K}$ be an AEC, let $M \in \mathbf{K}$, and let λ be a cardinal.

1. M is *λ-model-homogeneous* if for every $M_0 \leq M$, $M_0' \geq M_0$ with $\|M\| < \lambda$, there exists $f : M_0' \xrightarrow[M_0]{} M$. When $\lambda = \|M\|$, we omit it.
2. M is *universal* if for every $M' \in \mathbf{K}$ with $\|M'\| \leq \|M\|$, there exists $f : M' \to M$.

Definition 5.2.14. We say that an AEC $\mathbf{K}$ *has a monster model* if it has a model $\mathfrak{C}$ as above. Equivalently, it has amalgamation, joint embedding, and arbitrarily large models.

Remark 5.2.15. Even if $\mathbf{K}$ only has amalgamation and joint embedding, we can construct a monster model, but it may not be proper-class sized. If in addition joint embedding fails, for any $M \in \mathbf{K}$ we can construct a big model-homogeneous model $\mathfrak{C} \geq M$.

Note that if $\mathbf{K}$ were in fact an elementary class, then the monster model constructed here is the same as the classical concept.

When $\mathbf{K}$ has a monster model $\mathfrak{C}$, we can define a semantic notion of type[10] by working inside $\mathfrak{C}$ and specifying that $\bar{b}$ and $\bar{c}$ have the same type over A if and only if there exists an automorphism of $\mathfrak{C}$ taking $\bar{b}$ to $\bar{c}$ and fixing A. In fact, this can be generalized to arbitrary AECs:

Definition 5.2.16 (Galois types)**.** Let $\mathbf{K}$ be an AEC.

1. For an index set I, an *I-indexed Galois triple* is a triple $(\bar{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\bar{b} \in {}^I|N|$.

[9] To make sense of this, we have to work in Gödel-Von Neumann-Bernays set theory. Alternatively, we can simply ask for the monster model to be bigger than any sizes involved in our proofs. In any case, the way to make this precise is the same as in the elementary theory, so we do not elaborate.

[10] A semantic (as opposed to syntactic) notion of type is the only one that makes sense in a general AEC as there is no natural logic to work in. Even in AECs axiomatized in a logic such as $\mathbb{L}_{\omega_1,\omega}$, syntactic types do not behave as they do in the elementary framework; see the discussion of the Hart-Shelah example in Section 5.3.2.1.

2. We say that the I-indexed Galois triples $(\bar{b}_1, A_1, N_1)$, $(\bar{b}_2, A_2, N_2)$ are *atomically equivalent* and write $(\bar{b}_1, A_1, N_1) E_{\text{at}}^I (\bar{b}_2, A_2, N_2)$ if $A_1 = A_2$, and there exists $N \in \mathbf{K}$ and $f_\ell : N_\ell \underset{A}{\longrightarrow} N$ so that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$. When I is clear from context, we omit it.
3. Note that E_{at} is a symmetric and reflexive relation. We let E be its transitive closure.
4. For an I-indexed Galois triple $(\bar{b}, A, N)$, we let $\operatorname{gtp}(\bar{b}/A; N)$ (the *Galois type* of $\bar{b}$ over A in N) be the E-equivalence class of $(\bar{b}, A, N)$.
5. For $N \in \mathbf{K}$ and $A \subseteq |N|$, we let $\operatorname{gS}^I(A; N) := \{\operatorname{gtp}(\bar{b}/A; N) \mid \bar{b} \in {}^I|N|\}$. We also let $\operatorname{gS}^I(N) := \bigcup_{N' \geq N} \operatorname{gS}^I(N; N')$. When I is omitted, this means that $|I| = 1$, e.g. $\operatorname{gS}(N)$ is $\operatorname{gS}^1(N)$.
6. We can define restrictions of Galois types in the natural way: for $p \in \operatorname{gS}^I(A; N)$, $I_0 \subseteq I$ and $A_0 \subseteq A$, write $p \upharpoonright A_0$ for the restriction of p to A_0 and p^{I_0} for the restriction of p to I_0. For example, if $p = \operatorname{gtp}(\bar{b}/A; N)$ and $A_0 \subseteq A$, $p \upharpoonright A_0 := \operatorname{gtp}(\bar{b}/A_0; N)$ (this does not depend on the choice of representative for p).
7. Given $p \in \operatorname{gS}^I(M)$ and $f : M \cong M'$, we can also define $f(p)$ in the natural way.

Remark 5.2.17.

1. If $M \leq N$, then $\operatorname{gtp}(\bar{b}/A; M) = \operatorname{gtp}(\bar{b}/A; N)$. Similarly, if $f : M \cong_A N$, then $\operatorname{gtp}(\bar{b}/A; M) = \operatorname{gtp}(f(\bar{b})/A; N)$. Equivalence of Galois types is the coarsest equivalence relation with these properties.
2. If $\mathbf{K}$ has amalgamation, then $E = E_{\text{at}}$.
3. If $\mathfrak{C}$ is a monster model for $\mathbf{K}$, $\bar{b}_1, \bar{b}_2 \in {}^{<\infty}|\mathfrak{C}|$, $A \subseteq |\mathfrak{C}|$, then $\operatorname{gtp}(\bar{b}_1/A; \mathfrak{C}) = \operatorname{gtp}(\bar{b}_2/A; \mathfrak{C})$ if and only if there exists $f \in \operatorname{Aut}_A(\mathfrak{C})$ so that $f(\bar{b}_1) = \bar{b}_2$. When working inside $\mathfrak{C}$, we just write $\operatorname{gtp}(\bar{b}/A)$ for $\operatorname{gtp}(\bar{b}/A; \mathfrak{C})$, but in general, the model in which the Galois type is computed is important.
4. The cardinality of the index set is all that is important. However, when discussing type shortness later, it is convenient to allow the index set to be arbitrary.

When dealing with Galois types, one has to be careful about distinguishing between types over *models* and types over *sets*. Most of the basic definitions work the same for types over sets and models, and both require just amalgamation *over models* to make the transitivity of atomic equivalence work. Allowing types over sets gives slightly more flexibility in the definitions. For example, we can say what is meant to be $< \aleph_0$-tame or to be $(< \operatorname{LS}(\mathbf{K}))$-tame in $\mathbf{K}_{\geq \operatorname{LS}(\mathbf{K})}$. See the discussion around Definition 5.3.1.

On the other hand, several basic results–such as the construction of κ-saturated models–require amalgamation over the sort of object (set or model) desired in the conclusion. For instance, the following is true.

Proposition 5.2.18. *Suppose that* $\mathbf{K}$ *is an AEC with amalgamation*[11].

1. *The following are equivalent.*
   - A *is an amalgamation base*[12].
   - *For every* $p \in \mathrm{gS}^1(A; N)$ *and* $M \supseteq A$, *there is an extension of* p *to* M.
2. *The following are equivalent.*
   - $\mathbf{K}$ *has amalgamation over sets.*
   - *For every* M *and* κ, *there is an extension* $N \geq M$ *with the following property:*

     For every $A \subseteq |N|$ *and* $|M^*| \supseteq A$ *with* $|A| < \kappa$, *any* $p \in \mathrm{gS}^{<\kappa}(A; M^*)$ *is realized in* N.

A more substantial result is [She99, Claim 3.3], which derives a local character for splitting in stable AECs (see Lemma 5.2.23 below), but only in the context of Galois types over models.

One can give a natural definition of saturation in terms of Galois types.

Definition 5.2.19. A model $M \in \mathbf{K}$ is λ-*Galois-saturated* if for any $A \subseteq |M|$ with $|A| < \lambda$, any $N \geq M$, any $p \in \mathrm{gS}(A; N)$ is realized in M. When $\lambda = \|M\|$, we omit it.

Note the difference between this definition and Proposition 5.2.18.(2) above. When $\mathbf{K}$ does not have amalgamation or when $\lambda \leq \mathrm{LS}(\mathbf{K})$, it is not clear that this definition is useful. But if $\mathbf{K}$ has amalgamation and $\lambda > \mathrm{LS}(\mathbf{K})$, the following result of Shelah is fundamental:

Theorem 5.2.20. *Assume that* $\mathbf{K}$ *is an AEC with amalgamation and let* $\lambda > \mathrm{LS}(\mathbf{K})$. *Then* $M \in \mathbf{K}$ *is* λ-*Galois-saturated if and only if it is* λ-*model-homogeneous.*

5.2.1 Classification Theory

One theme of the classification theory of AECs is what Shelah has dubbed the "schizophrenia" of superstability (and other dividing lines) [She09b, p. 19]. Schizophrenia here refers to the fact that, in the elementary framework, dividing lines are given by several equivalent characterizations (e.g. stability is no order property or few types), typically with the existence of a definable, combinatorial object on the "high" or bad side and some good behavior of forking on the "low" or good side. However, this equivalence relies heavily on compactness or other ideas central to first-order and breaks down when dealing with general AECs. Thus, the search for stability, superstability, etc.

[11]Recall that this is defined to mean over models.

[12]This should be made precise, for example by considering the embedding of A inside a fixed monster model.

is in part a search for the "right" characterization of the dividing line and in part a search for equivalences between the different faces of the dividing line.

One can roughly divide approaches towards the classification of AECs into two categories: local approaches and global approaches[13]. Global approaches typically assume one or more structural properties (such as amalgamation or no maximal models) as well as a classification property (such as categoricity in a high-enough cardinal or Galois stability in a particular cardinality), and attempt to derive good behavior on a tail of cardinals. The local approach is a more ambitious strategy pioneered by Shelah in his book [She09b]. The idea is to first show (assuming e.g. categoricity in a proper class of cardinals) that the AEC has good behavior in some suitable cardinal λ. Shelah precisely defines "good behavior in λ" as having a good λ-frame (see Section 5.2.5). In particular, this implies that the class is superstable in λ. The second step in the local approach is to argue that good behavior in some λ transfers upward to λ^+ and, if the behavior is good enough, to all cardinals above λ. Having established global good behavior, one can rely on the tools of the global approach to prove the categoricity conjecture.

The local approach seems more general but comes with a price: increased complexity, and often the use of non-ZFC axioms (like the weak GCH: $2^\lambda < 2^{\lambda^+}$ for all λ), as well as stronger categoricity hypotheses. The two approaches are not exclusive. In fact in recent years, tools from local approach have been used and studied in a more global framework. We now briefly survey results in both approaches that do not use tameness.

5.2.2 Stability

Once Galois types have been defined, one can define *Galois-stability*:

Definition 5.2.21. An AEC $\mathbf{K}$ is *Galois-stable in* λ if for any $M \in \mathbf{K}$ with $\|M\| \leq \lambda$, we have $|\mathrm{gS}(M)| \leq \lambda$.

One can ask whether there is a notion like forking in stable AECs. The next sections discuss this problem in detail. A first approximation is *μ-splitting*:

Definition 5.2.22. Let $\mathbf{K}$ be an AEC, $\mu \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ has amalgamation in μ. Let $M \leq N$ both be in $\mathbf{K}_{\geq \mu}$. A type $p \in \mathrm{gS}^{<\infty}(N)$ *μ-splits* over M if there exists $N_1, N_2 \in \mathbf{K}_\mu$ with $M \leq N_\ell \leq N$, $\ell = 1, 2$, and $f : N_1 \cong_M N_2$ so that $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.

One of the early results was that μ-splitting has a local character properties in stable AECs:

Lemma 5.2.23. *Let* $\mathbf{K}$ *be an AEC,* $\mu \geq \mathrm{LS}(\mathbf{K})$*. Assume that* $\mathbf{K}$ *has amalgamation in* μ *and is Galois-stable in* μ*. For any* $N \in \mathbf{K}_{\geq \mu}$ *and* $p \in \mathrm{gS}(N)$*, there exists* $N_0 \in \mathbf{K}_\mu$ *with* $N_0 \leq N$ *so that* p *does not* μ*-split over* N_0*.*

[13]Monica VanDieren suggested set-theoretic scaffolding and model-theoretic scaffolding as alternate names for the local and global approaches.

With stability and amalgamation, we also get that there are unique non-μ-splitting extensions to universal models *of the same size.*

Theorem 5.2.24. *Let* $\mathbf{K}$ *be an AEC,* $\mu \geq \mathrm{LS}(\mathbf{K})$*. Assume that* $\mathbf{K}$ *has amalgamation in* μ *and is Galois-stable in* μ*. If* $M_0, M_1, M_2 \in \mathbf{K}_\mu$ *with* M_1 *universal over* M_0[14]*, then each* $p \in \mathrm{gS}(M_1)$ *that does not* μ*-split over* M_0 *has a unique extension* $q \in \mathrm{gS}(M_2)$ *that does not split over* M_0*. Moreover,* p *is algebraic if and only if* q *is.*

Note in passing that stability gives existence of universal extensions:

Lemma 5.2.25. *Let* $\mathbf{K}$ *be an AEC and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$ *be such that* $\mathbf{K}$ *has amalgamation in* λ *and is Galois-stable in* λ*. For any* $M \in \mathbf{K}_\lambda$*, there exists* $N \in \mathbf{K}_\lambda$ *which is universal over* M*.*

Similar to first-order model theory, there is a notion of an order property in AECs. The order property is more parametrized due to the lack of compactness. In the elementary framework, the order property is defined as the existence of a definable order of order type ω. However, the essence of it is that any order type can be defined. Thus the lack of compactness forces us to make the the order property in AECs longer in order to be able to build complicated orders:

Definition 5.2.26.

1. $\mathbf{K}$ has the *κ-order property of length α* if there exists $N \in \mathbf{K}$, $p \in \mathrm{gS}^{<\kappa}(\emptyset; N)$, and $\langle \bar{a}_i \in {}^{<\kappa}|M| : i < \alpha \rangle$ such that:
$$i < j \Leftrightarrow \mathrm{gtp}(\bar{a}_i \bar{a}_j / \emptyset; N) = p$$
2. $\mathbf{K}$ has the *κ-order property* if it has the κ-order property of all lengths.
3. $\mathbf{K}$ has the *order property* if it has the κ-order property for some κ.

From the presentation theorem, having the κ-order property of all lengths less than $h(\kappa)$ is enough to imply the full κ-order property. In this case, one can show that α above can be replaced by any linear ordering.

5.2.3 Superstability

A first-order theory T is superstable if it is stable on a tail of cardinals. One might want to adapt this definition to AECs, but it is not clear that it is enough to derive the property that we really want here: an analog of $\kappa(T) = \aleph_0$. A possible candidate is to say that a class is superstable if every type does not μ-split over a finite set. However, splitting is only defined for models, and, as remarked above, types over arbitrary sets are not too well-behaved. Instead, as with Galois types, we take an implication of the desired property as the new definition: no long splitting chains.

[14] That is, for every $M' \in \mathbf{K}_\mu$ with $M_0 \leq M'$, there exists $f : M' \xrightarrow[M_0]{} M_1$.

Definition 5.2.27. An AEC $\mathbf{K}$ is *μ-superstable* (or *superstable in μ*) if:

1. $\mu \geq \mathrm{LS}(\mathbf{K})$.
2. $\mathbf{K}_\mu$ is nonempty, has amalgamation[15], joint embedding, and no maximal models.
3. $\mathbf{K}$ is Galois-stable in μ.
4. for all limit ordinal $\delta < \mu^+$ and every increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ in $\mathbf{K}_\mu$ with M_{i+1} universal over M_i for all $i < \delta$, if $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ so that p does not μ-split over M_i.

If $\mathbf{K}$ is the class of models of a first-order theory T, then $\mathbf{K}$ is μ-superstable if and only if T is stable in every $\lambda \geq \mu$.

Remark 5.2.28. In (4), note that M_{i+1} is required to be universal over M_i, rather than just strong extension. For reasons that we do not completely understand, it unknown whether this variation follows from categoricity (see Theorem 5.2.36). On the other hand, it seems to be sufficient for many purposes. In the tame case, the good frames derived from superstability (see Theorem 5.5.22) will have this stronger property.

Another possible definition of superstability in AECs is the uniqueness of limit models:

Definition 5.2.29. Let $\mathbf{K}$ be an AEC and let $\mu \geq \mathrm{LS}(\mathbf{K})$.

1. A model $M \in \mathbf{K}_\mu$ is *(μ, δ)-limit* for limit $\delta < \mu^+$ if there exists a strictly increasing continuous chain $\langle M_i \in \mathbf{K}_\mu : i \leq \delta \rangle$ such that $M_\delta = M$ and for all $i < \delta$, M_{i+1} is universal over M_i. If we do not specify the δ, it means that there is one. We say that M is *limit over* N when such a chain exists with $M_0 = N$.
2. $\mathbf{K}$ has *uniqueness of limit models in* μ if whenever $M_0, M_1, M_2 \in \mathbf{K}_\mu$ and both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$.
3. $\mathbf{K}$ has *weak uniqueness of limit models in* μ if whenever $M_1, M_2 \in \mathbf{K}_\mu$ are limit models, then $M_1 \cong M_2$ (the difference is that the isomorphism is not required to fix M_0).

Limit models and their uniqueness have come to occupy a central place in the study of superstability of AECs. (μ^+, μ^+)-limit models are Galois-saturated, so even weak uniqueness of limit models in μ^+ implies that (μ^+, ω)-limit models are Galois-saturated. This tells us that Galois-saturated models can be built in fewer steps than expected, which is reminiscent of first-order characterizations of superstability. As an added benefit, the analysis of limit models can be carried out in a single cardinal (as opposed to Galois-saturated

[15]This requirement is not made in several other variations of the definition but simplifies notation. See the historical remarks for more.

models, which typically need smaller models) and, thus, lends itself well to the local analysis[16].

The following question is still open (the answer is positive for elementary classes):

Question 5.2.30. Let $\mathbf{K}$ be an AEC and let $\mu \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}_\mu$ is nonempty, has amalgamation, joint embedding, no maximal models, and is Galois-stable in μ, do we have that $\mathbf{K}$ has uniqueness of limit models in μ if and only if $\mathbf{K}$ is superstable in μ?

This phenomenon of having two potentially non-equivalent definitions of superstability that are equivalent in the first-order case is an example of the "schizophrenia" of superstability mentioned above.

Shelah and Villaveces [SV99] started the investigation of whether superstability implies the uniqueness of limit models. Eventually, VanDieren introduced a symmetry property for μ-splitting to show the following.

Theorem 5.2.31. *If $\mathbf{K}$ is a μ-superstable[17] AEC such that μ-splitting has symmetry, then $\mathbf{K}$ has uniqueness of limit models in μ.*

In fact, the full strength of amalgamation in μ is not needed, see [Vana] in this volume for more.

5.2.4 Categoricity

For an AEC $\mathbf{K}$, let us denote by $I(\mathbf{K}, \lambda)$ the number of non-isomorphic models in $\mathbf{K}_\lambda$. We say that $\mathbf{K}$ is *categorical in* λ if $I(\mathbf{K}, \lambda) = 1$. One of Shelah's motivation for introducing AECs was to make progress on the following test question:

Conjecture 5.2.32 (Shelah's categoricity conjecture for $\mathbb{L}_{\omega_1,\omega}$)**.** If a sentence $\psi \in \mathbb{L}_{\omega_1,\omega}$ is categorical in *some* $\lambda \geq \beth_{\omega_1}$, then it is categorical in *all* $\lambda' \geq \beth_{\omega_1}$.

Note that the lower bound is the Hanf number of this class. One of the best results toward the conjecture is:

Theorem 5.2.33. *Let $\psi \in \mathbb{L}_{\omega_1,\omega}$ be a sentence in a countable language. Assume*[18] $\mathbf{V} = \mathbf{L}$. *If ψ is categorical in all $\aleph_n$, $1 \leq n < \omega$, then ψ is categorical in all uncoutable cardinals.*

Shelah's categoricity conjecture for $\mathbb{L}_{\omega_1,\omega}$ can be generalized to AECs, either by requiring only "eventual" categoricity (Conjecture 5.1.1) or by asking for a specific Hanf number.

This makes a difference: using the axiom of replacement and the fact that

[16] For example, it gives a way to define what it means for a model of size $\mathrm{LS}(\mathbf{K})$ to be saturated.

[17] Recall that the definition includes amalgamation and no maximal models in μ.

[18] Much weaker set-theoretic hypotheses suffice.

every AEC $\mathbf{K}$ is determined by its restrictions to models of size at most $\mathrm{LS}(\mathbf{K})$, it is easy to see that Shelah's eventual categoricity conjecture is equivalent to the following statement: If an AEC is categorical in a proper class of cardinals, then it is categorical on a tail of cardinals. Thus requiring that the Hanf number can in some sense be explicitly computed makes sure that one cannot "cheat" and automatically obtain a free upward transfer.

When the Hanf number is H_1 (recall Notation 5.2.11), we call the resulting statement *Shelah's categoricity conjecture for AECs*. This is widely recognized as the main test question[19] in the study of AECs.

Conjecture 5.2.34. If an AEC $\mathbf{K}$ is categorical in *some* $\lambda > H_1$, then it is categorical in *all* $\lambda' \geq H_1$.

One of the milestone results in the global approach to this conjecture is Shelah's downward transfer from a successor in AECs with amalgamation.

Theorem 5.2.35. *Let* $\mathbf{K}$ *be an AEC with amalgamation. If* $\mathbf{K}$ *is categorical in a* successor $\lambda \geq H_2$*, then* $\mathbf{K}$ *is categorical in all* $\mu \in [H_2, \lambda]$.

The structure of the proof involves first deriving a weak version of tameness from the categoricity assumption (see Example 5.3.2.1.(2)). A striking feature of this result (and several other categoricity transfers) is the successor requirement, which is, of course, missing from similar results in the first-order case. Removing it is a major open question, even in the tame framework (see Shelah [She00, Problem 6.14]). We see at least three difficulties when working with an AEC categorical in a limit cardinal $\lambda > \mathrm{LS}(\mathbf{K})$:

1. It is not clear that the model of size λ should be Galois-saturated, see Question 5.2.37.
2. It is not clear how to transfer "internal characterizations" of categoricity such as no Vaughtian pairs or unidimensionality. In the first-order framework, compactness is a key tool to achieve this.
3. It is not clear how to even get that categoricity implies such an internal characterization (assuming $\lambda = \lambda_0^+$ is a successor, there is a relatively straightforward argument for the non-existence of Vaughtian pairs in λ_0). In the first-order framework, all the arguments we are aware of use in some way *primary models* but here we do not know if they exist or are well-behaved. For example, we cannot imitate the classical argument that primary models are primes (this relies on the compactness theorem).

Assuming tameness, the first two issues can be solved (see Theorem 5.5.44 and the proof of Theorem 5.5.50). It is currently not known how to solve the third in general, but adding the assumption that $\mathbf{K}$ has prime models over sets of the form $M \cup \{a\}$ is enough. See Theorem 5.5.47.

[19] It is not expected that solving it will produce a useful lemma in solving other problems. Rather, like Morley's Theorem, it is expected that the solution will necessitate the development of ideas that will be useful in solving other problems.

