The explanatory power of a new proof: Henkin's completeness proof

John Baldwin

January 16, 2017

Mancosu writes

> But explanations in mathematics do not only come in the form of proofs. In some cases explanations are sought in a major recasting of an entire discipline. ([Mancosu, 2008], 142)

This paper takes up both halves of that statement. On the one hand we provide a case study of the explanatory value of a particular milestone proof. In the process we examine how it began the recasting of a discipline.

Hafner and Mancosu take a broad view towards the nature of mathematical explanation. They argue that before attempting to establish a model of explanation, one should develop a 'taxonomy of recurrent types of mathematical explanation' ([Hafner and Mancosu, 2005], 221) and preparatory to such a taxonomy propose to examine in depth various examples of proofs. In [Hafner and Mancosu, 2005] and [Hafner and Mancosu, 2008], they study deep arguments in real algebraic geometry and analysis to test the models of explanation of Kitcher and Steiner. In their discussion of Steiner's model they challenge[1] the assertion [Resnik and Kushner, 1987] that Henkin's proof of the completeness theorem is explanatory, asking 'what the explanatory features of this proof are supposed to consist of?' As a model theorist the challenge to 'explain' the explanatory value of this fundamental argument is irresistible. In contrasting the proofs of Henkin and Gödel, we seek for the elements of Henkin's proofs that permit its numerous generalizations. In Section 2 we try to make this analysis more precise through Steiner's notion of characterizing property. And as we will see, when one identifies the characterizing property of the Henkin proof, rather than a characteristic property of an object in the statement of the theorem then one can find a variant of Steiner's model which applies in this situation. Key to this argument is identifying the family in which to evaluate the proof. We point out in Section 3 that this modification of Steiner depends on recognizing that 'explanatory' is not a determinate concept until one fills in the X in 'explanatory for X'. Thus, our goal is to establish that the Henkin proof is explanatory (contra [Hafner and Mancosu, 2005]) and moreover one can adapt Steiner's model to justify this claim.

This paper developed from a one page treatment in [Baldwin, 2016a] and the discussion of Henkin's role in the transformation of model theory in [Baldwin, 2017]. I thank Juliette Kennedy and Michael Lieberman for their comments on various drafts. And I thank Rami Grossberg and Jouko Väänänen for sources on the consistency property and pressing the important contributions to understanding the completeness theorem of Hintikka, Beth, and Smullyan.

[1]They write, '[In] (Resnik & Kushner 1987, p. 147), it is contended with some albeit rather vague reference to mathematical/logical practice that Henkins proof 'is generally regarded as really showing what goes on in the completeness theorem and the proof-idea has been used again and again in obtaining results about other logical systems?'

1 Comparing Gödel and Henkin on completeness

{compare}

There are two issues. In a comparative sense, how is Henkin's proof more explanatory than Gödel's? In an absolute sense why would one say that Henkin's proof is explanatory? We begin with the first. For this we analyze the different though equivalent statements of the theorem. We will see that one source of the greater explanatory value of Henkin's argument is simply his statement of the result. Then we examine the actual proofs and come to a similar conclusion. An historical issue affects the nomenclature in this paper. While there are some minor variants on Gödel's argument[2], Henkin's proof has become a motif in logic. Some of the essential attributes of what I am calling Henkin's proof were not explicit in the first publication; the most important were enunciated a bit later by Henkin; others arise in the long derivative literature.

1.1 Comparing the statements

Gödel's version of the completeness theorem for first order logic [Gödel, 1929] reads:

Theorem 1.1.1 (Gödel formulation)**.** *Every valid formula expressible in the restricted functional calculus*[3] *can be derived from the axioms by a finite sequence of formal inferences.*

But Henkin states his main theorem as

Theorem 1.1.2 (Henkin formulation)**.** *Let $\mathcal{S}_0$ be a particular system determined by some definite choice of primitive symbols.*

If Λ is a set of formulas of $\mathcal{S}_0$ in which no member has any occurrence of a free individual variable, and if Λ is consistent then Λ is simultaneously satisfiable in a domain of individuals having the same cardinal number as the set of primitive symbols of $\mathcal{S}_0$.