A key tool in the proof of Theorem 5.2.35 is the existence of Ehrenfeucht-Mostowski models which follow from the presentation theorem. In AECs with amalgamation and no maximal models, several structural properties can be derived below the categoricity cardinal. For example:

Theorem 5.2.36 (The Shelah-Villaveces theorem, [SV99])**.** *Let* $\mathbf{K}$ *be an AEC with amalgamation and no maximal models. Let* $\mu \geq \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *is categorical in a* $\lambda > \mu$*, then* $\mathbf{K}$ *is* μ*-superstable.*

Note that Theorem 5.2.36 fails to generalize to $\lambda \geq \mu$. In general, $\mathbf{K}$ may not even be Galois-stable in λ, see the Hart-Shelah example (Section 5.3.2.2 below). In the presence of tameness, the difficulty disappears: superstability can be transferred all the way up (see Theorem 5.5.22). This seems to be a recurring feature of the study of AECs without tameness: some structure can be established below the categoricity cardinal (using tools such as Ehrenfeucht-Mostowski models), but transferring this structure upward is hard due to the lack of locality. For example, in the absence of tameness the following question is open:

Question 5.2.37. Let $\mathbf{K}$ be an AEC with amalgamation and no maximal models. If $\mathbf{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathbf{K})$, is the model of size λ Galois-saturated?

It is easy to see that (if $\mathrm{cf}(\lambda) > \mathrm{LS}(\mathbf{K})$), the model of size λ is $\mathrm{cf}(\lambda)$-Galois-saturated. Recently, it has been shown that categoricity in a high-enough cardinal implies some degree of saturation:

Theorem 5.2.38. *Let* $\mathbf{K}$ *be an AEC with amalgamation and no maximal models. Let* $\lambda \geq \chi > \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *is categorical in* λ *and* $\lambda \geq h(\chi)$*, then the model of size* λ *is* χ*-Galois-saturated.*

What about the uniqueness of limit models? In the course of establishing Theorem 5.2.35, Shelah proves that categoricity in a successor λ implies weak uniqueness of limit models in all $\mu < \lambda$. Recently, VanDieren and the second author have shown:

Theorem 5.2.39. *Let* $\mathbf{K}$ *be an AEC with amalgamation and no maximal models. Let* $\mu \geq \mathrm{LS}(\mathbf{K})$*. If* $\mathbf{K}$ *is categorical in a* $\lambda \geq h(\mu)$*, then* $\mathbf{K}$ *has uniqueness of limit models in* μ*.*

5.2.5 Good frames

Roughly speaking, an AEC $\mathbf{K}$ *has a good* λ*-frame* if it is well-behaved in λ (i.e. it is nonempty, has amalgamation, joint embedding, no maximal model, and is Galois-stable, all in λ) and there is a forking-like notion for types of length one over models in $\mathbf{K}_\lambda$ that behaves like forking in superstable first-order theories[20]. In particular, it is λ-superstable. One motivation for good frames was the following (still open) question:

Question 5.2.40. If an AEC is categorical in λ and λ^+, does it have a model of size λ^{++}?

Now it can be shown that if $\mathbf{K}$ has a good λ-frame (or even just λ-superstable), then it has a model of size λ^{++}. Thus it would be enough to obtain a good frame to solve the question. Shelah has shown the following:

Theorem 5.2.41. *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.*
Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If:

1. *$\mathbf{K}$ is categorical in λ and λ^+.*
2. *$0 < I(\mathbf{K}, \lambda^{++}) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$[21].*

Then $\mathbf{K}$ has a good λ^+-frame.

Corollary 5.2.42. *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. If $\mathbf{K}$ is categorical in λ, λ^+, and λ^{++}, then $\mathbf{K}$ has a model of size λ^{+++}.*

Note the non-ZFC assumptions as well as the strong categoricity hypothesis[22]. We will see that this can be removed in the tame framework, or even already by making some weaker (but global) assumptions than tameness.

Recall from the beginning of Section 5.2.1 that Shelah's local approach aims to transfer good behavior in λ upward. The successor step is to turn a good λ-frame into a good λ^+ frame . Shelah says a good λ-frame is *successful* if it satisfies a certain (strong) technical condition that allows it to extend it to a good λ^+-frame.

Theorem 5.2.43. *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. If an AEC $\mathbf{K}$ has a good λ-frame $\mathfrak{s}$ and $0 < I(\mathbf{K}, \lambda^{++}) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$, then there exists a good λ^+-frame $\mathfrak{s}^+$ with underlying class the Galois-saturated models of size λ^+ (the ordering will also be different).*

The proof goes by showing that the weak GCH and few models assumptions imply that any good frame is successful.

So assuming weak GCH and few models in every λ^{+n}, one obtains an increasing sequence $\bar{\mathfrak{s}} = \mathfrak{s}, \mathfrak{s}^+, \mathfrak{s}^{++}, \ldots$ of good frames. One of the main results of Shelah's book is that the natural limit of $\bar{\mathfrak{s}}$ is also a good frame (the strategy

[20]There is an additional parameter, the set of *basic* types. These are a dense set of types over models of size λ such that forking is only required to behave well with respect to them. However, basic types play little role in the discussion of tameness (and eventually are eliminated in most cases even in general AECs), so we do not discuss them here, see the historical remarks.

[21]The cardinal $\mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ should be interpreted as $2^{\lambda^{++}}$; this is true when $\lambda \geq \beth_\omega$ and there is no example of inequality when $2^{\lambda^+} < 2^{\lambda^{++}}$.

[22]It goes without saying that the proof is also long and complex, see the historical remarks.

is to show that a good frame in the sequence is *excellent*). Let us say that a good frame is *ω-successful* if $\mathfrak{s}^{+n}$ is successful for all $n < \omega$. At the end of Chapter III of his book, Shelah claims the following result and promises a proof in [Sheb]:

Claim 5.2.0.1. *Assume $2^{\lambda^{+n}} < 2^{\lambda^{+(n+1)}}$ for all $n < \omega$. If an AEC $\mathbf{K}$ has an ω-successful good λ-frame, is categorical in λ, and $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$ (the class of $\lambda^{+\omega}$-Galois-saturated models in $\mathbf{K}$) is categorical in a $\lambda' > \lambda^{+\omega}$, then $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$ is categorical in all $\lambda'' > \lambda^{+\omega}$.*

Can one build a good frame in ZFC? In Chapter IV of his book, Shelah proves:

Theorem 5.2.44. *Let $\mathbf{K}$ be an AEC categorical in cardinals of arbitrarily high cofinality. Then there exists a cardinal λ such that $\mathbf{K}$ is categorical in λ and $\mathbf{K}$ has a good λ-frame.*

Theorem 5.2.45. *Let $\mathbf{K}$ be an AEC with amalgamation and no maximal models. If $\mathbf{K}$ is categorical in a $\lambda \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$, then there exists $\mu < \aleph_{\mathrm{LS}(\mathbf{K})^+}$ such that $\mathbf{K}^{\mu\text{-sat}}$ has a good μ-frame.*

The proofs of both theorems first get some tameness (and amalgamation in the first case), and then use it to define a good frame in λ by making use of the lower cardinals (as in Theorem 5.5.17).

Assuming amalgamation and weak GCH, Shelah shows that the good frame can be taken to be ω-successful. Combining this with Claim 5.2.0.1, Shelah deduces the eventual categoricity conjecture in AECs with amalgamation:

Theorem 5.2.46. *Assume Claim 5.2.0.1 and $2^\theta < 2^{\theta^+}$ for all cardinals θ. Let $\mathbf{K}$ be an AEC with amalgamation. If $\mathbf{K}$ is categorical in* some *$\lambda \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$, then $\mathbf{K}$ is categorical in* all *$\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$.*

Note that the first steps in the proof are again proving enough tameness to make the construction of an ω-successful good frame.

5.3 Tameness: what and where

5.3.1 What – Definitions and basic results

Syntactic types have nice locality properties: different types must differ on a formula and this difference can be seen by restricting the type to the finite[23] set of parameters in such a formula. Galois types do not necessarily have this

[23] Or larger if the logic allows infinitely many free variables.

property. Indeed, assuming the existence of a monster model $\mathfrak{C}$, this would imply a strong closure property on $\mathrm{Aut}(\mathfrak{C})$. Nonetheless, a generalization of this idea, called *tameness*, has become a key tool in the study of AECs.

For a set A, we write $P_\kappa A$ for the collection of subsets of A of size less than κ. We also define an analog notation for models: for $M \in \mathbf{K}_{\geq \kappa}$:

$$P^*_\kappa M := \{M_0 \in \mathbf{K}_{<\kappa} : M_0 \leq M\}$$

Definition 5.3.1. $\mathbf{K}$ is $< \kappa$-tame if, for all $M \in \mathbf{K}$ and $p \neq q \in \mathrm{gS}^1(M)$, there is $A \in P_\kappa|M|$ such that $p \upharpoonright A \neq q \upharpoonright A$.

For $\kappa > \mathrm{LS}(\mathbf{K})$, it is equivalent if we quantify over $P^*_\kappa M$ (models) rather than $P^*_\kappa|M|$ (sets). Quantifying over sets is useful to isolate notions such as $< \aleph_0$-tameness. Several parametrizations (e.g. of the length of type) and variations exist. Below we list a few that we use; note that, in all cases, writing "κ" in place of "$< \kappa$" should be interpreted as "$< \kappa^+$".

Definition 5.3.2. Suppose $\mathbf{K}$ is an AEC with $\kappa \leq \lambda$.

1. $\mathbf{K}$ is $(< \kappa, \lambda)$-*tame* if for any $M \in \mathbf{K}_\lambda$ and $p \neq q \in \mathrm{gS}^1(M)$, there is some $A \in P_\kappa|M|$ such that $p \upharpoonright A \neq q \upharpoonright A$.
2. $\mathbf{K}$ is $< \kappa$-*type short* if for any $M \in \mathbf{K}$, index set I, and $p \neq q \in \mathrm{gS}^I(M)$, there is some $I_0 \in P_\kappa I$ such that $p^{I_0} \neq q^{I_0}$.
3. $\mathbf{K}$ is κ-*local* if for any increasing, continuous $\langle M_i \in \mathbf{K} : i \leq \kappa\rangle$ and any $p \neq q \in \mathrm{gS}(M_\kappa)$, there is $i_0 < \kappa$ such that $p \upharpoonright M_{i_0} \neq q \upharpoonright M_{i_0}$.
4. $\mathbf{K}$ is κ-*compact* if for any increasing, continuous $\langle M_i : i \leq \kappa\rangle$ and increasing $\langle p_i \in \mathrm{gS}(M_i) : i < \kappa\rangle$, there is $p \in \mathrm{gS}(M)$ such that $p_i \leq p$ for all $i < \kappa$.
5. $\mathbf{K}$ is *fully* $<$ κ*-tame and -type short* if for any $M \in \mathbf{K}$, index set I, and $p \neq q \in \mathrm{gS}^I(M)$, there are $A \in P_\kappa|M|$ and $I_0 \in P_\kappa I$ such that $p^{I_0} \upharpoonright A \neq q^{I_0} \upharpoonright A$.

When κ is omitted, we mean that there exists κ such that the property holds at κ. For example, "$\mathbf{K}$ is tame" means that there exists κ such that $\mathbf{K}$ is $< \kappa$-tame. Note that definitions of locality and compactness implicitly assume κ is regular.

These types of properties are often called locality properties for AECs because they assert, in different ways, that Galois types are locally defined.

Each of these notions also has a *weak* version: weak $< \kappa$-tameness, etc. This variation means that the property holds when the domain is Galois-saturated.

A brief summary of the ideas is below. In each (and throughout this paper), "small" is used to mean "of size less than κ".

- $< \kappa$-tameness says that different types differ over some small subset of the domain.
- $< \kappa$-type shortness says that different types differ over some small subset of their length.

- κ-locality says that each increasing chain of Galois types of length κ has *at most* one upper bound.
- κ-compactness says that each increasing chain of Galois types of length κ has *at least* one upper bound.[24]

A combination of tameness and type shortness allows us to conceptualize Galois types as sets of smaller types.

There are several relations between the properties:

Proposition 5.3.3.

1. *For $\kappa > \mathrm{LS}(\mathbf{K})$, $< \kappa$-type shortness implies $< \kappa$-tameness.*
2. *$< \mathrm{cf}(\kappa)$-tameness implies κ-locality.*
3. *μ-locality for all $\mu < \lambda$ implies $(\mathrm{LS}(\mathbf{K}), \lambda)$-tameness.*
4. *μ-locality for all $\mu < \lambda$ implies λ-compactness.*

As discussed, one of the draws of working in a short and tame AEC is that Galois types behave much more like first-order syntactic types in the sense that a Galois type $p \in \mathrm{gS}(M)$ can be identified with the collection $\{p^{I_0} \upharpoonright M_0 : I_0 \in P_\kappa \ell(p) \text{ and } M_0 \in P^*_\kappa M\}$ of its small restrictions:

Proposition 5.3.4. $\mathbf{K}$ *is fully $< \kappa$-tame and -type short if and only if the map:*

$$p \in \mathrm{gS}(M) \mapsto \{p^{I_0} \upharpoonright M_0 : I_0 \in P_\kappa I, M_0 \in P^*_\kappa M\}$$

is injective.

In fact, one can see these small restrictions as formulas (this will be used later to generalize heir and coheir to AECs). This productive intuition can be made exact using *Galois Morleyization*. Start with an AEC $\mathbf{K}$ and add to the language an α-ary predicate R_p for each $N \in \mathbf{K}$, each $p \in \mathrm{gS}^\alpha(\emptyset; N)$, and each $\alpha < \kappa$. This gives us an infinitary language $\widehat{L}$. Then expand each $M \in \mathbf{K}$ to a $\widehat{L}$-structure $\widehat{M}$ by setting $R_p(\bar{a})$ to be true in M if and only if $\mathrm{gtp}(\bar{a}/\emptyset; M) = p$. We obtain a class $\widehat{K}^{<\kappa} := \{\widehat{M} \mid M \in \mathbf{K}\}$. $\widehat{K}$ has relations of infinite arity but it still behaves like an AEC. We call $\widehat{K}^{<\kappa}$ the $< \kappa$-*Galois Morleyization* of $\mathbf{K}$. The connection between tameness and $\widehat{K}$ is given by the following theorem:

Theorem 5.3.5. *Let* $\mathbf{K}$ *be an AEC. The following are equivalent:*

1. $\mathbf{K}$ *is fully $< \kappa$-tame and -type short.*
2. *The map $\mathrm{gtp}(\bar{b}/M; N) \mapsto tp_{qf\text{-}\mathbb{L}_{\kappa,\kappa}(\widehat{L})}(\bar{b}/\widehat{M}; \widehat{N})$ is an injection.*

[24] All AECs are ω-compact and global compactness statements have large cardinal strength; see [Shec, Section 2].

Here, the Galois type is computed in $\mathbf{K}$, and the type on the right is the (syntactic) quantifier-free $\mathbb{L}_{\kappa,\kappa}$-type in the language $\widehat{L}$. Note that the locality hypothesis in (1) can be weakened to $< \kappa$-tameness if in (2) we ask that $\ell(\bar{b}) = 1$. Several other variations are possible.

The Galois Morleyization gives a way to directly use syntactic tools (such as the results of stability theory inside a model, see for example [She09c, Chapter V.A]) in the study of tame AECs. See for example Theorem 5.5.14.

Another way to see tameness is as a topological separation principle: consider the set X_M of Galois types over M. For a fixed κ, we can give a topology on X_M with basis given by sets of the form $U_{p,A} := \{q \in \mathrm{gS}(M) \mid A \subseteq |M| \wedge q \upharpoonright A = p\}$, for p a Galois type over A and $|A| < \kappa$. This is the same topology as that generated by quantifier-free $\mathbb{L}_{\kappa,\kappa}$-formulas in the $< \kappa$-Galois Morleyization. Thus one can show:

Theorem 5.3.6. *Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. $\mathbf{K}$ is $(< \kappa, \lambda)$-tame if and only if for any $M \in \mathbf{K}_\lambda$, the topology on X_M defined above is Hausdorff.*

5.3.2 Where – Examples and counterexamples

5.3.2.1 Examples

Several "mathematically interesting" classes turn out to be tame. Moreover, there are several general ways to *derive* tameness from structural assumptions. We list some here, roughly in decreasing order of generality.

1. **Locality from large cardinals**
   Large cardinals κ allow the generalization of compactness results from first-order logic to $\mathbb{L}_{\kappa,\omega}$ in various ways (see, for instance, [Jec03, Lemma 20.2]). Since tameness is a weak form of compactness, these generalizations correspond to compactness results in AECs that can be "captured" by $\mathbb{L}_{\kappa,\omega}$. We state a simple version of these results here:

   Theorem 5.3.7. *Suppose $\mathbf{K}$ is an AEC with $\mathrm{LS}(\mathbf{K}) < \kappa$.*

   - *If κ is weakly compact, then $\mathbf{K}$ is $(< \kappa, \kappa)$-tame.*
   - *If κ is measurable, then $\mathbf{K}$ is κ-local.*
   - *If κ is strongly compact, then $\mathbf{K}$ is fully $< \kappa$-tame and -type short.*

   These results can be strengthened in various ways. First, they apply also to AECs that are explicitly axiomatized in $\mathbb{L}_{\kappa,\omega}$. The key fact is that ultraproducts by κ-complete ultrafilters preserve the AEC (the proof uses the presentation theorem, Theorem 5.2.9). Second, each large cardinal can be replaced by its "almost" version: for example, almost strongly compact means that, *for each $\delta < \kappa$*, $\mathbb{L}_{\delta,\delta}$ is κ-compact; equivalently, given a κ-complete filter, *for each $\delta < \kappa$*, it can be extended to a δ-*complete* ultrafilter. See [BU, Definition 2.1] for a full list of the "almost" versions.

Note that other structural properties (such as amalgamation) follow from the combination of large cardinals with categoricity. Thus these large cardinals make the development of a structure theory (culminating for example in the existence of well-behaved independence notion, see Corollary 5.5.54) much easier.

2. **Weak tameness from categoricity under amalgamation**
Recall from Section 5.3.1 that an AEC $\mathbf{K}$ is $(\chi_0, < \chi)$-weakly tame if for every *Galois-saturated* $M \in \mathbf{K}_{<\chi}$, every $p \neq q \in \mathrm{gS}(M)$, there exists $M_0 \leq M$ with $M_0 \in \mathbf{K}_{\leq \chi_0}$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$. It is known that, in AECs with amalgamation categorical in a sufficiently high cardinal, weak tameness holds below the categoricity cardinal. More precisely:

Theorem 5.3.8. *Assume that* $\mathbf{K}$ *is an AEC with amalgamation and no maximal models which is categorical in a* $\lambda > \mathrm{LS}(\mathbf{K})$.

(a) Let χ *be a limit cardinal such that* $\mathrm{cf}(\chi) > \mathrm{LS}(\mathbf{K})$. *If the model of size* λ *is* χ*-Galois-saturated, then there exists* $\chi_0 < \chi$ *such that* $\mathbf{K}$ *is* $(\chi_0, < \chi)$*-weakly tame.*

(b) If the model of size λ *is* H_1*-Galois-saturated, then there exists* $\chi_0 < H_1$ *such that whenever* $\chi \geq H_1$ *is so that the model of size* λ *is* χ*-Galois-saturated, we have that* $\mathbf{K}$ *is* $(\chi_0, < \chi)$*-weakly tame*[25].

Remark 5.3.9. The model in the categoricity cardinal λ is χ-Galois-saturated whenever $\mathrm{cf}(\lambda) \geq \chi$ (e.g. if λ is a successor) or (by Theorem 5.2.38) if[26] $\lambda \geq h(\chi)$.

The proof of Theorem 5.3.8 heavily uses Ehrenfeucht-Mostowski models to transfer the behavior below H_1 to a larger model that is generated by a nice enough linear order. Then the categoricity assumption is used to embed every model of size χ into such a model of size λ.

Theorem 5.3.8 is key to prove several of the categoricity transfers listed in Section 5.2.4.

3. **Tameness from categoricity and large cardinals**
The hypotheses in the last two examples can be combined advantageously.

Theorem 5.3.10. *Let* $\mathbf{K}$ *be an AEC and let* $\kappa > \mathrm{LS}(\mathbf{K})$ *be a measurable cardinal. If* $\mathbf{K}$ *is categorical in a* $\lambda \geq \kappa$*, then* $\mathbf{K}_{[\kappa,\lambda)}$ *has amalgamation and is* $(\kappa, < \lambda)$*-tame.*

In particular, if there exists a proper class of measurable cardinals and $\mathbf{K}$ is categorical in a proper class of cardinals, then $\mathbf{K}$ is tame. It is conjectured that the large cardinal hypothesis is not necessary. Note that the tameness here is "full", i.e. not the weak tameness in Theorem 5.3.8.

[25] Note that χ_0 does not depend on χ.

[26] A more clever application of Theorem 5.2.38 shows that it is enough to have $\lambda \geq \sup_{\theta < \chi} h(\theta^+)$.

4. **Tameness from stable forking**
   Suppose that the AEC $\mathbf{K}$ has amalgamation and a stable "forking-like" relation $\mathop{\smile\!\!\!\!\!\mid}$ (see Definition 5.5.7). That is, we ask that there is a notion "$p \in \mathrm{gS}(N)$ does not fork over M" for $M \leq N$ satisfying the usual monotonicity properties, uniqueness, and local character[27] : there exists a cardinal $\bar{\kappa} = \bar{\kappa}(\mathop{\smile\!\!\!\!\!\mid})$ such that for every $p \in \mathrm{gS}(N)$, there is $M \leq N$ of size less than $\bar{\kappa}$ such that p does not fork over M (see more on such relations in Section 5.5).

   Then, given any two types $p, q \in \mathrm{gS}(N)$ we can find $M \leq N$ over which both types do not fork over and so that $\|M\| < \bar{\kappa}$. If $p \upharpoonright M = q \upharpoonright M$, then uniqueness implies $p = q$. Thus, $\mathbf{K}$ is $(< \bar{\kappa})$-tame.

5. **Universal Classes**
   A *universal class* is a class $\mathbf{K}$ of structures in a fixed language $L(\mathbf{K})$ that is closed under isomorphism, substructure, and unions of increasing chains. In particular, $(\mathbf{K}, \subseteq)$ is an AEC with Löwenheim-Skolem-Tarski number $|L(\mathbf{K})| + \aleph_0$.

   In a universal class, any partial isomorphism extends uniquely to an isomorphism (just take the closure under the functions). This fact is key in the proof of:

   Theorem 5.3.11. *Any universal class is fully* $(< \aleph_0)$*-tame and short.*

   Thus, for instance, the class of locally finite groups (ordered with subgroup) is tame. Theorem 5.3.11 generalizes to any AEC $\mathbf{K}$ equipped with a notion of "generated by" which is (in a sense) canonical (for universal classes, this notion is just the closure under the functions). Note that this does not need to assume that $\mathbf{K}$ has amalgamation.

6. **Tame finitary AECs**
   A finitary AEC $\mathbf{K}$ is defined by several properties (including amalgamation and $\mathrm{LS}(\mathbf{K}) = \aleph_0$), but the key notion is that the strong substructure relation $\leq$ has *finite character*. This means that, for $M, N \in \mathbf{K}$, we have $M \leq N$ if and only if $M \subseteq N$ and:

   For every $\bar{a} \in {}^{<\omega}M$, we have that $\mathrm{gtp}(\bar{a}/\emptyset; M) = \mathrm{gtp}(\bar{a}/\emptyset; N)$.

   This means that there is a finitary test for when $\leq$ holds between two models that are already known to be members of $\mathbf{K}$. This definition is motivated by the observation that this condition holds for any AEC axiomatized in a countable fragment of $\mathbb{L}_{\omega_1,\omega}$ by the Tarski-Vaught test[28]. Homogeneous model theory can be seen as a special case of the study of finitary AECs. Hyttinen and Kësäla have shown that every $\aleph_0$-stable $\aleph_0$-tame finitary AEC is $(< \aleph_0)$-tame. These classes seem very amenable to some classification theory. For example, an $\aleph_0$-tame finitary AEC categorical in some uncountable λ is categorical in *all* $\lambda' \geq \min(\lambda, H_1)$.

[27] The extension property is not needed here.
[28] Kueker [Kue08] has asked whether any finitary AEC must be $\mathbb{L}_{\infty,\omega}$-axiomatizable.

Recent work has even developed some geometric stability theory in a larger class (Finite U-Rank classes, which included quasiminimal classes below) [HK16].

7. **Homogeneous model theory**
Homogeneous model theory takes place in the context of a large "monster model" (for a first-order theory T) that omits a set of types D, but is still as saturated as possible with respect to this omission. The notion of "as saturated as possible" is captured by requiring it to be sequentially homogeneous rather than model homogeneous. Note that the particular case when $D = \emptyset$ is the elementary case. In this context, amalgamation, joint embedding, and no maximal models hold for free and Galois types are first-order syntactic types. This identification means that the AEC of models of T omitting D (ordered with elementary substructure) is fully $(< \aleph_0)$-tame and short. Homogeneous model theory has a rich classification theory in its own right, with connections to continuous first-order logic (see the historical remarks).

8. **Averageable Classes**
Averageable classes are type omitting classes $\mathrm{EC}(T, \Gamma)$ (ordered with a relation $\leq$) that are nice enough to have a relativized ultraproduct that preserves the omission of types in Γ and satisfies enough of Łoś' Theorem to interact well with $\leq$. This relativized ultraproduct gives enough compactness to show that types are syntactic (and much more), which implies that an averageable class is fully $(< \aleph_0)$-tame and short. Examples of averageable classes include torsion modules over PIDs and densely ordered abelian groups with a cofinal and coinitial $\mathbb{Z}$-chain.

9. **Continuous first-order logic**
Continuous first-order logic can be studied in a fragment of $\mathbb{L}_{\omega_1,\omega}$ by using the infinitary logic to have a standard copy of $\mathbb{Q}$ and then studying dense subsets of complete metric spaces. Although the logic $\mathbb{L}_{\omega_1,\omega}$ is incompact, the fragment necessary to code this information is compact (as evidenced by the metric ultrapower and compactness results in continuous first-order logic), so the classes are fully $(< \aleph_0)$-tame and short. Beyond first-order, continuous model theory can be done in the so-called metric AECs, where a notion of tameness (d-tameness) can also be defined.

10. **Quasiminimal Classes**
A quasiminimal class is an AEC satisfying certain additional axioms; most importantly, the structures carry a pregeometry with certain nice properties. The axioms directly imply that Galois types over *countable* models are quantifier-free first-order types, and the excellence axiom can be used to transfer this to uncountable models. Therefore quasiminimal classes are $< \aleph_0$-tame. Examples of quasiminimal classes include covers of $\mathbb{C}^\times$ and Zilber fields with pseudoexponentiation. Note that it can be shown (from the countable closure axiom) that these classes are strictly

$\mathbb{L}_{\omega_1,\omega}(Q)$-definable. This gives important examples of categorical AECs that are not finitary.

11. **λ-saturated models of a superstable first-order theory**
Let T be a first-order superstable theory. We know that unions of increasing chains of λ-saturated models are λ-saturated and that models of size λ have saturated extensions of size at most $\lambda + 2^{|T|}$. Thus the class of λ-saturated models of T (ordered with elementary substructure) forms an AEC $\mathbf{K}^T_\lambda$ with $\mathrm{LS}(\mathbf{K}^T_\lambda) \leq \lambda + 2^{|T|}$. Furthermore, this class has a monster model and is fully $(< \aleph_0)$-tame and short.

12. **Superior AECs**
Superior AECs are a generalization of what some call excellent classes. An AEC is *superior* if it carries an axiomatic notion of forking for which one can state multi-dimensional uniqueness and extension properties. A combination of these gives some tameness:

Theorem 5.3.12. *Let* $\mathbf{K}$ *be a superior AEC with weak* $(\lambda, 2)$*-uniqueness and* λ*-extension for some* $\lambda \geq \mathrm{LS}(\mathbf{K}) + \bar{\kappa}(\mathbf{K})$*. Then* $\mathbf{K}$ *is* λ^+*-local. In particular, it is* (λ, λ^+)*-tame.*

13. **Hrushovski fusions**
Villaveces and Zambrano have studied Hrushovski's method of fusing pregeometries over disjoint languages as an AEC with strong substructure being given by self-sufficient embedding. They show that these classes satisfy a weakening of independent 3-amalgamation. This weakening is still enough to show, as with superior AECs, that the classes are $\mathrm{LS}(\mathbf{K})$-tame.

14. **${}^\perp N$ when N is an abelian group**
Given a module N, ${}^\perp N$ is the class of modules $\{M : \mathrm{Ext}^n(M, N) = 0$ for all $1 \ \leq n < \omega\}$. We make this into an AEC by setting $M \leq_\perp M'$ if and only if $M'/M \in {}^\perp N$. If N is an abelian group, then ${}^\perp N$ is set of all abelian groups that are p-torsion free for all p in some collection of primes P.

Theorem 5.3.13. *If* N *is an abelian group, then* ${}^\perp N$ *is* $< \aleph_0$*-tame.*

Moreover, such a ${}^\perp N$ is Galois-stable in exactly the cardinals $\lambda = \lambda^\omega$.

15. **Algebraically closed, rank one valued fields**
Let $\mathrm{ACVF}_\mathbb{R}$ be the $\mathbb{L}_{\omega_1,\omega}$-theory of an algebraically closed valued field such that the value group is Archimedean; equivalently, the value group can be embedded into $\mathbb{R}$. After fixing the characteristic, this AEC has a monster model and Galois types are determined by syntactic types. Thus the class is fully $< \aleph_0$-tame and -type short. This determination of Galois types can be seen either through algebraic arguments or the construction of an appropriate ultraproduct.

Such a class cannot have an uncountable ordered sequence, so it has the $\aleph_0$-order property of length α for every $\alpha < \omega_1$, but it does not have the $\aleph_0$-order property of length ω_1.

5.3.2.2 Counterexamples

Life would be too easy if all AECs were tame. Above we have seen that several natural mathematical classes are tame; in contrast, all the known counterexamples to tameness are pathological[29], with the most natural being the Baldwin-Shelah example of short exact sequences. We list the known ones below in increasing "strength".

1. **The Hart-Shelah example**
   The Hart-Shelah examples are a family of examples axiomatized by complete sentences in $\mathbb{L}_{\omega_1,\omega}$.

   Theorem 5.3.14. *For each $n < \omega$, there is an AEC $\mathbf{K}_n$ that is axiomatized by a complete sentence in $\mathbb{L}_{\omega_1,\omega}$ with $\mathrm{LS}(\mathbf{K}_n) = \aleph_0$ and disjoint amalgamation such that:*

   (a) $\mathbf{K}_n$ is $(\aleph_0, \aleph_{n-1})$-tame (in fact, the types are first-order syntactic);
   (b) $\mathbf{K}_n$ is categorical in $[\aleph_0, \aleph_n]$;
   (c) $\mathbf{K}_n$ is Galois-stable in μ for $\mu \in [\aleph_0, \aleph_{n-1}]$; and
   (d) Each of these properties is sharp. That is:
      i. $\mathbf{K}_n$ is not $(\aleph_0, \aleph_n)$-tame,
      ii. $\mathbf{K}_n$ is not categorical in $\aleph_{n+1}$[30].
      iii. $\mathbf{K}_n$ is not Galois-stable in $\aleph_n$.

   Each model $M \in \mathbf{K}_n$ begins with an index set I (called the spine); the direct sum $G := \oplus_{[I]^{n+2}}\mathbb{Z}_2$; $G^* \subseteq [I]^k \times G$ with a projection $\pi : G^* \to [I]^{n+2}$ such that each stalk $G^*_u = \pi^{-1}\{u\}$ has a regular, transitive action of G on it; and, similarly, $H^* = [I]^{n+2} \times H$ with a projection $\pi' : H^* \to [I]^k$ such that each stalk has an action of $\mathbb{Z}_2$ on it. So far, the structure described (along with the extra information required to code it) is well-behaved and totally categorical. Added to this is a $n+3$-ary relation $Q \subseteq (G^*)^{n+2} \times H^*$ such that $Q(u_1, \dots, u_{n+2}, v)$ is intended to code

   - there are exactly $n+3$ elements of I that make up the projections of $u_1, \dots, u_{n+2}, v$ (so each $(n+2)$-element subset shows up exactly once in the projections); and
   - the sum of the second coordinates evaluated at $\pi'(v)$ is equal to some fixed function of the $n+3$ elements of the projections.

   This coding allows one to "hide the zeros" and find nontameness at $\aleph_n$. The example shows that Theorem 5.2.33 is sharp.

   It should be noted that the ideas used in constructing the Hart-Shelah examples come from constructions that characterize various cardinals. Thus, although the construction takes place in ZFC, it still involves

[29] In the dictionary sense that they were constructed as counter-examples.

[30] Note that this follows from (1(d)i) by (the proof of) the upward categoricity transfer of Grossberg and VanDieren [GV06a].

set-theoretic ideas. Work in preparation by Shelah and Villaveces [SV] contains an extension of the Hart-Shelah example to larger cardinals, proving:

Theorem 5.3.15. *Assume the generalized continuum hypothesis. For each* λ *and* $k < \omega$, *there is* $\psi_k^\lambda \in \mathbb{L}_{(2^\lambda)^+,\omega}$ *that is categorical in* $\lambda^{+2}, \ldots, \lambda^{+(k-1)}$ *but not in* $\beth_{k+1}(\lambda)^+$.

As for the countable case, this example is likely not to be tame.

2. **The Baldwin-Shelah example**
The Baldwin-Shelah example **K** consists of several short exact sequences, each beginning with $\mathbb{Z}$.

$$\begin{tikzcd}
 & & H_j \arrow[dr] & & \\
0 \arrow[r] & \mathbb{Z} \arrow[ur] \arrow[r] \arrow[dr] & H_i \arrow[r] & G \arrow[r] & 0 \\
 & & H_k \arrow[ur] & &
\end{tikzcd}$$

Formally this consists of of sorts Z, G, I, and H with a projection $\pi : H \to I$ and group operations and embeddings such that each fiber $H_i := \pi^{-1}(\{i\})$ is a group that is in the middle of a short exact sequence.

The locality properties of Galois types over a model depend heavily on the group G used. The key observation is that, given $i, j \in I$, their Galois types are equal precisely when there is an isomorphism of the fibers $\pi^{-1}\{i\}$ and $\pi^{-1}\{j\}$ that commute with the rest of the short exact sequence. Thus, Baldwin and Shelah consider an $\aleph_1$-free, not free, not Whitehead group[31] G^* of size $\aleph_1$. With G^* in hand, we can construct a counterexample to $(\aleph_0, \aleph_1)$-tameness: set i_0 and i_1 to be in a short exact sequence that ends in G^* such that $\pi^{-1}\{i_0\} = G^* \oplus \mathbb{Z}$ and $\pi^{-1}\{i_1\} = H$ is not isomorphic to $G^* \oplus \mathbb{Z}$; such a group exists exactly because H is not Whitehead. Then, by the observation above, i_0 and i_1 have different types over the entire uncountable set G^*. However, any countable approximation G_0 of G^* will see that i_0 and i_1 have the same Galois type over it: the countable approximation will have that the fibers over i_0 and i_1 are both the middle of a short exact ending in G_0. By the choice of G^*, G_0 is free, thus Whitehead, so these fibers are both isomorphic to $G_0 \oplus \mathbb{Z}$. This isomorphism witnesses the equality of the Galois types of i_0 and i_1 over the countable approximation.

[31]It is a ZFC theorem that such a group exists at $\aleph_1$. Having such a group at κ (κ-free, not free, not Whitehead of size κ) is, in the words of Baldwin and Shelah, "sensitive to set theory". The known sensitivities are summarized in [Bon14c, Section 8], primarily drawing on work in [MS94] and [EM02].

Given a κ version G^*_κ of this group allows one to construct a counterexample to $(< \kappa, \kappa)$-tameness. Indeed, all that is necessary is that G^*_κ is 'almost Whitehead:' it is not Whitehead, but every strictly smaller subgroup of it is.

3. **The Shelah-Boney-Unger example**
While the Baldwin-Shelah example reveals a connection between tameness and set theory, the Shelah-Boney-Unger example shows an outright equivalence between certain tameness statements and large cardinals. For each cardinal $\sigma^\omega = \sigma$, there is an AEC $\mathbf{K}_\sigma$ that consists of an index predicate J with a projection $Q : H \to J$ such that each fiber $Q^{-1}\{j\}$ has a specified structure and a projection $\pi : H \to I$[32]. Given some partial order $(\mathcal{D}, \lhd)$ and set of functions $\mathcal{F}$ with domain $\mathcal{D}$, filtrations $\{M_{\ell,d} : d \in \mathcal{D}\}$ of a larger models $M_{\ell,\mathcal{D}}$, all from $\mathbf{K}_\sigma$, are built, for $\ell = 1, 2$. Similar to the Baldwin-Shelah example, types p_d and q_d are defined such that the types are equal if and only if there is a nice isomorphism between $M_{1,d}$ and $M_{2,d}$; the same is true of $p_\mathcal{D}$ and $q_\mathcal{D}$. Thus, various properties of type locality ($p_\mathcal{D} = q_\mathcal{D}$ following from $p_d = q_d$ for all $d \in \mathcal{D}$) is once more coded by "isomorphism locality".
In turn, the structure was built so that a nice isomorphism between $M_{1,\mathcal{D}}$ and $M_{2,\mathcal{D}}$ is equivalent to a combinatorial property $\#(\mathcal{D}, \mathcal{F})$.

Definition 5.3.16.

- Given functions f and g with the same domain, we define $f \leq^* g$ to hold if and only if there is some $e : \operatorname{ran} g \to \operatorname{ran} f$ such that $f = e \circ g$.
- Given a function f with a domain D that is partially ordered by $\leq_D$, we define $\operatorname{ran}^* f = \bigcap_{d \in D} \operatorname{ran}\left(f \restriction \{d' \in D : d \leq d'\}\right)$ to be the eventual range of f.
- $\#(\mathcal{D}, \mathcal{F})$ holds if and only if there are $f^* \in \mathcal{F}$ and a collection of nonempty finite sets $\{u_f \subseteq \operatorname{ran}^* f : f^* \leq^* f\}$ such that, given any e witnessing $f^* \leq^* f$, $e \restriction u_f$ is a bijection from u_f to u_{f^*}.

So $\#(\mathcal{D}, \mathcal{F})$ eventually puts some kind of structure on functions in $\mathcal{F}$. Shockingly, this principle can, under the right assumption on $\mathcal{D}$ and $\mathcal{F}$, define a very complete (ultra)filter on $\mathcal{D}$: for each f with $f \leq^* f^*$, set $i_f \in u_f$ to be the unique image of $\min u_f$ by some e witnessing $f^* \leq^* f$. Then, for $A \subseteq \mathcal{D}$,

$$A \in U \iff \exists d \in \mathcal{D}, f \in \mathcal{F} \left(f^{-1}\{i_f\} \cap \{d' \in \mathcal{D} : d \lhd d'\} \subseteq A\right)$$

Thus, we get that type locality in $\mathbf{K}_\sigma$ implies the existence of filters and ultrafilters used in the definitions of large cardinals; the converse is mentioned above.

This argument can be used to give the following theorems.

[32] Although there are two projections, they are used differently: the projection Q is a technical device to code isomorphisms of structures via equality of Galois types, while the interaction of (a fiber of) H and I is more interesting.

Theorem 5.3.17. *Let κ such that $\mu^\omega < \kappa$ for all $\mu < \kappa$.*

(a) If[33] *$\kappa^{<\kappa} = \kappa$ and every AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) < \kappa$ is $(< \kappa, \kappa)$-tame, then κ is almost weakly compact.*

(b) If every AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) < \kappa$ is κ-local, then κ is almost measurable.

(c) If every AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) < \kappa$ is $< \kappa$-tame, then κ is almost strongly compact.

We obtain a characterization of the statement "all AECs are tame" in terms of large cardinals.

Corollary 5.3.18. *All AECs are tame if and only if there is a proper class of almost strongly compact cardinals.*

Note that Corollary 5.3.18 says nothing about "well-behaved" classes of AECs such as AECs categorical in a proper class of cardinals. In fact, Theorem 5.3.10 shows that the consistency strength of the statement "all AECs are tame" is much higher than that of the statement "all unboundedly categorical AECs are tame".

5.4 Categoricity transfer in universal classes: an overview

In this section, we sketch a proof of Theorem 5.1.2, emphasizing the role of tameness in the argument:

Theorem 5.4.1. *Let $\mathbf{K}$ be a universal class. If $\mathbf{K}$ is categorical in cardinals of arbitrarily high cofinality*[34]*, then $\mathbf{K}$ is categorical on a tail of cardinals.*

The arguments in this section are primarily from Vasey [Vasf].

Note that (as pointed out in Section 5.2.4), we can replace the categoricity hypotheses of Theorem 5.4.1 by categoricity in a *single* "high-enough" cardinal of "high-enough" cofinality.

We avoid technical definitions in this section, instead referring the reader to Section 5.2 or Section 5.5.

So let $\mathbf{K}$ be a universal class categorical in cardinals of arbitrarily high cofinality. To prove the categoricity transfer, we first show that $\mathbf{K}$ has several structural properties that hold in elementary classes. As we have seen, amalgamation is one such property.

[33] The additional cardinal arithmetic here can be dropped at the cost of only concluding (κ is weakly compact)^L.

[34] This cofinality restriction is only used to obtain amalgamation. See the historical remarks for more.

5.4.1 Step 1: Getting amalgamation

It is not clear how to directly prove amalgamation in all cardinals, but Theorem 5.2.44 is a deep result of Shelah which says (since good frames must have amalgamation) that it holds for models of *some* suitable size[35].

This leads to a new fundamental question:

Question 5.4.2. If an AEC $\mathbf{K}$ has amalgamation in a cardinal λ, under what condition does it have amalgamation above λ?

One such condition is *excellence* (briefly, excellence asserts strong uniqueness of n-dimensional amalgamation results). However, it is open whether it follows from categoricity, even for classes of models of an uncountable first-order theory. Excellence also gives much more, and (for now) we are only interested in amalgamation. Another condition would be the existence of *large cardinals*. For example, a strongly compact κ with $\mathrm{LS}(\mathbf{K}) < \kappa \leq \lambda$ would be enough.

At that point, we recall a key theme in the study of tameness: when large cardinals appear in a model-theoretic result, tameness[36] can often replace them. For the purpose of an amalgamation transfer it is not clear that this suffices. For one thing, one can ask what tameness really means without amalgamation (of course, its definition makes sense, but how do we get a handle on the transitive closure of atomic equality, Definition 5.2.16.(2)). In the case of universal classes, this question has a nice answer: even without amalgamation, equality of Galois types is witnessed by an isomorphism and, in fact, tameness holds for free! This is Theorem 5.3.11. From its proof, we isolate a technical weakening of amalgamation:

Definition 5.4.3. An AEC $\mathbf{K}$ has *weak amalgamation* if whenever $\mathrm{gtp}(a_1/M; N_1) = \mathrm{gtp}(a_2/M; N_2)$, there exists $M_1 \in \mathbf{K}$ with $M \leq M_1 \leq N_1$ and $a_1 \in |M_1|$ such that (a_1, M, M_1) is *atomically equivalent* to (a_2, M, N_2).

It turns out that universal classes have weak amalgamation: we can take M_1 to be the closure of $|M_1| \cup \{a\}$ under the functions of N_1 and expand the definition of equality of Galois types.

We now rephrase Question 5.4.2 as follows:

Question 5.4.4. Let $\mathbf{K}$ be an AEC which has amalgamation in a cardinal λ. Assume that $\mathbf{K}$ is λ-tame and has weak amalgamation. Under what condition does it have amalgamation above λ?

To make progress, a characterization of amalgamation will come in handy (this lemma is reminiscent of Proposition 5.2.18.(1)):

[35] This is the only place where we use the cofinality assumptions on the categoricity cardinals.

[36] Or really, a "tameness-like" property like full tameness and shortness.

Lemma 5.4.5. *Let* $\mathbf{K}$ *be an AEC with weak amalgamation. Then* $\mathbf{K}$ *has amalgamation if and only if for any* $M \in \mathbf{K}$*, any Galois type* $p \in \mathrm{gS}(M)$*, and any* $N \geq M$*, there exists* $q \in \mathrm{gS}(N)$ *extending* p*.*

The proof is easy if one assumes that atomic equivalence of Galois types is transitive. Weak amalgamation is a weakening of this property, but allows us to iterate the argument (when atomic equivalence is transitive) and obtain full amalgamation.

Now, it would be nice if we could not only extend Galois types, but also extend them canonically. This is reminiscent of first-order *forking*, a basic property of which is that every type has a (unique under reasonable conditions) nonforking extension. Thus, out of the apparently very set-theoretic problem of obtaining amalgamation, forking, a model-theoretic notion, appears in the discussion. What is an appropriate generalization of forking to AECs? Shelah's answer is that the *bare-bone* generalization are the *good* λ*-frames*, see Section 5.2.5. There are several nonelementary setups where a good frame exists (see the next section). For example, Theorem 5.2.44 tells us that a good frame exists in our setup.

Still with the question of transferring amalgamation up in mind, one can ask whether it is possible to transfer an *entire good frame* up. In particular, given a notion of forking for models of size λ, is there one for models of size above λ? This is where tameness starts playing a very important role:

Theorem 5.4.6. *Let* $\mathbf{K}$ *be an AEC with amalgamation. Let* $\mathfrak{s}$ *be a good* λ*-frame with underlying AEC* $\mathbf{K}$*. Then* $\mathfrak{s}$ *extends to a good* $(\geq \lambda)$*-frame (i.e. all the properties hold for models in* $\mathbf{K}_{\geq \lambda}$*)* if and only if $\mathbf{K}$ *is* λ*-tame.*

We give a sketch of the proof in Theorem 5.5.26. Let us also note that not only do the properties of forking transfer, but also the structural properties of $\mathbf{K}$. Thus $\mathbf{K}_{\geq \lambda}$ has no maximal models (roughly, this is obtained using the extension property and the fact that nonforking extensions are nonalgebraic).

Even better, it turns out that not too much amalgamation is needed for the proof of the frame transfer to go through: weak amalgamation is enough! Moreover types can be extended by simply taking their nonforking extension. Thus we obtain:

Theorem 5.4.7. *Let* $\mathbf{K}$ *be an AEC with weak amalgamation. If there is a good* λ*-frame* $\mathfrak{s}$ *with underlying AEC* $\mathbf{K}$ *and* $\mathbf{K}$ *is* λ*-tame, then* $\mathfrak{s}$ *extends to a good* $(\geq \lambda)$*-frame. In particular* $\mathbf{K}_{\geq \lambda}$ *has amalgamation.*

Corollary 5.4.8. *Let* $\mathbf{K}$ *be a universal class categorical in cardinals of arbitrarily high cofinality. Then there exists* λ *such that* $\mathbf{K}_{\geq \lambda}$ *has amalgamation.*

Proof. By Theorem 5.2.44, there is a cardinal λ such that $\mathbf{K}$ has a good λ-frame with underlying class $\mathbf{K}_{\lambda}$. By Theorem 5.3.11, $\mathbf{K}$ is λ-tame and (it is easy to see), $\mathbf{K}$ has weak amalgamation. Now apply Theorem 5.4.7. □

Remark 5.4.9. Theorem 5.4.7 will be used even in the next steps, see the proof of Theorem 5.4.18.