We draw three distinctions between the two formulations. Gödel ([Gödel, 1929], 75) restates the result to be proved as: '*every valid logical expression is provable* ' and continues with the core statement, 'Clearly this can be expressed as *every valid expression is satisfiable or refutable*.' This 'clearly' is as close as Gödel comes to a definition of *valid*. More precisely, the effective meaning of 'ϕ is valid' in Godel's paper is '$\neg\phi$ is not satisfiable' and this double negation is essential. We discuss below the connections between his notion of valid and Tarski's. Henkin makes Godel's core assertion the stated theorem; the transfer to Gödel's original formulation is a corollary. Thus Henkin's proof gains explanatory value as the argument *directly* supports the actual statement of the theorem.

The last paragraph of [Gödel, 1929] extends the argument to *applied logic*. Henkin's 'definite choice of primitive symbols' amounts to fixing an applied axiom system. But still the full list of possible relation variables is present in Gödel's context while the modern scheme introduces additional relations only when needed and thus for Henkin, not at all; only constant symbols are added. Thus, he moves the focus to what becomes the basic stuff of model theory, first order logic in fixed vocabulary.

Finally, as Gödel observes, his argument is restricted to countable vocabularies; Henkin proves the results for uncountable languages.

As Franks [Franks, 2013] emphasizes, Gödel make a huge innovation in focusing on a duality between proof and truth rather than defining completeness in terms of 'descriptive completeness' [Detlefsen, 2014],

[2] [Robinson, 1951, Kreisel and Krivine, 1967]

[3] The term 'functional variables' used by both Gödel and Henkin are what are now thought of as relational variables (or relation symbols) and the functions that interpret them are relations on a set. They are not functions from the universe of a model to itself.

or Post completeness[4] , or 'as all that can be proved[5] (by any means)'. Henkin refines this insight of Gödel.

There is no objection below to the correctness of Godel's proof. There is an objection (not original here) to the casual reading that Gödel proved the theorem in the form (*) below. We draw two lessons from the difference in formulation.

Lesson 1: What hath Tarski wrought? The modern form[6] of the extended completeness theorem reads: for every vocabulary τ and every sentence $\phi \in \mathcal{L}(\tau)$

$$(*) \qquad \Sigma \vdash \phi \text{ if and only if } \Sigma \models \phi.$$

is possible only on the basis of [Tarski, 1935]. It requires a formal definition of logical consequence, $\models$. A conundrum that is often raised but seldom seriously asked is, 'Why did Tarski have to define truth if Gödel proved the completeness theorem earlier?' This query seems to rest on the thought that what Gödel meant by a quantified sentence being true in a structure is just the informal notion generated by the intuition that an existential quantification is true if there is a witness. That is not Gödel's notion. Gödel describes what in modern terms would be the definition of truth for atomic formulas in a structure. There is no *explicit* definition of validity and satisfaction is defined by example for π_2-formulas – that is, his argument invokes the natural intuitive notion of satisfaction for such formulas.

Gödel [Gödel, 1929] writes, 'Here, completeness is to mean that every valid formula expressible in the restricted functional calculus ...can be derived from the axioms by means of a finite sequence of formal inferences.' As we've seen 'valid' is taken as an understood notion. He rephrases the theorem as follows in the published version ([Gödel, 1930]),

> ...the question at once arises whether the initially postulated system of axioms and principles of inference is complete, that is, whether it actually suffices for the derivation of *every* logico-mathematical proposition, or whether, perhaps it is conceivable that there are true propositions (which may be even provable by means of other principles) that cannot be derived in the system under consideration. [Gödel, 1930]

In contrast, Henkin gives a semi-formal definition of 'satisfiable in a domain for an interpretation of the relation symbols on I and assignment[7] of variables to elements I (by an induction on quantifier complexity). He extends to 'valid in a domain' (satisfied by every assignment), and then defines valid as valid in all domains. He remarks in a footnote that this notion could be made more precise along the lines of [Tarski, 1935]. That is, working in naive set theory, he uses Tarski's inductive definition of truth.

Thus Gödel's formulation of the completeness theorem varies from the post-Tarski version in two ways.