5.4.2 Step 2: Global independence and orthogonality calculus

From the results so far, we see that we can replace $\mathbf{K}$ by $\mathbf{K}_{\geq \lambda}$ if necessary to assume without loss of generality that $\mathbf{K}$ is a universal class[37] categorical in a proper class of cardinals that has amalgamation. Other structural properties such as joint embedding and no maximal models follow readily. In fact, we have just pointed out that we can assume there is a good ($\geq \mathrm{LS}(\mathbf{K})$)-frame with underlying class $\mathbf{K}$. In particular, $\mathbf{K}$ is Galois-stable in all cardinals and has a superstable-like forking notion for types of length one.

What is the next step to get a categoricity transfer? The classical idea is to show that all big-enough models are Galois-saturated (note that by the above we have stability everywhere, so the model in the categoricity cardinal is Galois-saturated). Take M a model in a categoricity cardinal λ and p a nonalgebraic type over M. Assume that there exists $N > M$ of size λ such that p is omitted in N. If we can iterate this property λ^+-many times, we obtain a non λ^+-Galois-saturated models. If $\mathbf{K}$ was categorical in λ^+, this gives a contradiction. More generally, if we can iterate longer to find $N > M$ of size $\mu > \lambda$ such that N omits p and $\mathbf{K}$ is categorical in μ, we also get a contradiction. This is reminiscent of a Vaughtian pair argument and more generally of Shelah's theory of unidimensionality. Roughly speaking, a class is *unidimensional* if it has essentially only one Galois type. Then a model cannot have arbitrarily large extensions omitting the type. Conversely if the class is not unidimensional, then it has two "orthogonal" types and a model would be able to grow by adding more realizations of one type without realizing the other.

So we want to give a sense in which our class $\mathbf{K}$ is unidimensional. If $\mathbf{K}$ is categorical in a successor, this can be done much more easily than for the limit case using Vaughtian pairs. In fact a classical result of Grossberg and VanDieren for tame AECs says:

Theorem 5.4.10. *Suppose $\mathbf{K}$ has amalgamation and no maximal models. If $\mathbf{K}$ is a λ-tame AEC categorical in λ and λ^+, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.*

To study general unidimensionality, we will use a notion of orthogonality. As for forking, we focus on developing a theory of orthogonality for types of length one over models of a single size.

We already have a good ($\geq \mathrm{LS}(\mathbf{K})$)-frame available but for our purpose this is not enough. We will also use a notion of primeness:

Definition 5.4.11. We say an AEC $\mathbf{K}$ *has primes* if whenever $M \leq N$ are in $\mathbf{K}$ and $a \in |N| \backslash |M|$, there is a prime model $M' \leq N$ over $|M| \cup \{a\}$. This

[37] There is a small wrinkle here: if $\mathbf{K}$ is a universal class, $\mathbf{K}_{\geq \lambda}$ is not necessarily a universal class. We ignore this detail here since $\mathbf{K}_{\geq \lambda}$ will have enough of the properties of a universal class to carry the argument through.

means that if $\mathrm{gtp}(b/M; N') = \mathrm{gtp}(a/M; N)$, then[38] there exists $f : M' \underset{M}{\longrightarrow} N'$ so that $f(b) = a$. We call (a, M, M') a *prime triple*.

Note that this makes sense even if the AEC does not have amalgamation. Some computations give us that:

Proposition 5.4.12. *If* $\mathbf{K}$ *is a universal class, then* $\mathbf{K}$ *has primes.*

Definition 5.4.13. Let $\mathbf{K}$ be an AEC with a good λ-frame. Assume that $\mathbf{K}$ has primes (at least for models of size λ). Let $M \in \mathbf{K}_\lambda$ and let $p, q \in \mathrm{gS}(M)$. We say that p and q are *weakly orthogonal* if there exists a prime triple (a, M, M') such that $\mathrm{gtp}(a/M; M') = q$ and p has a unique extension to $\mathrm{gS}(M')$. We say that p and q are *orthogonal* if for any $N \geq M$, the nonforking extensions p', q' to N of p and q respectively are weakly orthogonal.

Orthogonality and weak orthogonality coincide assuming categoricity:

Theorem 5.4.14. *Let* $\mathbf{K}$ *be an AEC which has primes and a good* λ*-frame. Assume that* $\mathbf{K}$ *is categorical in* λ*. Then weak orthogonality and orthogonality coincide.*

We have arrived to a definition of unidimensionality (we say that a good λ-frame is categorical when the underlying class is categorical in λ):

Definition 5.4.15. Let $\mathbf{K}$ be an AEC which has primes and a categorical good λ-frame. $\mathbf{K}_\lambda$ is *unidimensional* if there does *not* exist $M \in \mathbf{K}$, and types $p, q \in \mathrm{gS}(M)$ such that p and q are orthogonal.

Theorem 5.4.16. *Let* $\mathbf{K}$ *be an AEC which has primes and a categorical good* λ*-frame. If* $\mathbf{K}$ *is unidimensional, then* $\mathbf{K}$ *is categorical in* λ^+*.*

Using the result of Grossberg and VanDieren, if in addition $\mathbf{K}$ is λ-tame, $\mathbf{K}$ will be categorical in every cardinal above λ. Therefore it is enough to prove unidimensionality. While step 2 was only happening locally in λ and did not use tameness, tameness will again have a crucial use in the next step.

5.4.3 Step 3: Proving unidimensionality

Let us make a slight diversion from unidimensionality. Recall that we work in a universal class $\mathbf{K}$ categorical in a proper class of cardinals with a lot of structural properties (amalgamation and existence of good frames, even global). We want to show that all big-enough models are Galois-saturated. Let M be a big model and assume it is not Galois-saturated, say it omits $p \in \mathrm{gS}(M_0)$, $M_0 \leq M$. Consider the class $\mathbf{K}_{\neg p}$ of all models $N \geq M_0$ that omit p. After adding constant symbols for M_0 and closing under isomorphisms,

[38] Why the formulation using Galois types? We have to make sure that the types of Ma in N and N' are the same.

this is an AEC. We would like to show that it has arbitrarily large models, for then this means that there exists a categoricity cardinal $\lambda > \|M_0\|$ and $N \in \mathbf{K}$ of size λ omitting p. This is a contradiction since we know that the model in the categoricity cardinal is Galois-saturated.

What are methods to show that a class has arbitrarily large models? A powerful one is again based on good frames: by definition, a good λ-frame has no maximal models in λ. If we can expand it to a good ($\geq \lambda$)-frame, then its underlying class $\mathbf{K}_{\geq \lambda}$ has no maximal models and hence models of arbitrarily large size. Recall that Theorem 5.4.7 gave mild conditions (tameness and weak amalgamation) under which a good frame can be transferred. Tameness was the key property there.

Moreover we know already that $\mathbf{K}$ itself has a good $\mathrm{LS}(\mathbf{K})$-frame, amalgamation, and is $\mathrm{LS}(\mathbf{K})$-tame. But it is not so clear that these properties transfer to $\mathbf{K}_{\neg p}$. Consider for example amalgamation: let $M_0 \leq M_\ell$, $\ell = 1, 2$, and assume that $p \in \mathrm{gS}(M_0)$ is omitted in both M_1 and M_2. Even if in $\mathbf{K}$ there exists amalgams of M_1 and M_2 over M_0, it is not clear that any such amalgams will omit p. Similarly, even if $q_1, q_2 \in \mathrm{gS}(M)$ are Galois types in $\mathbf{K}_{\neg p}$ and they are equal in $\mathbf{K}$, there is no guarantee that they will be equal in $\mathbf{K}_{\neg p}$ (the amalgam witnessing it may not be a member of $\mathbf{K}_{\neg p}$). So it is not clear that $\mathbf{K}_{\neg p}$ is tame.

However we are interested in universal classes, so consider the last property if $\mathbf{K}$ is a universal class. Say $q_1 = \mathrm{gtp}(a_1/M; N_1)$, $q_2 = \mathrm{gtp}(a_2/M; N_2)$. If q_1 is equal to q_2 in $\mathbf{K}$, then since $\mathbf{K}$ has primes there exists $M_1 \leq N_1$ containing a_1 and M and $f : M_1 \underset{M}{\longrightarrow} N_2$ so that $f(a_1) = a_2$. If N_1 omits p, then M_1 also omits p and so q_1 and q_2 are equal also in $\mathbf{K}_{\neg p}$. By the same argument, $\mathbf{K}_{\neg p}$ also has primes (in fact it is itself universal). Thus it has weak amalgamation. Similarly, since $\mathbf{K}$ is tame, $\mathbf{K}_{\neg p}$ is also tame[39].

The last problem to solve is therefore whether a good λ-frame in $\mathbf{K}$ is also a good λ-frame in $\mathbf{K}_{\neg p}$. This is where we use orthogonality calculus and unidimensionality. However the class we consider is slightly different than $\mathbf{K}_{\neg p}$: for $p \in \mathrm{gS}(M_0)$ nonalgebraic, we let $\mathbf{K}_{\neg^* p}$ be the class of M such that $M_0 \leq M$ and p has a *unique* extension to M (we add constant symbols for M_0 to make $\mathbf{K}_{\neg^* p}$ closed under isomorphisms). Note that $\mathbf{K}_{\neg^* p} \subseteq \mathbf{K}_{\neg p}$, as the unique extension must be the nonforking extension, which is nonalgebraic if p is. Using orthogonality calculus, we can show:

Theorem 5.4.17. *Let $\mathbf{K}$ be an AEC which has primes and a categorical good λ-frame $\mathfrak{s}$ for types at most λ. If $\mathbf{K}$ is* not *unidimensional, then there exists $M \in \mathbf{K}_\lambda$ and a nonalgebraic $p \in \mathrm{gS}(M)$ such that $\mathfrak{s}$ restricted to $\mathbf{K}_{\neg^* p}$ is still a good λ-frame.*

[39] A technical remark: if we only knew that $\mathbf{K}$ was *weakly* tame (i.e. tame for types over Galois-saturated models), we would not be able to conclude that $\mathbf{K}_{\neg p}$ was weakly tame: models that omit p of size larger than $|\mathrm{dom}(p)|$ are not Galois-saturated. Thus while many arguments in the study of tame AECs can be adapted to the weakly tame context, this one cannot.

We are ready to conclude:

Theorem 5.4.18. *Suppose that* $\mathbf{K}$ *is an AEC which has primes and a categorical good* λ*-frame for types at most* λ*. If* $\mathbf{K}$ *is categorical in some* $\mu > \lambda$ *and is* λ*-tame, then* $\mathbf{K}_\lambda$ *is unidimensional, and therefore* $\mathbf{K}$ *is categorical in all* $\mu' > \lambda$*.*

Proof. The last "therefore" follows from combining Theorem 5.4.16 and the Grossberg-VanDieren transfer (using tameness heavily) Theorem 5.4.10. To show that $\mathbf{K}_\lambda$ is unidimensional, suppose not. By Theorem 5.4.17, there exists $M \in \mathbf{K}_\lambda$ and a nonalgebraic $p \in \mathrm{gS}(M)$ such that there is a good λ-frame on $\mathbf{K}_{\neg^* p}$. By the argument above, $\mathbf{K}_{\neg^* p}$ is λ-tame and has weak amalgamation. But is $\mathbf{K}_{\neg^* p}$ an AEC? Yes! The only problematic part is if $\langle M_i : i < \delta \rangle$ is increasing in $\mathbf{K}_{\neg^* p}$, and we want to show that $M_\delta := \bigcup_{i<\delta} M_i$ is in $\mathbf{K}_{\neg^* p}$. Let $q_1, q_2 \in \mathrm{gS}(M_\delta)$ be extensions of p, we want to see that $q_1 = q_2$. By tameness, the good λ-frame of $\mathbf{K}$ transfers to a good $(\geq \lambda)$-frame. So we can fix $i < \delta$ such that q_1, q_2 do not fork over M_i. By definition of $\mathbf{K}_{\neg^* p}$, $q_1 \upharpoonright M_i = q_2 \upharpoonright M_i$. By uniqueness of nonforking extension, $q_1 = q_2$.

By Theorem 5.4.7 (using that $\mathbf{K}_{\neg^* p}$ is tame), $\mathbf{K}_{\neg^* p}$ has a good $(\geq \lambda)$-frame. In particular, it has arbitrarily large models. Thus, $\mathbf{K}$ has non-Galois-saturated models in every $\mu > \lambda$, hence cannot be categorical in any $\mu > \lambda$. □

We wrap up:

Proof of Theorem 5.4.1. Let $\mathbf{K}$ be a universal class categorical in cardinals of arbitrarily high cofinality.

1. Just because it is a universal class, $\mathbf{K}$ has primes and is $\mathrm{LS}(\mathbf{K})$-tame (recall Example 5.3.2.1.(5)).
2. By Theorem 5.2.44, there exists a good λ-frame on $\mathbf{K}$.
3. By the upward frame transfer (Theorem 5.4.7), $\mathbf{K}_{\geq\lambda}$ has amalgamation and in fact a good $(\geq \lambda)$-frame. This step uses λ-tameness.
4. By orthogonality calculus, if $\mathbf{K}_\lambda$ is not unidimensional then there exists a type p such that $\mathbf{K}_{\neg^* p}$ has a good λ-frame.
5. Since $\mathbf{K}$ has primes, $\mathbf{K}_{\neg^* p}$ is also λ-tame and has weak amalgamation, so by the upward frame transfer again (using tameness) it must have arbitrarily large models. So arbitrarily large models omit p, hence $\mathbf{K}$ has no Galois-saturated models of size above λ, so cannot be categorical above λ (by stability, $\mathbf{K}$ has a Galois-saturated model in every categoricity cardinal). This is a contradiction, therefore $\mathbf{K}_\lambda$ is unidimensional.
6. By Theorem 5.4.16, $\mathbf{K}$ is categorical in λ^+.
7. By the upward transfer of Grossberg and VanDieren (Theorem 5.4.10), $\mathbf{K}$ is categorical in all $\mu \geq \lambda$. This again uses tameness in a key way.

□

Remark 5.4.19. The proof can be generalized to abstract elementary classes which are tame and have primes. See Theorem 5.5.47.

5.5 Independence, stability, and categoricity in tame AECs

We have seen that good frames are a crucial tool in the proof of Shelah's eventual categoricity conjecture in universal classes. In this section, we give the precise definition of good frames in a more general axiomatic independence framework. We survey when good frames and more global independence notions are known to exist (i.e. the best known answers to Question 5.1.3).

We look at what can be said in both strictly stable and superstable AECs. Along the way we look at stability transfers, and the equivalence of various definitions of superstability in tame AECs.

Finally, we survey the theory of categorical tame AECs and give the best known approximations to Shelah's categoricity conjecture in this framework.

5.5.1 Abstract independence relations

To allow us to state precise results, we first fix some terminology. The terms used should be familiar to readers with experience in working with forking, either in the elementary or nonelementary context. One potentially unfamiliar notation: we sometimes refer to the pair $\mathfrak{i} = (\mathbf{K}, \perp)$ as an independence relation. This is particularly useful to deal with multiple classes as we can differentiate between the behavior of a possible forking relation on the class $\mathbf{K}$ compared to its behavior on the class $\mathbf{K}^{\lambda\text{-sat}}$ of λ-Galois-saturated models of $\mathbf{K}$.

Definition 5.5.1. An *independence relation* is a pair $\mathfrak{i} = (\mathbf{K}, \perp)$, where:

1. $\mathbf{K}$ is an AEC[40] with amalgamation (we say that $\mathfrak{i}$ is *on* $\mathbf{K}$ and write $\mathbf{K}_{\mathfrak{i}} = \mathbf{K}$).
2. $\perp$ is a relation on quadruples of the form (M, A, B, N), where $M \leq N$ are all in $\mathbf{K}$ and $A, B \subseteq |N|$. We write $\perp(M, A, B, N)$ or $A \underset{M}{\overset{N}{\perp}} B$ instead of $(M, A, B, N) \in \perp$.
3. The following properties hold:
   (a) Invariance: If $f : N \cong N'$ and $A \underset{M}{\overset{N}{\perp}} B$, then $f[A] \underset{f[M]}{\overset{N'}{\perp}} f[B]$.
   (b) Monotonicity: Assume $A \underset{M}{\overset{N}{\perp}} B$. Then:

[40] We may look at independence relations where $\mathbf{K}$ is not an AEC (e.g. it could be a class of Galois-saturated models in a strictly stable AEC).

i. Ambient monotonicity: If $N' \geq N$, then $A \underset{M}{\overset{N'}{\downfree}} B$. If $M \leq N_0 \leq N$ and $A \cup B \subseteq |N_0|$, then $A \underset{M}{\overset{N_0}{\downfree}} B$.

ii. Left and right monotonicity: If $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \underset{M}{\overset{N}{\downfree}} B_0$.

iii. Base monotonicity: If $A \underset{M}{\overset{N}{\downfree}} B$ and $M \leq M' \leq N$, $|M'| \subseteq B \cup |M|$, then $A \underset{M'}{\overset{N}{\downfree}} B$.

(c) Left and right normality: If $A \underset{M}{\overset{N}{\downfree}} B$, then $AM \underset{M}{\overset{N}{\downfree}} BM$.

When there is only one relation to consider, we sometimes write "$\downfree$ is an independence relation on $\mathbf{K}$" to mean "$(\mathbf{K}, \downfree)$ is an independence relation".

Definition 5.5.2. Let $\mathfrak{i} = (\mathbf{K}, \downfree)$ be an independence relation. Let $M \leq N$, $B \subseteq |N|$, and $p \in \mathrm{gS}^{<\infty}(B; N)$ be given. We say that p *does not* $\mathfrak{i}$*-fork over* M if whenever $p = \mathrm{gtp}(\bar{a}/B; N)$, we have that $\mathrm{ran}(\bar{a}) \underset{M}{\overset{N}{\downfree}} B$. When $\mathfrak{i}$ is clear from context, we omit it.

Remark 5.5.3. By the ambient monotonicity and invariance properties, this is well-defined (i.e. the choice of $\bar{a}$ and N does not matter).

An independence relation can satisfy several natural properties:

Definition 5.5.4 (Properties of independence relations)**.** Let $\mathfrak{i} = (\mathbf{K}, \downfree)$ be an independence relation.

1. $\mathfrak{i}$ has *disjointness* if $A \underset{M}{\overset{N}{\downfree}} B$ implies $A \cap B \subseteq |M|$.
2. $\mathfrak{i}$ has *symmetry* if $A \underset{M}{\overset{N}{\downfree}} B$ implies $B \underset{M}{\overset{N}{\downfree}} A$.
3. $\mathfrak{i}$ has *existence* if $A \underset{M}{\overset{N}{\downfree}} M$ for any $A \subseteq |N|$.
4. $\mathfrak{i}$ has *uniqueness* if whenever $M_0 \leq M \leq N_\ell$, $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$, $q_\ell \in \mathrm{gS}^{<\infty}(B; N_\ell)$, $q_1 \restriction M_0 = q_2 \restriction M_0$, and q_ℓ does not fork over M_0, then $q_1 = q_2$.
5. $\mathfrak{i}$ has *extension* if whenever $p \in \mathrm{gS}^{<\infty}(MB; N)$ does not fork over M and $B \subseteq C \subseteq |N|$, there exists $N' \geq N$ and $q \in \mathrm{gS}^{<\infty}(MC; N')$ extending p such that q does not fork over M.
6. $\mathfrak{i}$ has *transitivity* if whenever $M_0 \leq M_1 \leq N$, $A \underset{M_0}{\overset{N}{\downfree}} M_1$ and $A \underset{M_1}{\overset{N}{\downfree}} B$ imply $A \underset{M_0}{\overset{N}{\downfree}} B$.

7. $\mathfrak{i}$ has the $< \kappa$-*witness property* if whenever $M \leq N$, $A, B \subseteq |N|$, and $A_0 \mathop{\smile\!\!\!\!\!\mid}\limits_{M}^{N} B_0$ for all $A_0 \subseteq A$, $B_0 \subseteq B$ of size strictly less than κ, then $A \mathop{\smile\!\!\!\!\!\mid}\limits_{M}^{N} B$. The λ-*witness property* is the $(< \lambda^+)$-witness property.

The following cardinals are also important objects of study:

Definition 5.5.5 (Locality cardinals)**.** Let $\mathfrak{i} = (\mathbf{K}, \smile\!\!\!\!\!\mid)$ be an independence relation and let α be a cardinal.

1. Let $\bar{\kappa}_\alpha(\mathfrak{i})$ be the minimal cardinal $\mu \geq \alpha^+ + \text{LS}(\mathbf{K})^+$ such that for any $M \leq N$ in $\mathbf{K}$, any $A \subseteq |N|$ with $|A| \leq \alpha$, there exists $M_0 \leq M$ in $\mathbf{K}_{<\mu}$ with $A \mathop{\smile\!\!\!\!\!\mid}\limits_{M_0}^{N} M$. When μ does not exist, we set $\bar{\kappa}_\alpha(\mathfrak{i}) = \infty$.
2. Let $\kappa_\alpha(\mathfrak{i})$ be the minimal cardinal $\mu \geq \alpha^+ + \aleph_0$ such that for any regular $\delta \geq \mu$, any increasing continuous chain $\langle M_i : i \leq \delta \rangle$ in $\mathbf{K}$, any $N \geq M_\delta$, and any $A \subseteq |N|$ of size at most α, there exists $i < \delta$ such that $A \mathop{\smile\!\!\!\!\!\mid}\limits_{M_i}^{N} M_\delta$. When μ does not exist, we set $\kappa_{\alpha_0}(\mathfrak{i}) = \infty$.

We also let $\bar{\kappa}_{<\alpha}(\mathfrak{i}) := \sup_{\alpha_0 < \alpha} \bar{\kappa}_{\alpha_0}(\mathfrak{i})$. Similarly define $\kappa_{<\alpha}(\mathfrak{i})$. When clear, we may write $\kappa_\alpha(\smile\!\!\!\!\!\mid)$, etc., instead of $\kappa_\alpha(\mathfrak{i})$.

Definition 5.5.6. Let us say that an independence relation $\mathfrak{i}$ has *local character* if $\bar{\kappa}_\alpha(\mathfrak{i}) < \infty$ for all cardinals α.

Compared to the elementary framework, we differentiate between two local character cardinals, κ and $\bar{\kappa}$. The reason is that we do not in general (but see Theorem 5.5.41) know how to make sense of when a type does not fork over an arbitrary set (as opposed to a model). Thus we cannot (for example) define superstability by requiring that every type does not fork over a finite set: looking at unions of chains is a replacement.

We make precise when an independence relation is "like forking in a first-order stable theory":

Definition 5.5.7. We say that $\mathfrak{i}$ is a *stable independence relation* if it is an independence relation satisfying uniqueness, extension, and local character.

We could also define the meaning of a *superstable independence relation*, but here several nuances arise so to be consistent with previous terminology we will call it a *good* independence relation, see Definition 5.5.28.

As defined above, independence relations are *global objects*: they define an independence notion "p does not fork over M" for M of any size and p of any length. This is a strong requirement. In fact, the following refinement of Question 5.1.3 is still open:

Question 5.5.8. Let $\mathbf{K}$ be a fully tame and short AEC with amalgamation. Assume that $\mathbf{K}$ is categorical in a proper class of cardinals. Does there exists a λ and a stable independence relation $\mathfrak{i}$ on $\mathbf{K}_{\geq\lambda}$?

It is known that one can construct such an $\mathfrak{i}$ with local character and uniqueness, but proving that it satisfies extension seems hard in the absence of compactness. Note in passing that $\mathfrak{i}$ as above must be unique:

Theorem 5.5.9 (Canonicity of stable independence)**.** *If $\mathfrak{i}$ and $\mathfrak{i}'$ are stable independence relations on $\mathbf{K}$, then $\mathfrak{i} = \mathfrak{i}'$.*

As seen in Example 5.3.2.1.(4), we know that uniqueness and local character are enough to conclude some tameness and there are several relationships between the properties. We give one example:

Proposition 5.5.10. *Assume that $\mathop{\smile\!\!\!\!\!\!|}$ is a stable independence relation on $\mathbf{K}$.*

1. *$\mathop{\smile\!\!\!\!\!\!|}$ has symmetry, existence, and transitivity.*
2. *If $\mathbf{K}$ is fully $< \kappa$-tame and -type short, then $\mathop{\smile\!\!\!\!\!\!|}$ has the $< \kappa$-witness property.*
3. *For every α, $\kappa_\alpha(\mathop{\smile\!\!\!\!\!\!|}) \leq \bar{\kappa}_\alpha(\mathop{\smile\!\!\!\!\!\!|})$.*
4. *$\mathop{\smile\!\!\!\!\!\!|}$ has disjointness over sufficiently Galois-saturated models: if M is $\mathrm{LS}(\mathbf{K})^+$-Galois-saturated and $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$, then $A \cap B \subseteq |M|$.*

Proof sketch for (2). By symmetry and extension it is enough to show that for a given A, $A_0 \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$ for all $A_0 \subseteq A$ of size less than κ implies $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$. By extension, pick $N' \geq N$ and $A' \subseteq |N'|$ so that $A' \overset{N'}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$ and $\mathrm{gtp}(\bar{a}'/M_0; N') = \mathrm{gtp}(\bar{a}/M_0; N)$ (where $\bar{a}$, $\bar{a}'$ are enumerations of A and A' respectively). By the uniqueness property, $\mathrm{gtp}(\bar{a}' \upharpoonright I/M; N') = \mathrm{gtp}(\bar{a} \upharpoonright I/M_0; N)$ for all $I \subseteq \mathrm{dom}(\bar{a})$ of size less than κ. Now use by shortness this implies $\mathrm{gtp}(\bar{a}/M; N) = \mathrm{gtp}(\bar{a}'/M; N')$, hence by invariance $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$. $\square$

In what follows, we consider several approximations to Question 5.5.8 in the stable and superstable contexts. We also examine consequences on categorical AECs. It will be convenient to *localize* Definition 5.5.1 so that:

1. The relation $\mathop{\smile\!\!\!\!\!\!|}$ is only defined on types of certain lengths (that is, the size of the left hand side is restricted).
2. The relation $\mathop{\smile\!\!\!\!\!\!|}$ is only defined on types over domains of certain sizes (that is, the size of the right hand side and base is restricted).

More precisely:

Notation 5.5.11. *Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals. We say that $\mathfrak{i} = (\mathbf{K}, \mathop{\smile\!\!\!\!\!\!|})$ is a* $(< \alpha, \mathcal{F})$-independence relation *if it satisfies Definition 5.5.1 localized to types of length less than α and models in $\mathbf{K}_{\mathcal{F}}$ (so only amalgamation in $\mathcal{F}$ is required). We always require that $\theta \geq \alpha$ and $\lambda \geq \mathrm{LS}(\mathbf{K})$.*

$(\leq \alpha, \mathcal{F})$ means $(< \alpha^+, \mathcal{F})$, and if $\mathcal{F} = [\lambda, \lambda^+)$, then we say that $\mathfrak{i}$ is a $(\leq \alpha, \lambda)$-independence relation. Similar variations are defined as expected, e.g. $(\leq \alpha, \geq \lambda)$ means $(\leq \alpha, [\lambda, \infty))$.

We often say that $\mathfrak{i}$ is a $(< \alpha)$-ary independence relation on $\mathbf{K}_{\mathcal{F}}$ rather than a $(< \alpha, \mathcal{F})$-independence relation. We write α-ary rather than $(\leq \alpha)$-ary.

The properties in Definition 5.5.4 can be adapted to such localized independence relations. For example, we say that $\mathfrak{i}$ has *symmetry* if $A \overset{N}{\underset{M}{\perp}} B$ implies that for every $B_0 \subseteq B$ of size less than α, $B_0 \overset{N}{\underset{M}{\perp}} A$.

Using this terminology, we can give the definition of a *good λ-frame* (see Section 5.2.5), and more generally of a good $\mathcal{F}$-frame for $\mathcal{F}$ an interval of cardinals[41]:

Definition 5.5.12. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals. A *good $\mathcal{F}$-frame* is a 1-ary independence relation $\mathfrak{i}$ on $\mathbf{K}_{\mathcal{F}}$ such that:

1. $\mathfrak{i}$ satisfies disjointness, symmetry, existence, uniqueness, extension, transitivity, and $\kappa_1(\mathfrak{i}) = \aleph_0$.
2. $\mathbf{K}_{\mathcal{F}}$ has amalgamation in $\mathcal{F}$, joint embedding in $\mathcal{F}$, no maximal models in $\mathcal{F}$, and is Galois-stable in every $\mu \in \mathcal{F}$. Also of course $\mathbf{K}_{\mathcal{F}} \neq \emptyset$.

When $\mathcal{F} = [\lambda, \lambda^+)$, we talk of a good λ-frame, and when $\mathcal{F} = [\lambda, \infty)$, we talk of a good $(\geq \lambda)$-frame. As is customary, we may use the letter $\mathfrak{s}$ rather than $\mathfrak{i}$ to denote a good frame.

5.5.2 Stability

We compare results for stability in tame classes with those in general classes, summarized in Section 5.2.2. At a basic level, tameness strongly connects types over domains of different cardinalities. While a general AEC might be Galois-stable in λ but not in λ^+ (see the Hart-Shelah example in Section 5.3.2.2), this cannot happen in tame classes:

Theorem 5.5.13. *Suppose that $\mathbf{K}$ is an AEC with amalgamation which is λ-tame[42] and Galois-stable in λ. Then:*

1. *$\mathbf{K}$ is Galois-stable in λ^+.*
2. *$\mathbf{K}$ is Galois-stable in every $\mu > \lambda$ such that $\mu = \mu^\lambda$.*

There is also a partial stability spectrum theorem for tame AECs:

Theorem 5.5.14. *Let $\mathbf{K}$ be an AEC with amalgamation that is* $\mathrm{LS}(\mathbf{K})$*-tame. The following are equivalent:*

[41] Note that the definition here is different (but equivalent to) Shelah's notion of a *type-full* good λ-frame, see the historical remarks for more.

[42] For the first part, weak tameness suffices.

1. $\mathbf{K}$ *is Galois-stable in some cardinal* $\lambda \geq \mathrm{LS}(\mathbf{K})$.
2. $\mathbf{K}$ *does not have the order property (see Definition 5.2.26).*
3. *There are* $\mu \leq \lambda_0 < H_1$ *such that* $\mathbf{K}$ *is Galois-stable in any* $\lambda = \lambda^{<\mu} + \lambda_0$.

The proof makes heavy use of the Galois Morleyization (Theorem 5.3.5) to connect "stability theory inside a model" (results about formal, syntactic types within a particular model) to Galois types in an AEC. This allows the translation of classical proofs connecting the order property and stability.

This achieves two important generalizations from the elementary framework. First, it unites the characterizations of stability in terms of counting types and no order property from first-order, a connection still lacking in general AECs. Second, it gives one direction of the stability spectrum theorem by showing that, given stability in any one place, there are many stability cardinals, and some of the stability cardinals are given by satisfying some cardinal arithmetic above the first stability cardinal. Still lacking from this is a converse saying that the stability cardinals are exactly characterized by some cardinal arithmetic.

Another important application of the Galois Morleyization in stable tame AECs is that *averages* of suitable sequences can be analyzed. Roughly speaking, we can work inside the Galois Morleyization of a monster model and define the χ*-average over* A *of a sequence* $\mathbf{I}$ to be the set of formulas ϕ over A so that strictly less than χ-many elements of $\mathbf{I}$ satisfy ϕ. If χ is big-enough and under reasonable conditions on $\mathbf{I}$ (i.e. it is a Morley sequence with respect to nonsplitting), we can show that the average is complete and (if $\mathbf{I}$ is long-enough), realized by an element of $\mathbf{I}$. Unfortunately, a detailed study is beyond the scope of this paper, see the historical remarks for references.

Turning to independence relations in stable AECs, there are two main candidates. The first is the familiar notion of splitting (see Definition 5.2.22). Tameness simplifies the discussion of splitting by getting rid of the cardinal parameter: it is impossible for a type to λ^+-split over M and also not λ-split over M in a (λ, λ^+)-tame AEC, as the witness to λ^+-splitting could be brought down to size λ. This observation allows for a stronger uniqueness result in non-splitting. Rather than just having unique extensions in the same cardinality as in Theorem 5.2.24, we get a cardinal-free uniqueness result.

Theorem 5.5.15. *Suppose* $\mathbf{K}$ *is a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation and that* $M_0 \leq M_1 \leq M_2$ *are in* $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ *with* M_1 *universal over* M_0. *If* $p, q \in \mathrm{gS}(M_2)$ *do not split over* M_0 *and* $p \restriction M_1 = q \restriction M_1$, *then* $p = q$.

Proof sketch. If $p \neq q$, then there is a small $M^- \leq M_2$ with $p \restriction M^- \neq q \restriction M^-$; Without loss of generality pick M^- to contain M_0. By universality, we can find $f : M^- \xrightarrow[M_0]{} M_1$. By the nonsplitting,

$$p \restriction f(M^-) = f(p \restriction M^-) \neq f(q \restriction M^-) = q \restriction f(M^-)$$

Since $f(M^-) \leq M_1$, this contradicts the assumption they have equal restrictions. $\square$

Attempting to use splitting as an independence relation for $\mathbf{K}$ runs into the issue that several theorems require that the extension be *universal* (such as the above theorem). This can be mitigated by moving to the class of saturated enough models and looking at a localized version of splitting.

Definition 5.5.16.

1. Let $\mathbf{K}$ be an AEC with amalgamation. For $\mu > \mathrm{LS}(\mathbf{K})$, let $\mathbb{K}^{\mu\text{-sat}}$ denote the class of μ-Galois-saturated models in $\mathbf{K}_{\geq \mu}$.
2. Let $\mathbf{K}$ be an AEC with amalgamation and let $\mu \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is Galois-stable in μ. For $M_0 \leq M$ both in $\mathbf{K}^{\mu^+\text{-sat}}$ and $p \in \mathrm{gS}(M)$, we say that p *does not μ-fork over* M_0 if there exists $M_0' \leq M_0$ with $M_0' \in \mathbf{K}_\mu$ such that p does not μ-split over M_0' (see Definition 5.2.22).

Note that, by the μ^+-saturation of M_0, we have guaranteed that M_0 is a universal extension of M_0'. This gives us the following result.

Theorem 5.5.17. *Let $\mathbf{K}$ be an AEC with amalgamation and let $\mu \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is Galois-stable in μ and $\mathbf{K}$ is μ-tame. Let $\overset{\mu}{\smile}$ be the μ-nonforking relation restricted to the class $\mathbf{K}^{\mu^+\text{-}sat}$. Then*

1. $\overset{\mu}{\smile}$ *is a 1-ary independence relation that further satisfies disjointness, existence, uniqueness, and transitivity* when *all models are restricted to $\mathbf{K}^{\mu^+\text{-}sat}$ (in the precise language of Section 5.5.1, this says that $(\mathbf{K}^{\mu^+\text{-}sat}, \overset{\mu}{\smile})$ is an independence relation with these properties).*
2. $\overset{\mu}{\smile}$ *has set local character in $\mathbf{K}^{\mu^+\text{-}sat}$: Given $p \in \mathrm{gS}(M)$, there is $M_0 \in \mathbf{K}^{\mu^+\text{-}sat}$ such that $M_0 \leq M$ and p does not μ-fork over M_0.*
3. $\overset{\mu}{\smile}$ *has a local extension property: If $M_0 \leq M$ are both Galois-saturated and $\|M_0\| = \|M\| \geq \mu^+$ and $p \in \mathrm{gS}(M_0)$, then there exists $q \in \mathrm{gS}(M)$ extending p and not μ-forking over M_0.*

Proof sketch. Tameness ensures that μ-splitting and λ-splitting coincide when $\lambda \geq \mu$. The local extension property uses the extension property of splitting (see Theorem 5.2.24). Local character and uniqueness are also translations of the corresponding properties of splitting. Disjointness is a consequence of the moreover part in the extension property of splitting. Finally, transitivity is obtained by combining the extension and uniqueness properties of splitting. $\square$

The second candidate for an independence relation, drawing from stable first-order theories, is a notion of coheir, which we call $< \kappa$-satisfiability.

Definition 5.5.18. Let $M \leq N$ and $p \in \mathrm{gS}^{<\infty}(N)$.

1. We say that p is a $<$ *κ-satisfiable over* M if for every $I \subseteq \ell(p)$ and $A \subseteq |N|$ both of size strictly less than κ, we have that $p^I \restriction A$ is realized in M.

2. We say that p is a $< \kappa$-*heir over* M if for every $I \subseteq \ell(p)$ and every $A_0 \subseteq |M|$, $N_0 \leq N$, with $A_0 \subseteq |N_0|$ and I, A_0, N_0 all of size less than κ, there is some $f : N_0 \underset{A_0}{\longrightarrow} M$ such that

$$f(p^I \upharpoonright N_0) = p^I \upharpoonright f[N_0]$$

$< \kappa$-satisfiable is also called κ-coheir. As expected from first-order, these notions are dual[43] and they are equivalent under the κ-order property of length κ.

$< \kappa$-satisfiability turns out to be an independence relation in the stable context.

Theorem 5.5.19. *Let* $\mathbf{K}$ *be an AEC and* $\kappa > \mathrm{LS}(\mathbf{K})$. *Assume:*

1. $\mathbf{K}$ *has a monster model and is fully* $< \kappa$*-tame and -type short.*
2. $\mathbf{K}$ *does not have the* κ*-order property of length* κ.

Let $\perp$ *be the independence relation induced by* $< \kappa$*-satisfiability on the* κ*-Galois-saturated models of* $\mathbf{K}$. *Then* $\perp$ *has disjointness, symmetry, local character, transitivity, and the* κ*-witness property. Thus* if $\perp$ *also has extension, then it is a stable independence relation on the* κ*-Galois-saturated models of* $\mathbf{K}$.

If $\kappa = \beth_\kappa$, then it turns out that not having the κ-order property of length κ is equivalent to not having the order property, which by Theorem 5.5.14 is equivalent to stability.

Note that the conclusion gives already that the AEC is stable. Similarly, the $< \kappa$-satisfiability relation analyzes a type by breaking it up into its κ-sized components, so the tameness and type shortness assumptions seem natural[44].

Theorem 5.5.19 does not tell us if $< \kappa$-satisfiability has the extension property. At first glance, it seems to be a compactness result about Galois types. In fact:

Theorem 5.5.20. *Under the hypotheses of Theorem 5.5.19, if* κ *is a strongly compact cardinal, then* $< \kappa$*-satisfiability has the extension property.*

Extension also holds in some nonelementary classes (such as averageable classes) and we will see that it "almost" follows from superstability (see Section 5.5.4).

The existence of a reasonable independence notion for stable classes can be combined with averages to obtain a result on chains of Galois-saturated models:

Theorem 5.5.21. *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation. If* $\mathbf{K}$ *is Galois-stable in some* $\mu \geq \mathrm{LS}(\mathbf{K})$, *then there exists* $\chi < H_1$ *satisfying the following property:*

[43] A must be a model for the question to make sense.
[44] Although it is open if they are necessary.

If $\lambda \geq \chi$ is such that $\mathbf{K}$ is Galois-stable in μ for unboundedly many $\mu < \lambda$, then whenever $\langle M_i : i < \delta \rangle$ is a chain of λ-Galois-saturated models and $\mathrm{cf}(\delta) \geq \chi$, we have that $\bigcup_{i<\delta} M_i$ is λ-Galois-saturated.

Proof sketch. First note that Theorem 5.5.13.(2) and tameness imply that $\mathbf{K}$ is Galois-stable in stationary many cardinals. Then, develop enough of the theory of averages (and also investigate their relationship with forking) to be able to imitate Harnik's first-order proof [Har75]. $\square$

We will see that this can be vastly improved in the superstable case: the hypothesis that $\mathbf{K}$ be Galois-stable in μ for unboundedly many $\mu < \lambda$ can be removed and the Hanf number improved. Moreover, there is a proof of a version of the above theorem using only independence calculus and not relying on averages. Nevertheless, the use of averages has several other applications (for example getting solvability from superstability, see Theorem 5.5.23).

5.5.3 Superstability

As noted at the beginning of Section 5.2.1, Shelah has famously stated that superstability in AECs suffers from "schizophrenia". However superstability is much better behaved in tame AECs than in general. Recall Definition 5.2.27 which gave a definition of superstability in a single cardinal using local character of splitting. Recall also that there are several other local candidates such as the uniqueness of limit models (Definition 5.2.29) and the existence of a good frame (Section 5.2.5 and Definition 5.5.12). Theorem 5.5.21 suggests another definition saying that the union of a chain of μ-Galois-saturated models is μ-Galois-saturated. As noted before, it is unclear whether these definitions are equivalent cardinal by cardinal, that is, μ-superstability and λ-superstability for $\mu \neq \lambda$ are potentially different notions and it is not easy to combine them. With tameness, this difficulty disappears:

Theorem 5.5.22. *Assume that $\mathbf{K}$ is μ-superstable, μ-tame, and has amalgamation. Then for every $\lambda > \mu$:*

1. *$\mathbf{K}$ is λ-superstable.*
2. *If $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-Galois-saturated models, then $\bigcup_{i<\delta} M_i$ is λ-Galois-saturated.*
3. *There is a good λ-frame with underlying class $\mathbf{K}^{\lambda\text{-}sat}$.*
4. *$\mathbf{K}$ has uniqueness of limit models in λ. In fact, $\mathbf{K}$ also has uniqueness of limit models in μ.*

Proof sketch. Fix $\lambda > \mu$. We can first prove an approximation to (3) by defining forking as in Definition 5.5.16 and following the proof of Theorem 5.5.17. We obtain an independence relation $\mathfrak{i}$ on 1-types whose underlying class is $\mathbf{K}^{\lambda\text{-sat}}$ (at that point we do not yet know yet if it is an AEC), and which satisfies all the properties from the definition of a good frame[45] (including the structural properties on $\mathbf{K}$) except perhaps symmetry.

Still we can use this to prove that $\mathbf{K}$ satisfies (4) in Definition 5.2.27. Using just this together with uniqueness, we can show that $\mathbf{K}$ is Galois-stable in λ. Joint embedding follows from amalgamation and no maximal models holds by a variation on a part of the proof of Theorem 5.4.7. Therefore (1) holds: $\mathbf{K}$ is λ-superstable. We can prove the symmetry property of the good λ-frame by proving that a failure of it implies the order property. This also give the symmetry property for splitting, and hence by Theorem 5.2.31 the condition (4), uniqueness of limit models in λ, holds. Uniqueness of limit models can in turn be used to obtain (2), hence the underlying class of $\mathfrak{i}$ is really an AEC so (3) holds. $\square$

Strikingly, a converse to Theorem 5.5.22 holds. That is, several definitions of superstability are eventually equivalent in the tame framework:

Theorem 5.5.23. *Let* $\mathbf{K}$ *be a tame AEC with a monster model and assume that* $\mathbf{K}$ *is Galois-stable in unboundedly many cardinals. The following are equivalent:*

1. *For all high enough λ, the union of a chain of λ-Galois-saturated models is λ-Galois-saturated.*
2. *For all high enough λ, $\mathbf{K}$ has uniqueness of limit models in λ.*
3. *For all high enough λ, $\mathbf{K}$ has a superlimit model of size λ.*
4. *There is θ such that, for all high enough λ, $\mathbf{K}$ is (λ, θ)-solvable.*
5. *For all high enough λ, $\mathbf{K}$ is λ-superstable.*
6. *For all high enough λ, there is $\kappa = \kappa_\lambda \leq \lambda$ such that there is a good λ-frame on $\mathbf{K}_\lambda^{\kappa\text{-}sat}$.*

Any of these equivalent statements also implies that $\mathbf{K}$ *is Galois-stable in all high enough λ.*

Note that the "high enough" threshold can potentially vary from item to item. Also, note that the stability assumption in the hypothesis is not too important: in several cases, it follows from the assumption and, in others (such as the uniqueness of limit models), it is included to ensure that the condition is not vacuous. Finally, if $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame, we can add in each of that $\lambda < H_1$ in each of the conditions (except in (4) where we can say that $\theta < H_1$).

Superlimit models and solvability both capture the notion of the AEC $\mathbf{K}$ having a "categorical core", a sub-AEC $\mathbf{K}_0$ that is categorical in some κ. In the case of superlimits, $M \in \mathbf{K}_\kappa$ is superlimit if and only if M is universal[46] and the class of models isomorphic to M generates a nontrivial AEC. That is, the class:

$$\{N \in \mathbf{K}_{\geq \kappa} \mid \forall N_0 \in P^*_{\kappa^+}(N) \exists N_1 \in P^*_{\kappa^+} N : N_0 \leq N_1 \wedge N_1 \cong M\}$$

[45] Note that tameness was crucial to obtain the uniqueness property.
[46] That is, every model of size κ embeds into M.

is an AEC with a model of size κ^{+}[47]. (λ, κ)-solvability further assumes that this superlimit is isomorphic to $\mathrm{EM}_{L(\mathbf{K})}(I, \Phi)$ for some proper Φ of size κ and *any* linear order I of size λ.

Note that although we did not mention them in Section 5.2, superlimits and especially solvable AECs play a large role in the study of superstability in general AECs (see the historical remarks).

The proof that superstability implies solvability relies on a characterization of Galois-saturated models using averages (essentially, a model M is Galois-saturated if and only if for every type $p \in \mathrm{gS}(M)$, there is a long-enough Morley sequence $\mathbf{I}$ inside M whose average is p). We give the idea of the proof that a union of Galois-saturated models being Galois-saturated implies superstability. This can also be used to derive superstability from categoricity in the tame framework (without using the much harder proof of the Shelah-Villaveces Theorem 5.2.36).

Lemma 5.5.24. *Let $\mathbf{K}$ be an AEC with a monster model. Assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and let $\kappa = \beth_\kappa > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is Galois-stable in κ. Assume that for all $\lambda \geq \kappa$ and all limit δ, if $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-Galois-saturated models, then $\bigcup_{i<\delta} M_i$ is λ-Galois-saturated.*

Then $\mathbf{K}$ is κ^+-superstable.