1. Gödel's definition of 'satisfiability in a structure' depends on the ambient deductive system. Specifically, the deductive system must support the existence of a π_2-prenex normal form for each non-refutable sentence. We will observe below various logics satisfying completeness theorems that fail this condition.

---

[4] A set of sentences Σ is Post complete if for every sentence ϕ either $\Sigma \cup \{\phi\}$ or $\Sigma \cup \{\neg\phi\}$ is inconsistent.

[5] Franks [Franks, 2010] quotes Gentzen's goal as follows, 'Our formal definition of provability, and, more generally, our choice of the forms of inference will seem appropriate only if it is certain that a sentence q is 'provable' from the sentences $p_1, \ldots p_v$ if and only if it represents informally a consequence of the p's. ([Gentzen, 1932],p. 33). This is an entirely syntactic conception of completeness and so distinct from Gödel.

[6] Apparently the first statement of the 'extended completeness theorem' in this form is in [Robinson, 1951] (Compare [Dawson, 1993], page 24). When $\models$ was introduced is unclear. A clear statement recommending its use appears in the preface to [Addison et al., 1965] but earlier published uses are hard to find. {dawrob}

[7] $(\forall x)A(x)$ is satisfiable just if $A(a)$ is true for each $a \in I$.

2. Gödel does not assert the modern form of the extended completeness theorem (he hasn't defined semantic consequence). Rather he says (Theorem IX), 'Every denumerably infinite set of formulas of the restricted predicate calculus either is satisfiable (that is, all formulas of the system are simultaneously satisfied) or some finite subset is refutable.'

   He proves it by using the compactness theorem for *countable* sets of formulas (implicitly in [Gödel, 1929] and explicitly[8] in [Gödel, 1930]).

Lesson 2: The modern concept of vocabulary. Gödel's emphasis is on the provability of every *valid* formula in the restricted predicate calculus. This calculus has infinitely many relation symbols of each finite arity. In contrast, modern model theory specifies a vocabulary using primitive symbols directly connected to the specific subject. Many mathematicians had adopted this practice (e.g., Pasch, Hilbert, Noether, van der Warden) as geometry became more carefully formalized and notions of groups, rings, and fields developed in the first third of the twentieth century. But the incorporation of this requirement into logic took place later[9].

Henkin's proof of the completeness theorem was a crucial step towards the modern conception of vocabulary[10]. In general he specifies that a system contains 'for each number $n = 1, 2, \ldots$ a set of functional symbols (relation symbols[11]) of degree n which may be separated into variables and constants'. Henkin makes the modern convention of a fixed vocabulary (e.g. symbols $+, \times, 0, 1$ for rings) completely explicit in [Henkin, 1953], which is the published version of the 'algebra' portion of his 1947 thesis, where he used the old notation. In contrast to modern practice, both authors treat the extensional treatment of equality as an 'add-on'; since they are dealing with a purely relational language they avoid the complication of including in the axioms the requirement that equality is a congruence for each function symbol in the language. In [Henkin, 1953], he applies the completeness theorem to theories with function symbols and equality with no explanation of the shift from a relational language. Modern versions of Henkin's proof require that the equivalence relation of equality is a *congruence* (preserved by operations of the vocabulary). If fact, this extension turns out to be central, for example, in the discussion of the omitting types theorem below.

A key distinction[12] between Gödel and Henkin is that Henkin's proof adds only constants to the vocabulary while to settle the question for a particular sentence Gödel draws on finitely many relation symbols that do not appear in the given sentence.

Before stating the theorem, Henkin restricts the context on page 161 of [Henkin, 1949] with 'Let S_0 be a particular system determined by some definite choice of primitive symbols.' This apparently minor remark is central to changing the viewpoint from logic as an analysis of reasoning to model theory as a mathematical tool. Henkin emphasizes this aspect of a second difference between his description of the setting from Gödel's.

> In the first place an important property of formal systems which is associated with completeness can now be generalized to systems containing a non-denumerable infinity of primitive symbols. While this is not of especial interest when formal systems are considered as logics–i.e., as means for analyzing the structure of languages– it leads to interesting applications in the field of abstract algebra. [Henkin, 1949]

[8]This historical distinction has been emphasized by Franks in [Franks, 2013].