Proof sketch. By tameness and Theorem 5.5.13.(1), we have that $\mathbf{K}$ is Galois-stable in κ^+. Thus, we only have to show Definition 5.2.27.(4), that there are no long splitting chains. There is a Galois-saturated model in κ^+ and, by a back and forth argument, it is enough to show Definition 5.2.27.(4) when all the models are κ^+-Galois-saturated.

Let $\delta < \kappa^{++}$ be a limit ordinal and let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of Galois-saturated models in $\mathbf{K}_{\kappa^+}$; that we can make the models at limit stages Galois-saturated crucially uses the assumption. Let $p \in \mathrm{gS}(M_\delta)$. We need to show that there is $i < \delta$ such that p does not κ^+-split over M_i. By standard means, one can show that there is an $i < \delta$ such that p is $< \kappa^+$ satisfiable in M_i. Tameness gives the uniqueness of $< \kappa$-satisfiability, which allows us to conclude that p is $< \kappa$-satisfiable in M_i, which in turn implies that p does not κ^+-split over M_i, as desired. □

Remark 5.5.25. From the argument, we obtain the following intriguing consequence in first-order model theory[48]: if T is a stable first-order theory, $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain of $\aleph_1$-saturated models (so M_i is $\aleph_1$-saturated also for limit i), then for any $p \in S(M_\delta)$, there exists $i < \delta$ so that p does not fork over M_i. This begs the question of whether any such chain exists in strictly stable theories.

[47] An equivalent definition: $M \in \mathbf{K}_\kappa$ is superlimit if and only if it is universal, has a proper extension isomorphic to it, and for any limit $\delta < \kappa^+$, and any increasing continuous chain $\langle M_i : i \leq \delta \rangle$, if $M_i \cong M$ for all $i < \delta$, then $M_\delta \cong M$.

[48] Hence showing that perhaps the study of AEC can also lead to new theorems in first-order model theory.

We now go back to the study of good frames. One can ask when instead of a good λ-frame, we obtain a good $(\geq \lambda)$-frame (i.e. forking is defined for types over models of all sizes). It turns out that the proof of Theorem 5.5.22 gives a good $(\geq \mu^+)$-frame on $\mathbf{K}^{\mu^+\text{-sat}}$. This still has the disadvantage of looking at Galois-saturated models. The next result starts from a good μ-frame and shows that μ-tameness can transfer it up (note that this was already stated as Theorem 5.4.6):

Theorem 5.5.26. *Assume* $\mathbf{K}$ *is an AEC with* $\mathrm{LS}(\mathbf{K}) \leq \lambda$ *and* $\mathfrak{s}$ *is a good* λ*-frame on* $\mathbf{K}$*. If* $\mathbf{K}$ *has amalgamation, then* $\mathbf{K}$ *is* λ*-tame if and only if there is a good* $(\geq \lambda)$*-frame* $\geq \mathfrak{s}$ *on* $\mathbf{K}$ *that extends* $\mathfrak{s}$*.*

Proof sketch. That tameness is necessary is discussed in Example 5.3.2.1.(4).

For the other direction, it is easy to check that if there is any way to extends forking to models of size at least λ, the definition must be the following:

$p \in \mathrm{gS}(M)$ does not fork over M_0 if and only if there exists $M_0' \leq M_0$ with $M_0' \in \mathbf{K}_\lambda$ and $p \restriction M'$ does not fork over M_0' for all $M' \leq M$ with $M' \in \mathbf{K}_\lambda$.

Several frame properties transfer without tameness; however, the key properties of uniqueness, extension, stability, and symmetry can fail. λ-tameness can be easily seen to be equivalent to the transfer of uniqueness from $\mathfrak{s}$ to $\geq \mathfrak{s}$. Using uniqueness, extension and stability can easily be shown to follow. Symmetry is harder and the proof goes through independent sequences (see below and the historical remarks).

As an example, we show how to prove the extension property. Note that one of the key difficulties in proving extension in general is that upper bounds of types need not exist; while this is trivial in first-order, such AECs are called compact (see Definition 5.3.2). To solve this problem, we use the forking machinery of the frame to build a chain of types with a canonical extension at each step. This canonicity provides the existence of types.

Let $M \in \mathbf{K}_{\geq \lambda}$ and let $p \in \mathrm{gS}(M)$. Let $N \geq M$. We want to find a nonforking extension of p to N. By local character and transitivity, without loss of generality $M \in \mathbf{K}_\lambda$. We now work by induction on $\mu := \|N\|$. If $\mu = \lambda$, we know that p can be extended to N by definition of a good frame, so assume $\mu > \lambda$. Write $N = \bigcup_{i<\mu} N_i$, where $N_i \in \mathbf{K}_{\lambda+|i|}$. By induction, let $p_i \in \mathrm{gS}(N_i)$ be the nonforking extension of p to N_i. Note that by uniqueness $p_j \restriction N_i = p_i$ for $i \leq j < \mu$. We want to take the "direct limit" of the p_i's: build $\langle f_i : i < \mu \rangle$, $\langle N_i' : i < \mu \rangle$, $\langle a_i : i < \mu \rangle$ such that $p_i = \mathrm{gtp}(a_i/N_i; N_i')$, $f_i : N_i' \underset{N_i}{\longrightarrow} N_{i+1}'$ such that $f_i(a_i) = a_{i+1}$. If this can be done, then taking the direct limit of the system induced by $\langle f_i, a_i N_i' : i < \mu \rangle$, we obtain a_μ, N_μ' such that $\mathrm{gtp}(a_\mu/N_\mu; N_\mu')$ is a nonforking extension of p. How can we build such a system? The base and successor cases are no problem, but at limits, we want to take the direct limit and prove that everything is still preserved. This will be the case because of the local character and uniqueness property.

□

This should be compared to Theorem 5.2.43 which achieves the more modest goal of transferring $\mathfrak{s}$ to λ^+ (over Galois-saturated models and with a different ordering) with assumptions on the number of models and some non-ZFC hypotheses.

An interesting argument in the proof of Theorem 5.5.26 is the transfer of the symmetry property. One could ignore that issue and use that failure of the order property implies symmetry, however this would make the argument non-local in the sense that we require knowledge about the AEC near the Hanf of λ to conclude good property at λ. A more local (but harder) approach is to study *independent sequences*.

Given a good $(\geq \lambda)$-frame and $M_0 \leq M \leq N$, we want to say that a sequence $\langle a_i \in N : i < \alpha \rangle$ is independent in (M_0, M, N) if and only if $\mathrm{gtp}(a_i / |M| \cup \{a_j : j < i\}; N)$ does not fork over M_0. However, forking behaves better for types over models so instead, we require that there is a sequence of models $M \leq N_i \leq N$ growing with the sequence $\langle a_i : i < \alpha \rangle$ such that $a_i \in |N_{i+1}| \backslash |N_i|$ and require $\mathrm{gtp}(a_i / N_i; N)$ does not fork over M_0.

The study of independent sequences shows that under tameness they themselves form (in a certain technical sense) a good frame. That is, from an independence relation for types of length one, we obtain an independence relation for types *of independent sequences* of all lengths. One other ramification of the study of independence sequence is the isolation of a good notion of dimension: inside a fixed model, any two infinite maximal independent sets must have the same size.

Theorem 5.5.27. *Let $\mathbf{K}$ be an AEC, $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is λ-tame and has amalgamation. Let $\mathfrak{s}$ be a good $(\geq \lambda)$-frame on $\mathbf{K}$. Let $M_0 \leq M \leq N$ all be in $\mathbf{K}_{\geq \lambda}$.*

1. *Symmetry of independence: For a fixed set I, I is independent in (M_0, M, N) if and only if* all *enumerations are independent in (M_0, M, N).*
2. *Let $p \in \mathrm{gS}(M)$. Assume that I_1 and I_2 are independent in (M_0, M, N) and every $a \in I_1 \cup I_2$ realizes p. If both I_1 and I_2 are $\subseteq$-maximal with respect to that property and I_1 is infinite, then $|I_1| = |I_2|$.*

5.5.4 Global independence and superstability

Combined with Theorem 5.5.22, Theorem 5.5.26 shows that every tame superstable AEC has a good $(\geq \lambda)$-frame. It is natural to ask whether this frame can also be extended in the other direction: to types of length larger than one. More precisely, we want to build a superstability-like global independence relation (i.e. the global version of a good frame):

Definition 5.5.28. We say an independence relation $\mathop{\smile}\limits^{\vert}$ on $\mathbf{K}$ is *good* if:

1. $\mathbf{K}$ is an AEC with amalgamation, joint embedding, and arbitrarily large models.

2. $\mathbf{K}$ is Galois-stable in all $\mu \geq \mathrm{LS}(\mathbf{K})$.
3. $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$ has disjointness, symmetry, existence, uniqueness, extension, transitivity, and the $\mathrm{LS}(\mathbf{K})$-witness property.
4. For all cardinals $\alpha > 0$:
   (a) $\bar{\kappa}_\alpha(\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}) = (\alpha + \mathrm{LS}(\mathbf{K}))^+$.
   (b) $\kappa_\alpha(\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}) = \alpha^+ + \aleph_0$.

We say that an AEC $\mathbf{K}$ is *good* if there exists a good independence relation on $\mathbf{K}$.

We would like to say that if $\mathbf{K}$ is a $\mathrm{LS}(\mathbf{K})$-superstable AEC with amalgamation that is fully tame and short, then there exists λ such that $\mathbf{K}_{\geq\lambda}$ is good. At present, we do not know if this is true (see Question 5.5.8). All we can conclude is a weakening of good:

Definition 5.5.29. We say an independence relation $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$ is *almost good* if it satisfies all the conditions of Definition 5.5.28 except it only has the following weakening of extension: If $p \in \mathrm{gS}^\alpha(M)$ and $N \geq M$, we can find $q \in \mathrm{gS}^\alpha(N)$ extending p and not forking over M provided that at least one of the following conditions hold:

1. M is Galois-saturated.
2. $M \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$.
3. $\alpha < \mathrm{LS}(\mathbf{K})^+$.

An AEC $\mathbf{K}$ is *almost good* if there is an almost good independence relation on $\mathbf{K}$.

Remark 5.5.30. Assume that $\mathfrak{i}$ is an independence relation on $\mathbf{K}$ which satisfies all the conditions in the definition of good except extension, and it has extension for types over Galois-saturated models. Then we can restrict $\mathfrak{i}$ to $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ and obtain an almost good independence relation. Thus extension over Galois-saturated models is the important condition in Definition 5.5.29.

We can now state a result on existence of global independence relation:

Theorem 5.5.31. *Let $\mathbf{K}$ be a fully $\mathrm{LS}(\mathbf{K})$-tame and short AEC with amalgamation. Let $\lambda := \left(2^{\mathrm{LS}(\mathbf{K})}\right)^{+4}$. If $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable, then $\mathbf{K}^{\lambda\text{-sat}}$ is almost good.*

We try to describe the proof. For simplicity, we will work with $<\kappa$-satisfiability, so will obtain a Hanf number approximately equal to a fixed point of the beth function. The better bound is obtained by looking at splitting but this makes the proof somewhat more complicated. So let $\kappa = \beth_\kappa > \mathrm{LS}(\mathbf{K})$. We know that the $<\kappa$-satisfiability independence relation is an independence relation on $\mathbf{K}^{\kappa\text{-sat}}$ with uniqueness, local character, and symmetry (but not extension). Let $\mathfrak{i}$ denote this relation independence relation. Furthermore we can show that $\kappa_1(\mathfrak{i}) = \aleph_0$. In fact, $\mathfrak{i}$ restricted to types of length one induces a good κ-frame $\mathfrak{s}$ on $\mathbf{K}_\kappa^{\kappa\text{-sat}}$. We would like to extend $\mathfrak{s}$ to types of length at most κ.

To do this, we need to make use of the notion of domination and successful frames[49]:

Definition 5.5.32. Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation on $\mathbf{K}$. Work inside a monster model[50].

1. For $M \leq N$ κ-Galois-saturated and $a \in |N|$, *a dominates N over M* if for any B, $a \mathop{\smile\!\!\!\!\!\!|}_M B$ implies $N \mathop{\smile\!\!\!\!\!\!|}_M B$.
2. $\mathfrak{s}$ is *successful* if for every Galois-saturated $M \in \mathbf{K}_\kappa$, every nonalgebraic type $p \in \mathrm{gS}(M)$, there exists $N \geq M$ and $a \in |N|$ with $N \in \mathbf{K}_\kappa$ Galois-saturated such that a dominates N over M.
3. $\mathfrak{s}$ is *ω-successful* if $\mathfrak{s}^{+n}$ is successful for all $n < \omega$. Here, $\mathfrak{s}^{+n}$ is the good κ^{+n} induced on the Galois-saturated models of size κ^{+n} by $< \kappa$-satisfiability.

An argument of Makkai and Shelah [MS90, Proposition 4.22] shows that $\mathfrak{s}$ is successful (in fact ω-successful), and a deep result of Shelah shows that if $\mathfrak{s}$ is successful, then we can extend $\mathfrak{s}$ to a κ-ary independence relation $\mathfrak{i}'$ which has extension, uniqueness, symmetry, and for all $\alpha \leq \kappa$, $\kappa_\alpha(\mathfrak{i}') = \alpha^+ + \aleph_0$. This completes the first step of the proof. Note that we have taken $\mathfrak{i}$ (which was built on $< \kappa$-satisfiability), restricted it to 1-types and then "lengthened" it to κ-ary types. However, we do not necessarily get $< \kappa$-satisfiability back! We do get, however, an independence relation with a better local character property.

From ω-successfulness, we could extend the frame $\mathfrak{s}$ to models of size κ^{+n}. Now we would like to extend $\mathfrak{i}'$ to models of all sizes above κ. However, the continuity of $\mathfrak{i}'$ is not strong enough. The missing property is:

Definition 5.5.33. An independence relation $\mathfrak{i} = (\mathbf{K}, \mathop{\smile\!\!\!\!\!\!|})$ has *full model continuity* if for any limit ordinal δ, for any increasing continuous chain $\langle M_i^\ell : i \leq \delta \rangle$ with $\ell < 4$, and $M_i^0 \leq M_i^k \leq M_i^3$ for $k = 1, 2$ and $i \leq \delta$, if $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_i^2$ for all $i < \delta$, then $M_\delta^1 \overset{M_\delta^3}{\underset{M_\delta^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_\delta^2$.

Let us say that $\mathfrak{i}$ is *fully good* [*almost fully good*] if it is good [almost good] and has full model continuity. As before, $\mathbf{K}$ is *[almost] fully good* if it there is an [almost] fully good independence relation on $\mathbf{K}$.

[49]Note that the definitions here do not coincide with Shelah's, although they are equivalent in our context. The equivalence uses tameness again, including a result of Adi Jarden. See the historical remarks for more.

[50]So if $\mathfrak{C}$ is the monster model, $a \mathop{\smile\!\!\!\!\!\!|}_M B$ means $a \overset{\mathfrak{C}}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$.

Another powerful result of Shelah [She09b, III.8.19] connects ω-successful good frames with full model continuity. Suppose that $\mathfrak{s}$ is an ω-successful good κ-frame (as we have). We do not know that $\mathfrak{i}'$ defined above has full model continuity, but it we move to the (still ω-successful) good κ^{+3}-frame $\mathfrak{s}^{+3}$ and "lengthen" this to an independence relation $\mathfrak{i}'_{+3}$ on κ^{+3}-ary types, then $\mathfrak{i}'_{+3}$ has full model continuity!

This allows us to transfer all of the nice properties of $\mathfrak{i}'_{+3}$ to a κ^{+3}-ary independence relation $\mathfrak{i}''$ on models of all sizes above κ^{+3}. To get a truly global independence relation, we can define an independence relation $\mathfrak{i}'''$ on types of *all* lengths by specifying that $p \in \mathrm{gS}^{\alpha}(M)$ do not $\mathfrak{i}'''$-fork over $M_0 \leq M$ if and only if $p \upharpoonright I$ does not $\mathfrak{i}''$-fork over M_0 for every $I \subseteq \alpha$ with $|I| \leq \kappa^{+3}$. With some work, we can show that $\mathfrak{i}'''$ is almost fully good (thus "fully" can be added to the conclusion of Theorem 5.5.31).

What about getting the extension over property over all models (not just the Galois-saturated models). It is known how to do it by making one more locality hypothesis:

Definition 5.5.34 (Type locality)**.**

1. Let δ be a limit ordinal, and let $\bar{p} := \langle p_i : i < \delta \rangle$ be an increasing chain of Galois types, where for $i < \delta$, $p_i \in \mathrm{gS}^{\alpha_i}(M)$ and $\langle \alpha_i : i \leq \delta \rangle$ are increasing. We say $\bar{p}$ is *κ-type-local* if $\mathrm{cf}(\delta) \geq \kappa$ and whenever $p, q \in \mathrm{gS}^{\alpha_\delta}(M)$ are such that $p^{\alpha_i} = q^{\alpha_i} = p_i$ for all $i < \delta$, then $p = q$.
2. We say $\mathbf{K}$ is *κ-type-local* if every $\bar{p}$ as above is κ-type-local.

We think of κ-type-locality as the dual to κ-locality (Definition 5.3.2.(3)) in the same sense that shortness is the dual to tameness.

Remark 5.5.35. If κ is a regular cardinal and $\mathbf{K}$ is $< \kappa$-type short, then $\mathbf{K}$ is κ-type-local. In particular, if $\mathbf{K}$ is fully $< \aleph_0$-tame and -type short, then $\mathbf{K}$ is $\aleph_0$-type-local.

Remark 5.5.36. If there is a good λ-frame on $\mathbf{K}$, then $\mathbf{K}_\lambda$ is $\aleph_0$-local (use local character and uniqueness), and thus assuming λ-tameness $\mathbf{K}$ is $\aleph_0$-local. This is used in the transfer of a good λ-frame to a good ($\geq \lambda$)-frame. Unfortunately, an analog for this fact is missing when looking at $\aleph_0$-type-locality, i.e. it is not clear that even a fully good AEC is $\aleph_0$-type-local.

Using type-locality, we can start from a fully good $\mathrm{LS}(\mathbf{K})$-ary independence relation on $\mathbf{K}$ and prove extension for types of all lengths. Thus we obtain the following variation of Theorem 5.5.31:

Theorem 5.5.37. *Let $\mathbf{K}$ be a fully $\mathrm{LS}(\mathbf{K})$-tame and short AEC with amalgamation. Assume that $\mathbf{K}$ is $\aleph_0$-type-local. Let $\lambda := \left(2^{\mathrm{LS}(\mathbf{K})}\right)^{+4}$. If $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable, then $\mathbf{K}^{\lambda\text{-sat}}$ is fully good.*

Remark 5.5.38. It is enough to assume that $\aleph_0$-type-locality holds "densely" in a certain technical sense. See the historical remarks.

Finally, we know of at least two other ways to obtain extension: using total categoricity and large cardinals. We collect all the results of this section in a corollary:

Corollary 5.5.39. *Let* $\mathbf{K}$ *be an AEC. Assume that* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-superstable and fully* $\mathrm{LS}(\mathbf{K})$*-tame and short.*

1. *If* $\kappa > \mathrm{LS}(\mathbf{K})$ *is a strongly compact cardinal, then* $\mathbf{K}^{\kappa\text{-sat}}$ *is fully good.*
2. *If either* $\mathbf{K}$ *is* $\aleph_0$*-type-local (e.g. it is fully* $(<\aleph_0)$*-tame and short) or* $\mathbf{K}$ *is totally categorical, then* $\mathbf{K}^{\lambda\text{-sat}}$ *is fully good, where* $\lambda := \left(2^{\mathrm{LS}(\mathbf{K})}\right)^{+4}$.

Proof sketch. By Theorem 5.5.31, $\mathbf{K}^{\lambda\text{-sat}}$ is almost good, and in fact (as we have discussed) almost fully good. If $\mathbf{K}$ is totally categorical, all the models are Galois-saturated and hence by definition of almost fully good, $\mathbf{K}$ is fully good. If $\mathbf{K}$ is $\aleph_0$-type-local, then apply Theorem 5.5.37. Finally, if $\kappa > \mathrm{LS}(\mathbf{K})$ is strongly compact, then the extension property for $<\kappa$-satisfiability holds (see Theorem 5.5.20) and using a canonicity result similar to Theorem 5.5.9 one can conclude that $\mathbf{K}^{\kappa\text{-sat}}$ is fully good. □

Since the existence of a strongly compact cardinal implies full tameness and shortness (see Theorem 5.3.7), we can state a version of the first part of Corollary 5.5.39 as follows:

Theorem 5.5.40. *If* $\mathbf{K}$ *is an AEC which is superstable in every* $\mu \geq \mathrm{LS}(\mathbf{K})$ *and* $\kappa > \mathrm{LS}(\mathbf{K})$ *is a strongly compact cardinal, then* $\mathbf{K}^{\lambda\text{-sat}}$ *is fully good, where* $\lambda := \left(2^{\mathrm{LS}(\mathbf{K})}\right)^{+4}$.

Note that in all of the results above, we are restricting ourselves to classes of sufficiently saturated models. This is related to the fact that the uniqueness property is required in the definition of a good independence relation, i.e. all types must be stationary. But what if we relax this requirement? Can we obtain an independence relation that specifies what it means to fork over an arbitrary set? A counterexample of Shelah [HL02, Section 4] shows that this cannot be done in general. However this is possible for universal classes:

Theorem 5.5.41. *If* $\mathbf{K}$ *is an almost fully good universal class, then:*

1. $\mathbf{K}$ *is fully good.*
2. *We can define* $A \overset{N}{\underset{A_0}{\perp}} B$ *(for* A_0 *an arbitrary set) to hold if and only if* $\mathrm{cl}^N(A_0 A) \overset{N}{\underset{\mathrm{cl}^N(A_0)}{\perp}} \mathrm{cl}^N(A_0 B)$*. Here* cl^N *is the closure under the functions of* N*. This has the expected properties (extension, existence, local character).*
3. *This also has the finite witness property:* $A \overset{N}{\underset{A_0}{\perp}} B$ *if and only if* $A' \overset{N}{\underset{A_0}{\perp}} B'$ *for all* $A' \subseteq A$, $B' \subseteq B$ *finite.*

Remark 5.5.42. It is enough to assume that $\mathbf{K}$ admits intersections, i.e. for any $N \in \mathbf{K}$ and any $A \subseteq |M|$, $\bigcap\{M \leq N \mid A \subseteq |M|\} \leq N$.

5.5.5 Categoricity

One of the first marks made by tame AEC was the theorem by Grossberg and VanDieren [GV06a] that tame AECs (with amalgamation) satisfy an *upward* categoricity transfer from a successor (see Theorem 5.4.10). Combining it with Theorem 5.2.35, we obtain that tame AECs satisfy Shelah's eventual categoricity conjecture from a successor:

Theorem 5.5.43. *Let* $\mathbf{K}$ *be an* H_2*-tame AEC with amalgamation. If* $\mathbf{K}$ *is categorical in* some successor $\lambda \geq H_2$, *then* $\mathbf{K}$ *is categorical in* all $\lambda' \geq H_2$.