[9]The abstract notion of 'structure' (for a given vocabulary) was first formalized in 1935 by Birkhoff [Birkhoff, 1935]; both Tarski [Tarski, 1946] and Robinson [Robinson, 1952] refer to that paper Robinson specifies a vocabulary for the particular topic. Tarski certainly has arrived at the modern formulation by [Tarski, 1954] and [Tarski and Vaught, 1956].

[10]I use the word vocabulary; similarity type or one use of 'language' are synomyms.

[11]The article is part of his five year project of revising and publishing his thesis on both second order logic and the theory of types; the function variables appear to accommodate these extensions.

[12]Henkin's procedure preserves for example ω-stability of the initial theory and that must fail for some theories using Gödel's proof.

Henkin (Corollary 2) uses the uncountable vocabulary to deduce the full force of the Löwenheim-Skolem-Tarski theorem: a consistent first order theory has models in every infinite cardinality[13]

1.2 Contrasting the proofs

Hafner and Mancosu [Hafner and Mancosu, 2005] list a number of mottos for explanation that they found in the mathematical literature. Simplifying a bit, they center around the notion of 'deep reason'. In this section we try to identify the 'deep reason' behind Henkin's proof; in the next we fill this out with further explication of its generalizability. Here are outlines of the two proofs.

Gödel:

G1. Citing the 1928 edition of [Hilbert and Ackermann, 1938], Gödel notes that an arbitrary formula may be assumed to be in prenex normal form, π_n. That is, n alternating blocks of universal and existential quantifiers followed by a quantifier-free matrix.

G2. By adding additional relation symbols, Gödel Skolemizes[14] the entire logic and makes every formula equivalent to a π_2-formula[15].

G3. Then he shows that every π_2-formula is either refutable or satisfiable in a countably infinite structure[16].

Note that steps 1) and 2) are entirely syntactic and are in fact are theorem schemas - patterns for proofs within the system.

Henkin:

I describe Henkin's proof in more detail using some of the expository enhancements (e.g. the term 'Henkin theory') that have been made in the more than half century the argument has been the standard. T is a *Henkin theory* if[17] for every formula $\phi(x)$, there is a *witness constant* c_ϕ such that

$$T \vdash (\exists x)\phi(x) \rightarrow \phi(c_\phi).$$

H1. Every syntactically consistent theory can be extended to a Post-complete Henkin theory.

H2. Every Post-complete syntactically consistent theory with the witness property has a canonical *term model*.

Henkin's vital insight is the separation of the problem into two parts, 1) extend the given theory to one that is complete and satisfies certain additional syntactic properties such that 2) there is a functor from the theories in H1 to structures that realize the original theory. Both of these steps are metamathematical. We report below Henkin's account of his roundabout way of arriving at the necessity of this extension of a theory to a larger one.

[13] Note that the proof of completeness for countable vocabularies uses only König's infinity lemma [Smullyan, 1966], while the existence of arbitrarily large models via Henkin's argument uses the Boolean prime ideal theorem and the full Löwenheim-Skolem-Tarski theorem requires the axiom of choice.

[14] See `http://mathoverflow.net/questions/45487/compactness-theorem-for-first-order-logic` for Blass's outline of a proof using Skolem functions and reduction to propositional logic (nominally for compactness).

[15] Contrary to contemporary terminology where Skolem implies Skolem *function*, each formula $\phi(\mathbf{x}, \mathbf{y})$ is replaced by a relation symbol $F_{\phi(\mathbf{x},\mathbf{y})}(\mathbf{x}, \mathbf{y})$ such that $(\exists \mathbf{x}, \mathbf{y})F_{\phi(\mathbf{x},\mathbf{y})}(\mathbf{x}, \mathbf{y}) \wedge \forall \mathbf{x}[(\exists \mathbf{x}, \mathbf{y})\phi(\mathbf{x}, \mathbf{y}) \rightarrow (\exists \mathbf{y})F_{\phi(\mathbf{x},\mathbf{y})}(\mathbf{x}, \mathbf{y}) \wedge \phi(\mathbf{x}, \mathbf{y})$. This difference is why he can reduce only to π_2 and not universal formulas.

[16] In fact, he constructs the model as a subset of the natural numbers using the arithmetic of the natural numbers for the construction.

[17] [Chang and Keisler, 1973] say 'has the witness property'.

Henkin's proof explains the actual argument: finding a model of an irrefutable sentence. The two steps each contribute to that goal.

H1. The extension of an arbitrary consistent T to one satisfying the witnessing property depends precisely on the axioms and rules of inference of the logic.

The Post-completeness is obtained by an inductive construction: add ϕ_α or $\neg\phi_\alpha$ at stage α to ensure that each sentence is decided.

H2 To construct the model, consider the set of witnesses, M and show that after modding out by the equivalence relation cEd if and only $T \vdash c = d$, the structure $M' = M/E$ satisfies T. More precisely, show by induction on formulas that for any formula $\phi(\mathbf{c})$,

$$T \vdash \phi(\mathbf{c}) \text{ if and only if } M' \models \phi(\mathbf{c}).$$

For this he uses both the Post-completeness and the witness property from H1.

Karp [Karp, 1959] generalized the completeness result to the infinitary logic[18] $L_{\omega_1,\omega}$. This argument has been the source of many generalizations of the result with names that emphasize the goal of step H2 while modifying the argument in H1 to obtain an appropriate 'Henkin theory', such as the *consistency property*[19] [Smullyan, 1963, Makkai, 1969] and with the theorem renamed as *the model existence theorem* in [Keisler, 1971].

My original impetus [Baldwin, 2016a] for discussing Henkin's proof was to give a serious example of how mathematical induction is used in abstract mathematics. There are too many investigations of 'explanation' to list that center around the extremely elementary uses of mathematical induction. But in more advanced mathematics the main use of induction is as proof tool to study objects defined by generalized inductive definition. This includes not only such algebraic constructions as the closure of a set to a subgroup or in a logic, the set of formulas in logic or theorems of a theory, but constructions as in the Henkin proof: truth in a structure, completing a theory and fulfilling the witness property. These constructions are followed by the complementary proof by induction that the canonical structure is in fact a model. A similar pattern of inductive definition is central in Gödel's proof. But, more is on the syntactic side taking place as metatheorems on prenex normal form and equivalences between formulas. However, still another inductive definition is a piece of his proof that a π_2-sentence is satisfiable.

There are several fundamental distinctions between the foundational outlooks of Gödel and Henkin.

Gödel works in a background theory of naive set theory and studies a single system of logic with predicates of arbitrary order; this is essential to the proof. He has a definition of truth for atomic *formulas, which is extended by deductive rules to determine truth in a structure for arbitrary sentences.*

Henkin (by 1951) works in a background theory of naive set theory and studies the first order logic of each vocabulary. The proof for each vocabulary adds only constant *symbols. He has a* uniform definition of truth *in a structure for each vocabulary that has no dependence on the deductive rules of the logic.*

This second view underlies modern model theory. While technically, one could incorporate Gödel's argument into the modern framework (by adding the additional predicates *ad hoc*) this is not only cumbersome but raises the question of what these new predicates have to do with the original topic – something more to explain.

[18] The logic allows countable conjunctions but quantifies over only finite sequences of variable. Proofs can have infinitely many hypotheses.

[19] The name 'consistency property' is apparently introduced in [Smullyan, 1963]. Smullyan remarks that his method takes from Henkin that only constants need to be added to the original vocabulary; Makkai, who is proving preservation and interpolation theorems, carefully lays out the connection of his proof with [Smullyan, 1963].

2 Generalization and Steiner

{Steiner}

In Steiner's seminal [Steiner, 1978], he first rejects the view that 'a proof is more explanatory than another because more general' by specifying more carefully what is involved in an explanatory generalization.

Steiner proposed the notion of *characterizing property* to clarify 'explanatory'.