Recall that categoricity implies superstability below the categoricity cardinal (Theorem 5.2.36). A powerful result is that assuming tameness, superstability also holds above, while this need not be true without tameness; recall the discussion after Theorem 5.2.36. In particular, Question 5.2.38 has a positive answer: the model in the categoricity cardinal is Galois-saturated.

Theorem 5.5.44. *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation and no maximal models. If* $\mathbf{K}$ *is categorical in some* $\lambda > \mathrm{LS}(\mathbf{K})$*, then:*

1. $\mathbf{K}$ *is superstable in every* $\mu \geq \mathrm{LS}(\mathbf{K})$.
2. *For every* $\mu > \mathrm{LS}(\mathbf{K})$*, there is a good* μ*-frame with underlying class* $\mathbf{K}^{\mu\text{-}sat}$.
3. *The model of size* λ *is Galois-saturated.*

Proof.

1. By Theorem 5.2.36, $\mathbf{K}$ is superstable in $\mathrm{LS}(\mathbf{K})$. Now apply Theorem 5.5.22.
2. As above, using Theorem 5.5.22.
3. $\mathbf{K}$ is λ-superstable, so in particular Galois-stable in λ. It is not hard to build a μ^+-Galois-saturated model in λ for every $\mu < \lambda$ so the result follows from categoricity.

□

Theorem 5.5.44 allows one to show that a tame AEC categorical in some cardinal is categorical in a closed unbounded set of cardinals of a certain form. This already plays a key role in Shelah's proof of Theorem 5.2.35. The key is what we call Shelah's omitting type theorem, a refinement of Morley's omitting type theorem. Note that a version of this theorem is also true without tameness, but removing the tameness assumption changes the condition on p being omitted to requiring that the small approximations to p be omitted[51].

[51] In the sense that each element omits *some* small approximation of p.

Theorem 5.5.45 (Shelah's omitting type theorem)**.** *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation. Let* $M_0 \leq M$ *and let* $p \in \mathrm{gS}(M_0)$*. Assume that* p *is omitted in* M*. If* $\|M_0\| \geq \mathrm{LS}(\mathbf{K})$ *and* $\|M\| \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}(\|M_0\|)$*, then there is a non-*$\mathrm{LS}(\mathbf{K})^+$*-Galois-saturated model in every cardinal.*

Corollary 5.5.46. *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation and no maximal models. If* $\mathbf{K}$ *is categorical in* some $\lambda > \mathrm{LS}(\mathbf{K})$*, then* $\mathbf{K}$ *is categorical in all cardinals of the form* $\beth_\delta$*, where* $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$ *divides* δ*.*

Proof. Let δ be divisible by $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$. If there is a model $M \in \mathbf{K}_{\beth_\delta}$ which is not Galois-saturated, then by Shelah's omitting type theorem we can build a non $\mathrm{LS}(\mathbf{K})^+$-Galois-saturated model in λ. This contradicts Theorem 5.5.44. $\square$

In section 5.4, a categoricity transfer was proven *without* assuming that the categoricity cardinal is a successor. As hinted at there, this generalizes to tame AECs that *have primes* (recall from Definition 5.4.11 that an AEC has primes if there is a prime model over every set of the form $M \cup \{a\}$):

Theorem 5.5.47. *Let* $\mathbf{K}$ *be an AEC with amalgamation and no maximal models. Assume that* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-tame and has primes. If* $\mathbf{K}$ *is categorical in some* $\lambda > \mathrm{LS}(\mathbf{K})$*, then* $\mathbf{K}$ *is categorical in all* $\lambda' \geq \min(\lambda, H_1)$*.*

Remark 5.5.48. A partial converse is true: if a fully tame and short AEC with amalgamation and no maximal models is categorical on a tail, then it has primes on a tail.

We deduce Shelah's categoricity conjecture in homogeneous model theory (see Section 5.3.2.1.(7)):

Corollary 5.5.49. *Let* D *be a homogeneous diagram in a first-order theory* T*. If* D *is categorical in a* $\lambda > |T|$*, then* D *is categorical in all* $\lambda' \geq \min(\lambda, h(|T|))$*.*

Using a similar argument, we can also get rid of the hypothesis that $\mathbf{K}$ has primes if the categoricity cardinal is a successor. This allows us to obtain a downward transfer for tame AECs which improves on Theorem 5.5.43 (there H_1 was H_2). The price to pay is to assume more tameness.

Theorem 5.5.50. *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-tame AEC with amalgamation and no maximal models. If* $\mathbf{K}$ *is categorical in a* successor $\lambda > \mathrm{LS}(\mathbf{K})^+$*, then* $\mathbf{K}$ *is categorical in all* $\lambda' \geq \min(\lambda, H_1)$*.*

Proof sketch. Let us work in a good $(\geq \mathrm{LS}(\mathbf{K})^+)$-frame $\mathfrak{s}$ on $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ (this exists by Theorems 5.2.36, 5.5.22, and 5.5.26). As in Section 5.4.2, we can define what it means for two types p and q to be orthogonal (written $p \perp q$) and say that $\mathfrak{s}$ is μ*-unidimensional*[52] if no two types are orthogonal. We can show

[52]In this framework, this definition need not be equivalent to categoricity in the next successor but we use it for illustrative purpose.

that $\mathfrak{s}$ is μ-unidimensional if and only if $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in μ^+, and argue by studying the relationship between forking and orthogonality that $\mathfrak{s}$ is unidimensional in some μ if and only if it is unidimensional in all μ' (this uses tameness, since we are moving across cardinals). Thus $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in every successor cardinal, hence also (by a straightforward argument using Galois-saturated models) in every limit. We conclude by using a version of Morley's omitting type theorem to transfer categoricity in $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ to categoricity in $\mathbf{K}$ (this is where the H_1 comes from). $\square$

What if we do not want to assume that the AEC has primes or that it is categorical in a successor? Then the best known results are essentially Shelah's results from Section 5.2.5. We show how to obtain a particular case using the results presented in this section:

Theorem 5.5.51. *Assume Claim 5.2.0.1 and $2^\theta < 2^{\theta^+}$ for every cardinal θ. Let $\mathbf{K}$ be a fully $\mathrm{LS}(\mathbf{K})$-tame and short AEC with amalgamation and no maximal models. If $\mathbf{K}$ is categorical in some $\lambda \geq H_1$, then $\mathbf{K}$ is categorical in all $\lambda' \geq H_1$.*

Proof sketch. By Theorem 5.2.36, $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable. By the proof of Theorem 5.5.31, we can find an ω-successful good μ-frame (see Definition 5.5.32) on $\mathbf{K}_\mu^{\mu\text{-sat}}$ for some $\lambda < H_1$. By Claim 5.2.0.1, $\mathbf{K}^{\mu^{+\omega}\text{-sat}}$ is categorical in every $\mu' > \mu^{+\omega}$. Using a version of Morley's omitting type theorem, we get that $\mathbf{K}$ must be categorical on a tail of cardinals, hence in a successor above H_1. By Theorem 5.5.50, $\mathbf{K}$ is categorical in all $\lambda' \geq H_1$. $\square$

Remark 5.5.52. Slightly different arguments show that the locality assumption can be replaced by only $\mathrm{LS}(\mathbf{K})$-tameness. Moreover, it can be shown that categoricity in some $\lambda > \mathrm{LS}(\mathbf{K})$ implies categoricity in all $\lambda' \geq \min(\lambda, H_1)$.

The proof shows in particular that almost fully good independence relations can be built in fully tame and short categorical AECs:

Theorem 5.5.53. *Let $\mathbf{K}$ be a fully $\mathrm{LS}(\mathbf{K})$-tame and -type short AEC with amalgamation and no maximal models. If $\mathbf{K}$ is categorical in a $\lambda \geq \left(2^{\mathrm{LS}(\mathbf{K})}\right)^{+4}$, then:*

1. *$\mathbf{K}_{\geq \min(\lambda, H_1)}$ is almost fully good.*
2. *If $\mathbf{K}$ is fully $< \aleph_0$-tame and -type short, then $\mathbf{K}_{\geq \min(\lambda, H_1)}$ is fully good.*

Proof. As in the proof above. Note that by Corollary 5.5.46, $\mathbf{K}$ is categorical in H_1. $\square$

Using large cardinals, we can remove all the hypotheses except categoricity:

Corollary 5.5.54. *Let $\mathbf{K}$ be an AEC. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be a strongly compact cardinal. If $\mathbf{K}$ is categorical in a $\lambda \geq h(\kappa)$, then:*

1. *$\mathbf{K}_{\geq\lambda}$ is fully good.*
2. *If $2^\theta < 2^{\theta^+}$ for every cardinal θ and Claim 5.2.0.1 hold, then*[53] *$\mathbf{K}$ is categorical in all $\lambda' \geq h(\kappa)$.*

Proof sketch. By a result similar to Theorem 5.3.10, $\mathbf{K}_{\geq\kappa}$ has amalgamation and no maximal models. By Theorem 5.3.7, $\mathbf{K}$ is fully $< \kappa$-tame and -type short. Now Corollary 5.5.39 gives the first part. Theorem 5.5.51 gives the second part. □

5.6 Conclusion

The classification theory of tame AECs has progressed rapidly over the last several years. The categoricity transfer results of Grossberg and VanDieren indicated that tameness (along with amalgamation, etc.) is a powerful tool to solve questions that currently seem out of reach for general AECs.

Looking to the future, there are several open question and lines of research that we believe deserve to be further explored.

1. **Levels of tameness**
   This problem is less grandiose than other concerns, but still concerns a basic unanswered question about tameness: are there nontrivial relationships between the parametrized versions of tameness in Definition 5.3.2? For example, does κ-tameness for α-types imply κ-tameness for β-types when $\alpha < \beta$? This question reveals a divide in the tameness literature: some results only use tameness for 1-types (such as the categoricity transfer of Grossberg and VanDieren Theorem 5.4.10 and the deriving a frame from superstability Theorem 5.5.22), while others require full tameness and shortness (such as the development of $< \kappa$-satisfiability Theorem 5.5.19). Answering this question would help to unify these results.

   Another stark divide is revealed by examining the list of examples of tame AECs in Section 5.3.2.1: the list begins with general results that give some form of locality at a cardinal λ. However, once the list reaches concrete classes of AECs, every example turns out to be $< \aleph_0$-tame (often this is a result of a syntactic characterization of Galois types, but not always). This suggests the question of what lies between or even if the general results can be strengthened down to $< \aleph_0$-tameness. For the large cardinal results, this seems impossible: the large cardinal κ should give no information about the low properties of $\mathbf{K}$ below it because

[53]The same conclusion holds assuming only that κ is a measurable cardinal. Moreover, if $\mathbf{K}$ is axiomatized by an $\mathbb{L}_{\kappa,\omega}$ theory, we can replace $h(\kappa)$ with $h(\kappa + \mathrm{LS}(\mathbf{K}))$ and do not need to assume that $\kappa > \mathrm{LS}(\mathbf{K})$.

this cardinal disappears in $\mathbf{K}_\kappa$. The other results also seem unlikely to have this strengthening, but no counter example is known. Indeed, the following is still open: Is there an AEC $\mathbf{K}$ that is $\aleph_0$-tame but not $< \aleph_0$-tame?

2. **Dividing lines**
This direction has two prongs. The first prong is to increase the number of dividing lines. So far, the classification of tame AECs (and AECs in general) has focused on the superstable or better case with a few forays into strictly stable [BG, BVa]. Towards the goal of proving Shelah's Categoricity Conjecture, this focus makes sense. However, this development pales in comparison to the rich structure of classification theory in first-order[54]. Exploring the correct generalizations of NIP, NTP_2, etc. may help fill out the AEC version of this map. It might be that stronger locality hypotheses than tameness will have to be used: as we have seen already in the superstable case, it is only known how to prove the existence of a global independence relation assuming full $(< \aleph_0)$-tameness and shortness.

   The other prong is to turn classification results into true dividing lines. In the first-order case, dividing lines correspond to nice properties of forking on one side and to chaotic non-structure results on the other. In AECs, the non-structure side of dividing lines is often poorly developed and most results either revolve around the order property or use non-ZFC combinatorial principle. While these combinatorial principles seem potentially necessary in arbitrary AECs[55], a reasonable hope is that tame AECs will allow the development of stronger ZFC nonstructure principles. For example, Shelah claims that in AECs with amalgamation, the order property (or equivalently in the tame context stability, see Theorem 5.5.14) implies many models on a tail of cardinals. However there is no known analog for superstability: does unsuperstability imply many models?

3. **Interaction with other fields**
Historically, examples have not played a large role in the study of AECs. Examples certainly exist because $\mathbb{L}_{\kappa,\omega}$ sentences provide them, but the investigation of specific classes is rarely carried out[56]. A better understanding of concrete examples would help advance the field in two ways. First, nontrivial applications would help provide more motivation for

[54] Part of this structure is represented graphically at `http://forkinganddividing.com` by Gabe Conant.

[55] For instance, result the statement "Categoricity in λ and less than the maximum number of models in λ^+ implies λ-AP" holds under weak diamond, but fails under Martin's axiom [She09b, Conclusion I.6.13].

[56] A large exception to this is the study of quasiminimal classes (see Example 5.3.2.1.(10)) by Zilber and others, which are driven by questions from algebra.

exploring AECs[57]. Second, interesting applications can help drive the isolation of new AEC properties that might, *a priori*, seem strange.

This interaction has the potential to go the other way as well: one can attempt to study a structure or a class of structures by determining where the first-order theory lies amongst the dividing lines and using the properties of forking there. However, if the class is not elementary, then the first-order theory captures new structures that have new definable objects. These definable objects might force the elementary class into a worse dividing line. However, AECs offer the potential to look at a narrower, better behaved class. For instance, an interesting class might only have the order property up to some length λ[58] or only be able to define short and narrow (but infinite) trees. Looking at the first-order theory loses this extra information and looking at the class as an AEC might move it from an unstable elementary class to a stable AEC.

4. **Reverse mathematics of tameness**
   The compactness theorem of first-order logic is equivalent to a weak version of the axiom of choice (Tychonoff's theorem for Hausdorff spaces). If we believe that tameness is a natural principle, then maybe the first-order version of "tameness" is also, in the choiceless context, equivalent to some topological principle: what is this principle?
5. **How "natural" is tameness?**
   We have seen that all the known counterexamples of tameness are pathological. Is this a general phenomenon? Are there natural mathematical structures that are, in some sense, well-behaved and should be amenable to a model-theoretic analysis, but are not tame? Would this example then satisfy a weaker version of tameness?
6. **Categoricity and tameness**
   We have seen that tameness helps with Shelah's Categoricity Conjecture, but there are still unanswered questions about eliminating the successor assumption and amalgamation property. For example, does the categoricity conjecture hold in fully $< \aleph_0$-tame and -type short AECs with amalgamation?

   Going the other way, what is the impact of categoricity on tameness? Grossberg has conjectured that amalgamation should follow from high enough categoricity. Does something like this hold for tameness?
7. **On stable and superstable tame AECs**
   From the work discussed in this survey, several more down to earth questions arise:
   (a) Can one build a global independence relation in a fully tame and short superstable AEC? See also Question 5.5.8.

[57] It should be noted that some prominent AEC theorists disagree with this as a motivating principle.

[58] Like example the algebraically closed valued fields of rank one, Example 5.3.2.1.(15).

(b) Is there a stability spectrum theorem for tame AECs (i.e. a converse to Theorem 5.5.14)?
(c) In superstable tame AECs, can one develop the theory of forking further, say by generalizing geometric stability theory to that context?

5.7 Historical remarks

5.7.1 Section 5.2

Abstract elementary classes were introduced by Shelah [She87a]; see Grossberg [Gro02] for a history. Shelah [She87a] (an updated version appears in [She09b, Chapter I]) contains most of the basic results in this Section 5.2, including Theorem 5.2.9. Notation 5.2.11 is due to Baldwin and is used in [Bal09, Chapter 14]. Galois types are implicit in [She87b] where Theorem 5.2.20 also appears. Existence of universal extensions (Lemma 5.2.25) is also due to Shelah and has a similar proof ([She99, I.2.2.(4)]).

Splitting (Definition 5.2.22) is introduced by Shelah in [She99, Definition 3.2]. Lemma 5.2.23 is [She99, Claim 3.3]. The extension and uniqueness properties of splitting (Theorem 5.2.24) are implicit in [She99] but were first explicitly stated by VanDieren [Van06, I.4.10, I.4.12]. The order property is first defined for AECs in [She99, Definition 4.3].

Definition 5.2.27 is implicit already in [SV99], but there amalgamation in μ is not assumed (only a weak version: the density of amalgamation bases). It first appears[59] explicitly (and is given the name "superstable"[60]) in [Gro02, Definition 7.12]. Limit models appear in [SK96, Definition 4.1] under the name "(θ, σ)-saturated". The "limit" terminology is used starting in [SV99]. The reader should consult [GVV] for a history of limit models and especially the question of uniqueness. Theorem 5.2.31 is due to VanDieren [Vanb].

Shelah's eventual categoricity conjecture can be traced back to a question of o [Ło54] which eventually became Morley's categoricity theorem [Mor65]. See the introduction to [Vasf] for a history. Conjecture 5.2.32 appears as an open problem in [She78]. Theorem 5.2.33 is due to Shelah [She83a, She83b]. Conjecture 5.2.34 appears as [She00, Question 6.14]. Theorem 5.2.35 is the main result of [She99]. Theorem 5.2.36 appears in [SV99, Theorem 2.2.1] under GCH but without amalgamation. Assuming amalgamation (but in ZFC),

[59]With minor variations: joint embedding and existence of a model in μ is not required.

[60]This can be seen as a somewhat unfortunate naming, as there are several potentially non-equivalent definitions of superstability in AECs. Some authors use "no long splitting chains", but this omits the conditions of amalgamation, no maximal models, and joint embedding in μ. Perhaps it is best to think of the definition as a weak version of having a good μ-frame.

the proof is similar, see [GV, Corollary 6.3]. The proof of Shelah and Villaveces omits some details. A clearer version will appear in [BGVV]. An easier proof exists if the categoricity cardinal has high-enough cofinality, see [She99, Lemma 6.3]. Question 5.2.37 is stated explicitly as an open problem in [Bal09, Problem D.1.(2)]. Theorems 5.2.38 and 5.2.39 are due to VanDieren and the second author [VV, Section 7].

Good frames are the main concept in Shelah's book on classification theory for abstract elementary classes [She09b]. The definition appears at the beginning of Chapter II there, which was initially circulated as Shelah [She09a]. There are some minor differences with the definition we give here, see the notes for Section 5.5 for more. Question 5.2.40 originates in the similar question Baldwin asked for $L(Q)$ [Fri75, Question 21]. For AECs, this is due to Grossberg (see the comments around [She01a, Problem 5]). A version also appears as [She00, Problem 6.13]. Theorem 5.2.41 is due to Shelah [She09c, Theorem VI.0.2]. A weaker version with the additional hypothesis that the weak diamond ideal is not λ^{++}-saturated appears is proved in Shelah [She01a], see [She09b, Theorem II.3.7]. Corollary 5.2.42 is the main result of [She01a]. Theorem 5.2.43 is the main result of [She09b, Chapter II]. Claim 5.2.0.1 is implicit in [She09b, Discussion III.12.40] and a proof should appear in [Sheb]. Theorem 5.2.44 is due to Shelah and appears in [She09b, Section IV.4]. Shelah claims that categoricity in a proper class of cardinals is enough but this is currently unresolved, see [BVb] for a more in-depth discussion. Theorems 5.2.44, 5.2.45, and 5.2.46 appear in [She09b, Section IV.7]. However we have not fully checked Shelah's proofs. A stronger version of Theorem 5.2.44 has been shown by VanDieren and the second author in [VV, Section 7], while [Vasc, Section 11] gives a proof of Theorems 5.2.45 and 5.2.46 (with alternate proofs replacing the hard parts of Shelah's argument).

5.7.2 Section 5.3

The version of Definition 5.3.1 using types over sets is due to the second author [Vas16b, Definition 2.22]. Type-shortness was isolated by the first author [Bon14c, Definition 3.3]. Locality and compactness appear in [BS08]. Proposition 5.3.4 is folklore. As for Proposition 5.3.3, the first part appears as [Bon14c, Theorem 3.5], the second and third first appear in Baldwin and Shelah [BS08], and the third is implicit in [She99] and a short proof appears in [Bal09, Lemma 11.5].

In the framework of AECs, the Galois Morleyization was introduced by the second author [Vas16b] and Theorem 5.3.5 is proven there. After the work was completed, we were informed that a 1981 result of Rosick [Ros81] also uses a similar device to present concrete categories as syntactic classes. That tameness can be seen as a topological principle (Theorem 5.3.6) appears in Lieberman [Lie11b].

On Section 5.3.2.1:

1. Tameness could (but historically was not) also have been extracted from

the work of Makkai and Shelah on the model theory of $L_{\kappa,\omega}$, κ a strongly compact [MS90]. There the authors prove that Galois types are, in some sense, syntactic [MS90, Proposition 2.10][61]. The first author [Bon14c] generalized these results to AECs and later observations in [BTR, BU] slightly weakened the large cardinal hypotheses.

2. Theorem 5.3.8 is due to Shelah. The first part appears essentially as [She99, II.2.6] and the second is [She09b, IV.7.2]. The statement given here appears as [Vasc, Theorem 8.4].
3. Theorem 5.3.10 is essentially [She01b, Conclusion 3.7].
4. This is folklore and appears explicitly on [GK, p. 15].
5. The study of the classification theory of universal classes starts with [She87b] (an updated version appears as [She09c, Chapter V]), where Shelah claims a main gap for this framework (the details have not fully appeared yet). Theorem 5.3.11 is due to the first author [Bonc]. A full proof appears in [Vasf, Theorem 3.7].
6. Finitary AECs were introduced by Hyttinen and Kesl [HK06]. That $\aleph_0$-Galois-stable $\aleph_0$-tame finitary AECs are $< \aleph_0$-tame is Theorem 3.12 there. The categoricity conjecture for finitary AECs appears in [HK11]. The beginning of a geometric stability theory for finitary AECs appears in [HK16].
7. Homogeneous model theory was introduced by Shelah in [She70]. See [GL02] for an exposition. The classification theory of this context is well developed, see, for instance [Les00, HS00, HS01, BL03, HLS05]. For connections with continuous logic, see [BB04, SU11].
8. Averageable classes are introduced by the first author in [Bona].
9. A summary of continuous first-order logic in its current form can be found in [BYBHU08]. Metric AECs were introduced in [HH09] and tameness there is first defined in [Zam12].
10. Quasiminimal classes were introduced by Zilber [Zil05]. See [Kir10] for an exposition and [BHHKK14] for a proof of the excellence axiom (and hence of tameness).
11. That the λ-saturated models of a first-order superstable theory forms an AEC is folklore. That it is tame is easy using that the Galois types are the same as in the original first-order class.
12. Superior AECs are introduced in [GK].
13. Hrushovski fusions are studied as AECs in [ZV].
14. This appears in [BET07].
15. This is analyzed in [Bonb].

[61] This was another motivation for developing the Galois Morleyization.

The Hart-Shelah example appears in [HS90], where the authors show that it is categorical in $\aleph_0, \dots, \aleph_n$ but not in $\aleph_{n+1}$. The example was later extensively analyzed by Baldwin and Kolesnikov [BK09] and the full version of Theorem 5.3.14 appears there. The Baldwin-Shelah example appears in [BS08]. The Shelah-Boney-Unger example was first introduced by Shelah [Shec] for a measurable cardinal and adapted by Boney and Unger [BU] for other kinds of large cardinals.