> We have, then, this result: an explanatory proof depends on a characterizing property of something mentioned in the theorem: if we 'deform' the proof, substituting the characterizing property of a related entity, we get a related theorem. A characterizing property picks out one from a family ['family' is undefined in the essay]; an object might be characterized variously if it belongs to distinct families. ([Steiner, 1978], 147)

Resnik and Kushner assess Henkin's proof as follows"

> The proof is generally regarded as really showing what goes on in the completeness theorem and the proof-idea has been used again and again in obtaining results about other logical systems. Yet again, it is not easy to identify the characterizing property on which it depends. [Resnik and Kushner, 1987]

We elaborate below what we think is the source of the 'generally regarded'. To do this, we modify Steiner's notion. He notes on page 143 that the characteristic property refers to a property of the theorem, not the proof, and thus is an absolute rather than relative evaluation of the explanatory value of the proof. This requirement seems not to fit the statement of the completeness theorem where both Godel's statement and the modern statement (*) obscure the crucial point: the construction of a countermodel . Thus, we require that the characterizing property should not be required to be something 'mentioned in the theorem[20]' but of something mentioned in the proof or the theorem. Thus we modify Steiner's model to require the existence of a characterizing property of a proof that appears in a family of arguments to qualify a proof as 'explanatory'. The characterizing property should distinguish this argument from others.

Thus, there is an immediate division into two families of proofs of the completeness theorem. In the Gödel-Henkin style the main lemma is a proof that $\neg\phi$ is not refutable then ϕ has a model. In contrast the main lemma of the Herbrand style asserts that if $\neg\phi$ is not satisfiable then ϕ is provable. Smullyan [Smullyan, 1963] introduces the notion of a consistency property, from which he can construct proofs in each style of the completeness theorem[21] A consistency property Γ is for him a collection of finite sets of sentences such that Γ is closed under a set of operations (e.g. adding a Henkin witness). He shows any consistency property a model and distinguishes his argument from Henkin (footnote 13) because there is no requirement that every sentence is decided. The Herbrand style was developed independently by Beth and Hintikka as the method of semantic tableaux or model sets. Smullyan [Smullyan, 1966] introduces his method of *analytic tableaux*, which generalizes both [Hintikka, 1955] and [Beth, 1959]. In this line a (natural deduction) proof system is developed such that a proof of ϕ is proved to terminate if ϕ is valid because if the proof does not terminate a model of $\neg\phi$ is constructed. We discuss the Herbrand style no further although further work on this topic would be valuable; our goal here is to distinguish within the Gödel-Henkin style.

Gödel's proof certainly did not arise as a generalization. But, it was clear from the first that first order logic (the restricted functional calculus) admitted of many variants in the formulation of a deductive system:

[20] One might argue that the deformable objects are the proof systems and the notion of model. But this still seems to miss the crux of Henkin's argument.

[21] They are applications B for Gödel-Henkin and C for Herbrand on page 829 of [**?**].

one feature of the theorem is as a precise statement of the equivalence of these variants. The very proof of Gödel was built on Bernay's proof of completeness of propositional calculus. So the idea that distinct logics could be complete was there. Unlike Gödel, Henkin, as he completed the work, already made generalizations to other logics; he proves completeness for three logics: first order, the interpretation of 2nd order with the Henkin semantics, and a theory of types of infinite order, again with respect to the Henkin semantics. In the insightful [Henkin, 1996], published nine years after [Resnik and Kushner, 1987], Henkin explains the relationship between the three results. He reports that he worked for more than a year on a related issue: trying to show (roughly speaking[22]) that in Church's theory of types there was no uniform way to assign a choice function for non-empty sets of reals. After he had almost given up, he realized the key idea of inductively extending both the axioms as well as the collection of functions named. His insight that *both* the axioms and the names must be extended led to the proof of the three completeness theorems. While in the unsolved[23] target problem, functions were to be named, when writing up the completeness results ([Henkin, 1996],155) he discovered that in both the first order and the finite type case, he could add only constants – a crucial point for later model theory. Still later, he realized, as is now standard, that the constants can be added first and only their properties defined in the induction.

We observed above that Henkin's vital insight is the separation of the problem into two parts: i) extend the given theory to one satisfying conditions that ii) permit the construction of a canonical model. Both the sufficient conditions on the base theory and the conditions on the complete theory that supports the construction are quite general and adaptable. In this generality we see that it applies not only to proving completeness for a family of logics but to the other uses of the model construction technique.

Here are the first order conditions. Henkin and later adapters can quickly state rules and axioms for a logic; the key for H1 is to isolate rules that prove that if T is consistent then for any formula $\phi(x)$ and any constant c that does not appear in T, $T \cup \{(\exists x)\phi(x) \rightarrow \phi(c)\}$ is consistent. For H2 the logic must satisfy equality axioms guaranteeing that the equality relation is a congruence in the sense of universal algebra.

In fact, later authors reverted to Skolem's standpoint[24] and give completely semantic versions of the Henkin construction, solely for convenience in [Marker, 2002] and as a step towards a completeness theorem for infinitary logic in [Keisler, 1971].

Resnik and Kushner [Resnik and Kushner, 1987] 'think (though in correspondence Steiner disagrees) that the proof does not make clear that when we apply the proof-idea to second order logic, we must change the sense of model to allow for non-standard models nor that when we apply it to modal logics, we must use many maximally consistent sets, etc.' This seems to misunderstand the effect of Henkin semantics for second order logic. The Henkin semantics interpret second order as a many-sorted first order system and then a similar proof applies. Clearly, there is no completeness theorem for second order logic with the full semantics.

From the view point of a mathematician-model theorist, the witness property appears to be key. But from the standpoint of modal and intuitionistic logic there are a different set of properties of 'propositional logics' which are key. For example, De Jongh[25] identifies the crucial syntactic property to a get canonical model for intuitionistic propositional logics as the disjunction property. As [Resnik and Kushner, 1987] noted, the concept of model has widened from a first order structure to a Kripke frame; this only emphasizes the depth of Henkin's innovation.- Similarly in [Cintual and Noguera, 2015] the authors stress the role of the *term model* in extending the Henkin proof to many-valued logics and in general to algebraizable logics.

[22] A more detailed description is given on page 148 of [Henkin, 1996].

[23] It developed that by work of Gödel and Feferman that the conjecture Henkin had been attacking was independent.

[24] Gödel refers to the great similarity between his argument and [Skolem, 1967]; the distinction is that Skolem ignores the deductive standpoint. See the notes to [Gödel, 1929].

[25] Slide 27 of his tutorial in the 2008 Logic Days in Lisbon `https://staff.fnwi.uva.nl/d.h.j.dejongh/teaching/il/lisbonslides.pdf`.

Intuitionistic logics do not have a prenex normal form theorem so Gödel's proof could not adapted to this case.

The characterizing property of the Henkin proof is the systematic extension of the given theory to a complete theory whose canonical model satisfies the original theory. In order to apply Steiner's test, one must step far enough back to recognize the characteristic property. As we will see below as we reformulate this property to accomodate the wider applicability of the key notion, 'satisfy' can be replaced by a wide variety of stronger conditions (which always include 'satisfy').

An abstract formulation of the completeness theorem was popular in the late 70's: the set of first order validities is recursively enumerable. This formulation was discussed by the authors but did not make it into [Barwise and Feferman, 1985]. A reason is that it does not persist so nicely to infinitary logics. There are (infinitary) proof systems for the logics $L_{\kappa,\omega}$ and their completeness is proved by adaptations of the Henkin method. This extends to logics that add the Q-quantifier[26]. But these systems do not demonstrate that the validities are recursively enumerable or even Borel[27]. This weakness in the proof system also prevents these infinitary logics from satisfying the full compactness theorem and thus they fail the upward-Löwenheim Skolem theorem. This notion of completeness of course is foreign to Gödel who requires finite proofs and naturally his proof could not be adapted to these contexts.

Henkin already points out that his proof (unlike Gödel's) generalizes easily to uncountable vocabularies. So the first order theory of R-modules can be developed uniformly regardless of the cardinality of the ring R.