5.7.3 Section 5.4

The categoricity transfer for universal classes is due to the second author [Vasf][62]. This section presents a proof incorporating ideas from the later paper [Vase]. If not mentioned otherwise, results and definitions there are due to the second author.

Lemma 5.4.5 is folklore when atomic equivalence is transitive but is [Vasf, Theorem 4.14] in the general case. As for Theorem 5.4.6, one direction is folklore. The other direction (tameness implies that the good frame can be extended) is due to the authors, see the notes on Theorem 5.5.26 below. The version with weak amalgamation (Theorem 5.4.7) is due to the second author.

Theorem 5.4.10 is due to Grossberg and VanDieren [GV06a]. Definition 5.4.11 is due to Shelah [She09b, Definition III.3.2]. The account of orthogonality and unidimensionality owes much to Shelah's development in [She09b, Sections III.2,III.6] but differs in some technical points explained in details in [Vase]. Theorem 5.4.16 is due to Shelah [She09b, III.2.3]. Theorem 5.4.17 is due to Shelah with stronger hypotheses [She09b, Claim III.12.39] and to the second author as stated [Vase, Theorem 2.15].

5.7.4 Section 5.5

Question 5.1.3 is implicit in [She99, Remark 4.9]. A more precise statement appears as [BGKV16, Question 7.1].

The presentation of abstract independence given here appears in [Vasa][63]. The definition of a good frame given here (Definition 5.5.12) also appears in [Vasa, Definition 8.1]. Compared to Shelah's original definition ([She09b, Definition II.2.1]), the definition given here is equivalent [Vasa, Remarks 3.5, 8.2] except for three minor differences:

- The existence of a superlimit model is not required. This has been adopted in most subsequent works on good frames, including e.g. [JS13].
- Shelah's definition requires forking to be defined for types over models only. However it is possible to close the definition to types over sets (see for example [BGKV16, Lemma 5.4]).

[62] Previous version of this preprint claimed the full categoricity conjecture but gaps have been found and a complete solution has been delayed to a sequel.

[63] There independence relations are not required to satisfy base monotonicity.

- Shelah defines forking only for a subclass of all types which he calls *basic*. They are required to satisfy a strong density property (if $M < N$, then there is a basic type over M realized in N). If the basic types are all the (nonalgebraic) types, Shelah calls the good frame *type-full*. In the tame context, a type-full good frame can always be built (see [GV, Remark 3.10]). Even in Theorem 5.2.41, the frame can be taken to be type-full (see [She09b, Claim III.9.6]). The bottom line is that in all cases where a good frames is known to exist, a type-full one also exists.

Question 5.5.8 appears (in a slightly different form) as [BGKV16, Question 7.1]. Theorem 5.5.9 is Corollary 5.19 there[64]. As for Proposition 5.5.10, all are folklore except (2) which appears as [Vasa, Lemma 4.5] and symmetry which in this abstract framework is [BGKV16, Corollary 5.18] (in the first-order case, the result is due to Shelah [She78] and uses the same method of proof: symmetry implies failure of the order property).

Galois stability was defined for the first time in [She99]. The second part of Theorem 5.5.13 is due to Grossberg and VanDieren [GV06b, Corollary 6.4]. Later the argument was refined by Baldwin, Kueker, and VanDieren [BKV06] to prove the first part. Theorem 5.5.14 is due to the second author [Vas16b, Theorem 4.13].

Averages in the nonelementary framework were introduced by Shelah (for stability theory inside a model) in [She87b], see [She09c, Chapter V.A]. They are further used in the AEC framework in [She09b, Chapter IV]. The Galois Morleyization is used by the authors to translate Shelah's results from stability theory inside a model to tame AECs in [BVc, Section 5]. They are further used in [GV].

That tameness can be used to obtain a global uniqueness property for splitting (Theorem 5.5.15) is due to Grossberg and VanDieren [GV06b, Theorem 6.2]. $< \kappa$-satisfiability was introduced as κ-coheir in the AEC framework by Grossberg and the first author [BG]. This was strongly inspired from the work of Makkai and Shelah [MS90] on coheir in $L_{\kappa,\omega}$, κ a strongly compact. A weakening of Theorem 5.5.19 appears in [BG], assuming that coheir has the extension property. The version stated here is due to the second author and appears as [Vas16b, Theorem 5.15]. Theorem 5.5.20 is [BG, Theorem 8.2]. The definition of μ-forking (5.5.16) is due to the second author and appears in [Vas16a]. Theorem 5.5.17 is proven in [Vasa, Section 7]. Theorem 5.5.21 is due to the authors [BVc, Theorem 6.10].

Theorem 5.5.22.(1) is due to the second author [Vasa, Proposition 10.10]. Theorem 5.5.22.(2) is due to VanDieren and the second author [VV, Corollary 6.10] (an eventual version appears in [BVc], and an improvement of VanDieren [Van16] can be used to obtain the full result). Theorem 5.5.22.(3-4) are also due to VanDieren and the second author [VV], although (3) and (4) were

[64] Of course the general idea of looking at forking as an abstract independence relation which turns out to be canonical is not new (see for example Lascar's proof that forking is canonical in superstable theories [Las76, Theorem 4.9]).

observed by the second author in [Vas16a] in the categorical case (i.e. when we know that the union of a chain of λ-Galois-saturated models is λ-Galois-saturated).

Theorem 5.5.23 and Remark 5.5.25 are due to Grossberg and the second author [GV]. The notion of a superlimit model appears already in Shelah's original paper on AECs [She87a] (see [She09b, Chapter I]). Shelah introduces solvability in [She09b, Definition IV.1.4]. Lemma 5.5.24 appears in [GV, Lemma 3.8]. When κ is strongly compact, it can be traced back to Makkai-Shelah [MS90, Proposition 4.12].

Theorem 5.5.26 is due to the authors and appears in full generality in [BVd]. Rami Grossberg told us that he privately conjectured the result in 2006 and told it to Adi Jarden and John Baldwin (see also the account in the introduction to [Jar16]). In [Bon14b, Theorem 8.1], the first author proved the theorem with an additional assumption of tameness for *two* types used to transfer symmetry. Later, [BVd] showed that symmetry transfer holds without this extra assumption. At about the same time as [BVd] was circulated, Adi Jarden gave a proof of symmetry from tameness assuming an extra property called the continuity of independence (he also showed that this property followed from the existence property for uniqueness triples). The argument in [BVd] shows that the continuity of independence holds under tameness and hence also completes Jarden's proof.

Independent sequences were introduced by Shelah in the AEC framework [She09b, Definition III.5.2]. A version of Theorem 5.5.27 for models of size λ is proven as [She09b, III.5.14] with the assumption that the frame is weakly successful. This is weakened in [JS12], showing that the so-called continuity property of independence is enough. In [BVd], the continuity property is proven from tameness and hence Theorem 5.5.27 follows, see [BVd, Corollary 6.10].

Definition 5.5.28 is due to the second author [Vasa, Definition 8.1]. The definition of almost good (Definition 5.5.29) is implicit there and made explicit in [Vasf, Definition A.2]. Fully good and almost fully good are also defined there. Theorem 5.5.31 and Theorem 5.5.37 are due to the second author. A statement with a weaker Hanf number is the main result of [Vasa] (the proof uses ideas from Shelah [She09b, Chapter III] and Adi Jarden [Jar16]). The full result is proven in [Vasf, Appendix A]. What it means for a frame to be successful (Definition 5.5.32) is due to Shelah [She09b, Definition III.1.1] but we use the equivalent definition from [Vasa, Section 11]. Type locality (Definition 5.5.34) is introduced by the second author in [Vasa, Definition 14.9]. Corollary 5.5.39 and Theorem 5.5.40 is implicit in [Vasa] (with the Hanf number improvement in [Vasf, Appendix A]). Theorem 5.5.41 is due to the second author [Vasf, Appendix C].

Theorem 5.5.43 appears in [GV06c]. A version of Theorem 5.5.44 is already implicit in [Vasa, Section 10]. Shelah's omitting type theorem (Theorem 5.5.45) appears in its AEC version as [She99, Lemma II.1.6] and has its roots in [MS90, Proposition 3.3], where a full proof already appears. Corollary 5.5.46

is due to the second author [Vasc, Theorem 9.8]. The categoricity conjecture for tame AECs with primes (Theorem 5.5.47) is due to the second author (the result as stated here was proven in a series of papers [Vasf, Vase, Vasc], see [Vasc, Corollary 10.9]). The converse from Remark 5.5.48 is stated in [Vasb]. The categoricity conjecture for homogeneous model theory is more or less implicit in [She70] and is made explicit by Hyttinen in [Hyt98] (when the language is countable, this is due to Lessmann [Les00][65]). More precisely, Hyttinen proves that categoricity in a $\lambda > |T|$ with $\lambda \neq |T|^{+\omega}$ implies categoricity in all $\lambda' \geq \min(\lambda, h(|T|))$. Corollary 5.5.49 is stronger (as it includes the case $\lambda = |T|^{+\omega}$) and is due to the second author [Vase, Theorem 0.2]. Theorem 5.5.50 is due to the second author [Vasc, Corollary 10.6]. Theorem 5.5.51 is due to the second author (although the main idea is due to Shelah, and the only improvement given by tameness is the Hanf number, see Theorem 5.2.46). With full tameness and shortness, a weaker version appears in [Vasa, Theorem 1.6], and the full version using only tameness is [Vasc, Corollary 11.9]. The second part of Corollary 5.5.54 also appears there.

5.7.5 Section 5.6

Several of these questions have been in the air and there is some overlap with the list [Bal09, Appendix D]. The question about the length of tameness appears in the first author's Ph.D. thesis [Bon14a]. A question about examples of tameness and nontameness appear in [Bal09, Appendix D.2]. Whether failure of superstability implies many models is conjectured in [She99] (see the remark after Claim 5.5 there) and further discussed at the end of [GV].

The idea of exploring the reverse mathematics of tameness (and the specific question of what tameness corresponds to if compactness is the Tychnoff theorem for Hausdorff spaces) was communicated to the second author by Rami Grossberg. That tameness follows from categoricity was conjectured by Grossberg and VanDieren [GV06a, Conjecture 1.5]. That one can build a global independence relation in a fully tame and short superstable AEC is conjectured by the second author in [Vasa, Section 15].

[65] In that case, a stronger statement can be proven: if D is categorical in some uncountable cardinal, then it is categorical in all uncountable cardinals.

Bibliography

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BB04] Alexander Berenstein and Steven Buechler, *Simple stable homogeneous expansions of hilbert spaces*, Annals of Pure and Applied Logic **128** (2004), 75–101.

[BET07] John T. Baldwin, Paul C. Eklof, and Jan Trlifaj, *$^{\perp}N$ as an abstract elementary class*, Annals of Pure and Applied Logic **149** (2007), 25–39.

[BG] Will Boney and Rami Grossberg, *Forking in short and tame AECs*, Preprint. URL: `http://arxiv.org/abs/1306.6562v9`.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BGVV] Will Boney, Rami Grossberg, Monica VanDieren, and Sebastien Vasey, *The Shelah-Villaveces theorem*, In preparation.

[BHHKK14] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä, and Jonathan Kirby, *Quasiminimal structures and excellence*, Bulletin of the London Mathematical Society **46** (2014), no. 1, 155–163.

[BK09] John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*, Israel Journal of Mathematics **170** (2009), 411–443.

[BKL] John T. Baldwin, Martin Koerwien, and Michael C. Laskowski, *Disjoint amalgamation in locally finite AEC*, The Journal of Symbolic Logic, To appear. URL: `http://www.math.uic.edu/~jbaldwin/pub/BKLstreamlined1.pdf`.

[BKV06] John T. Baldwin, David Kueker, and Monica VanDieren, *Upward stability transfer for tame abstract elementary classes*, Notre Dame Journal of Formal Logic **47** (2006), no. 2, 291–298.

[BL03] Steven Buechler and Olivier Lessmann, *Simple homogeneous models*, Journal of the American Mathematical Society **16** (2003), no. 1, 91–121.

[Bona] Will Boney, *The Γ-ultraproduct and averageable classes*, Preprint. URL: `http://arxiv.org/abs/1511.00982v2`.

[Bonb] Will Boney, *Some model theory of classically valued fields*, In preparation.

[Bonc] Will Boney, *Tameness in groups and similar AECs*, In preparation.

[Bon14a] Will Boney, *Advances in classification theory for abstract elementary classes*, Ph.D. thesis, 2014.

[Bon14b] Will Boney, *Tameness and extending frames*, Journal of Mathematical Logic **14** (2014), no. 2, 1450007.

[Bon14c] Will Boney, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BS08] John T. Baldwin and Saharon Shelah, *Examples of non-locality*, The Journal of Symbolic Logic **73** (2008), 765–782.

[BTR] Andrew Brooke-Taylor and Jií Rosick, *Accessible images revisited*, Proceedings of the American Mathematical Society, To appear. URL: `http://arxiv.org/abs/1506.01986v2`.

[BU] Will Boney and Spencer Unger, *Large cardinal axioms from tameness*, Preprint. URL: `http://arxiv.org/abs/1509.01191v2`.

[BVa] Will Boney and Monica VanDieren, *Limit models in strictly stable abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1508.04717v3`.

[BVb] Will Boney and Sebastien Vasey, *Categoricity and infinitary logics*, Preprint. URL: `http://arxiv.org/abs/1508.03316v2`.

[BVc] Will Boney and Sebastien Vasey, *Chains of saturated models in AECs*, Preprint. URL: `http://arxiv.org/abs/1503.08781v3`.

[BVd] Will Boney and Sebastien Vasey, *Tameness and frames revisited*, Preprint. URL: `http://arxiv.org/abs/1406.5980v5`.

[BYBHU08] Itay Ben-Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), vol. 2, Cambridge University Press, 2008, pp. 315–427.

[EM02] Paul C. Eklof and Alan Mekler, *Almost free modules: set-theoretic methods*, revised ed., North Holland Mathematical library, vol. 46, North-Holland, 2002.

[Fri75] Harvey M. Friedman, *One hundred and two problems in mathematical logic*, The Journal of Symbolic Logic **40** (1975), no. 2, 113–129.

[GK] Rami Grossberg and Alexei Kolesnikov, *Superior abstract elementary classes are tame*, Preprint. URL: `http://www.math.cmu.edu/~rami/AtameP.pdf`.

[GL02] Rami Grossberg and Olivier Lessmann, *Shelah's stability spectrum and homogeneity spectrum in finite diagrams*, Archive for Mathematical Logic **41** (2002), no. 1, 1–31.

[Gro] Rami Grossberg, *A course in model theory I*, A book in preparation.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GV] Rami Grossberg and Sebastien Vasey, *Superstability in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1507.04223v3`.

[GV06a] Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 2, 181–201.

[GV06b] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GV06c] Rami Grossberg and Monica VanDieren, *Shelah's categoricity conjecture from a successor for tame abstract elementary classes*, The Journal of Symbolic Logic **71** (2006), no. 2, 553–568.

[GVV] Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly, To appear. URL: `http://arxiv.org/abs/1507.02118v1`.

[Har75] Victor Harnik, *On the existence of saturated models of stable theories*, Proceedings of the American Mathematical Society **52** (1975), 361–367.

[HH09] Åsa Hirvonen and Tapani Hyttinen, *Categoricity in homogeneous complete metric spaces*, Archive for Mathematical Logic **48** (2009), 269–322.

[HK06] Tapani Hyttinen and Meeri Kesälä, *Independence in finitary abstract elementary classes*, Annals of Pure and Applied Logic **143** (2006), 103–138.

[HK11] Tapani Hyttinen and Meeri Kesälä, *Categoricity transfer in simple finitary abstract elementary classes*, The Journal of Symbolic Logic **76** (2011), no. 3, 759–806.

[HK16] Tapani Hyttinen and Kaisa Kangas, *Quasiminimal structures, groups and Zariski-like geometries*, Annals of Pure and Applied Logic **167** (2016), no. 6, 457–505.

[HL02] Tapani Hyttinen and Olivier Lessmann, *A rank for the class of elementary submodels of a superstable homogeneous model*, The Journal of Symbolic Logic **67** (2002), no. 4, 1469–1482.

[HLS05] Tapani Hyttinen, Olivier Lessmann, and Saharon Shelah, *Interpreting groups and fields in some nonelementary classes*, Journal of Mathematical Logic **5** (2005), no. 1, 1–47.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over P for first order T or categoricity for $\phi \in \mathbb{L}_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \dots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[HS00] Tapani Hyttinen and Saharon Shelah, *Strong splitting in stable homogeneous models*, Annals of Pure and Applied Logic **103** (2000), 201–228.

[HS01] Tapani Hyttinen and Saharon Shelah, *Main gap for locally saturated elementary submodels of a homogeneous structure*, The Journal of Symbolic Logic **66** (2001), no. 3, 1286–1302.

[Hyt98] Tapani Hyttinen, *Generalizing Morley's theorem*, Mathematical Logic Quarterly **44** (1998), 176–184.

[Jar16] Adi Jarden, *Tameness, uniqueness triples, and amalgamation*, Annals of Pure and Applied Logic **167** (2016), no. 2, 155–188.

[Jec03] Thomas Jech, *Set theory*, 3rd ed., Springer-Verlag, 2003.

[JS12] Adi Jarden and Alon Sitton, *Independence, dimension and continuity in non-forking frames*, The Journal of Symbolic Logic **78** (2012), no. 2, 602–632.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Annals of Pure and Applied Logic **164** (2013), 135–191.

[Kir10] Jonathan Kirby, *On quasiminimal excellent classes*, The Journal of Symbolic Logic **75** (2010), no. 2, 551–564.

[Kue08] David W. Kueker, *Abstract elementary classes and infinitary logics*, Annals of Pure and Applied Logic **156** (2008), 274–286.

[Las76] Daniel Lascar, *Ranks and definability in superstable theories*, Israel Journal of Mathematics **23** (1976), no. 1, 53–87.

[Les00] Olivier Lessmann, *Ranks and pregeometries in finite diagrams*, Annals of Pure and Applied Logic **106** (2000), 49–83.

[Lie11a] Michael J. Lieberman, *Category-theoretic aspects of abstract elementary classes*, Annals of Pure and Applied Logic **162** (2011), no. 11, 903–915.

[Lie11b] Michael J. Lieberman, *A topology for Galois types in abstract elementary classes*, Mathematical Logic Quarterly **57** (2011), no. 2, 204–216.

[Mor65] Michael Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[MS94] Menachem Magidor and Saharon Shelah, *When does almost free imply free?*, Journal of the American Mathematical Society **7** (1994), no. 4, 769–830.

[Ros81] Jií Rosick, *Concrete categories and infinitary languages*, Journal of Pure and Applied Algebra **22** (1981), no. 3, 309–339.

[Shea] Saharon Shelah, *A.E.C. with not too many models*, Preprint. URL: `http://arxiv.org/abs/1302.4841v2`.

[Sheb] Saharon Shelah, *Eventual categoricity spectrum and frames*, Paper number 842 on Shelah's publication list. Preliminary draft from Oct. 3, 2014 (obtained from the author).

[Shec] Saharon Shelah, *Maximal failure of sequence locality in A.E.C.*, Preprint. URL: `http://arxiv.org/abs/0903.3614v3`.

[She70] Saharon Shelah, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970), no. 1, 69–118.

[She78] Saharon Shelah, *Classification theory and the number of non-isomorphic models*, Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1978.

[She83a] Saharon Shelah, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in \mathbb{L}_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), no. 3, 214–240.

[She83b] Saharon Shelah, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in \mathbb{L}_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), no. 4, 241–273.

[She87a] Saharon Shelah, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] Saharon Shelah, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She99] Saharon Shelah, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She00] Saharon Shelah, *On what I do not understand (and have something to say), model theory*, Math. Japonica **51** (2000), 329–377.

[She01a] Saharon Shelah, *Categoricity of an abstract elementary class in two successive cardinals*, Israel Journal of Mathematics **126** (2001), 29–128.

[She01b] Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, when κ is a measurable cardinal. Part II*, Fundamenta Mathematica **170** (2001), 165–196.

[She09a] Saharon Shelah, *Categoricity in abstract elementary classes: going up inductively*, Classification theory for abstract elementary classes, College Publications, 2009, Paper number 600 in Shelah's publication list, pp. 224–377.

[She09b] Saharon Shelah, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09c] Saharon Shelah, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[SK96] Saharon Shelah and Oren Kolman, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, when κ is a measurable cardinal. Part I*, Fundamentae Mathematica **151** (1996), 209–240.

[SU11] Saharon Shelah and Alexander Usvyatsov, *Model theoretic stability and categoricity for complete metric spaces*, Israel Journal of Mathematics **182** (2011), 157–198.

[SV] Saharon Shelah and Andrés Villaveces, *Categoricity may fail late*, Preprint. URL: `http://arxiv.org/abs/math/0404258v1`.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Vana] Monica VanDieren, *A characterization of uniqueness of limit models in categorical abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1511.09112v2`.

[Vanb] Monica VanDieren, *Superstability and symmetry*, Annals of Pure and Applied Logic, To appear. URL: `http://arxiv.org/abs/1507.01990v4`.

[Van06] Monica VanDieren, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van16] Monica VanDieren, *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 4, 395–407.

[Vasa] Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic, To appear. URL: `http://arxiv.org/abs/1503.01366v6`.

[Vasb] Sebastien Vasey, *Building prime models in fully good abstract elementary classes*, Mathematical Logic Quarterly, To appear. URL: `http://arxiv.org/abs/1509.07024v6`.

[Vasc] Sebastien Vasey, *Downward categoricity from a successor inside a good frame*, Preprint. URL: http://arxiv.org/abs/1510.03780v4.

[Vasd] Sebastien Vasey, *Saturation and solvability in abstract elementary classes with amalgamation*, Preprint. URL: http://arxiv.org/abs/1604.07743v2.

[Vase] Sebastien Vasey, *Shelah's eventual categoricity conjecture in tame AECs with primes*, Preprint. URL: http://arxiv.org/abs/1509.04102v3.

[Vasf] Sebastien Vasey, *Shelah's eventual categoricity conjecture in universal classes. Part I*, Preprint. URL: http://arxiv.org/abs/1506.07024v9.

[Vasg] Sebastien Vasey, *Shelah's eventual categoricity conjecture in universal classes. Part II*, Preprint. URL: http://arxiv.org/abs/1602.02633v1.

[Vas16a] Sebastien Vasey, *Forking and superstability in tame AECs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[Vas16b] Sebastien Vasey, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[VV] Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Preprint. URL: http://arxiv.org/abs/1508.03252v3.

[Zam12] Pedro H. Zambrano, *A stability transfer theorem in d-tame metric abstract elementary classes*, Mathematical Logic Quarterly **58** (2012), 333–341.

[Zil05] Boris Zilber, *A categoricity theorem for quasi-minimal excellent classes*, Logic and its applications (Andreas Blass and Yi Zhang, eds.), Contemporary Mathematics, American Mathematical Society, 2005, pp. 297–306.

[ZV] Pedro H. Zambrano and Andrés Villaveces, *Hrushovski fusions and tame abstract elementary classes*, Preprint.

[Ło54] Jerzy Łoś, *On the categoricity in power of elementary deductive systems and some related problems*, Colloquium Mathematicae **3** (1954), no. 1, 58–62.