The functorial aspect of the Henkin construction is best illustrated by generalizations that require more than mere existence of the canonical model. The *omitting types theorem* is the most basic: add to the requirements in the construction of the Henkin theory that for each $\phi(\boldsymbol{a}, x)$, for each non-principal p there is a p-omitting witness (i.e. the consistency of $\exists x\phi(\boldsymbol{a}, x) \wedge \neg\sigma(x)$ for some $\sigma(x) \in p$). This requirement is easily established as one just has to arrange that each term (in the vocabulary with new constants) omits the type. Somewhat more exotic is the use of Henkin's method in [Baldwin and Lachlan, 1971] to prove that an $\aleph_1$-categorical but not $\aleph_0$-categorical theory cannot have finitely many countable models. Still more exotic is the modification of the method by [Baldwin and Laskowski, 2017] to construct atomic models in the continuum by an ω-step Henkin construction. Hodges [Hodges, 1985] provides a plethora of examples in algebra, in exploring the quantifier complexity of definable sets in first order theories, and in various logics.

Earlier than any of these examples, the fundamental idea of the Henkin construction, systematically extend the given theory to a complete theory whose canonical model satisfies a desired property, was regimented by Abraham Robinson's concepts of finite and infinite forcing [Robinson, 1970] and expounded for students in [Hodges, 1985]. Very recently, the method is extended to construct counterexamples in functional analysis, e.g. of specific types of C^*-algebras. The authors (model theorists and analysts) of [Farah et al., 2016] write, 'We describe a way of constructing C^*-algebras (and metric structures in general) by Robinson forcing (also known as the Henkin construction).'

We have given a myriad of examples where the key idea of Henkin's proof is applied, by deforming (i.e. finding the appropriate) notions of derivation and canonical model but in each case following the Henkin template of extending a given theory to a complete theory admitting a canonical model which satisfies the original theory and perhaps further requirements (e.g. omitting types). In a fundamental sense Henkin's argument is explanatory because he has identified the key features connecting the hypothesis and conclusion, modifying both the syntactic and the semantic component. Moreover, the proof is explanatory by the variant of Steiner's criterion obtained by looking for a characterizing property within the proof.

[26] $M \models (Qx)\phi(x)$ means there are uncountably many solutions for ϕ in M.

[27] In fact, the validities of $L_{\omega_1,\omega}$ are Σ_1-definable on the hereditarily countable sets ($\langle H(\omega_1), \epsilon\restriction H(\omega_1)\rangle$ and Σ_2 on $\langle H(\kappa), \epsilon\restriction H(\kappa)\rangle$ for $L_{\omega_1,\omega}$ when κ is uncountable (page 328 of [Dickmann, 1985]).

3 Explanation for who?

{who}

The explanatory value of a proof, concept, theory can only be evaluated in terms of the intended audience. We give three accounts of the explanatory value of Henkin's proof, first for undergraduates taking a first course in logic, the second for contemporary research logicians, and the third for research logicians in the early 1950's.

We would motivate the argument for undergraduates by asking, since the goal is to construct a model of a theory T, what properties of an extension of T would allow the equivalence relation (guaranteed by equality axioms) of provable equality on the constants to quotient to a model. The first goal is that the quotient is actually a structure. For this we need the equality axioms for preserving functions. To ensure the structure is a model of T, do an induction on quantifiers, noting the hard case is solved by the witness property. Of course, this will be somewhat more convincing for students with a background in algebra.

The undergraduate argument is reinforced for cognoscenti by observing the various applications that follow the same pattern. See what conditions are needed on a theory T for the logic in question to guarantee that the quotient is a model. For $L_{\omega_1,\omega}$ (logic allowing countable conjunctions) an additional rule of inference is needed; if $\psi \to \phi$ for each ϕ in a countable set Φ, infer $\psi \to \bigwedge \Phi$. To build a theory whose 'Henkin' model would satisfy an infinitary sentence, Makkai [Makkai, 1969] extended Smullyan's notion of a consistency property to guarantee the theory was closed under infinite conjunction.

The reader will have noticed that our argument for the identity of the characterizing property depended on this second account and invoked an even wider family of generalized arguments. None of this was known in 1949. Indeed, there is no immediate recognition in the reviews of Henkin's paper of the significance of the change in vocabulary. Only the concrete example of the uncountable Löwenheim-Skolem theorem is even noticed. While the fixed vocabulary approach to first order logic appears in [Robinson, 1951] and [Henkin, 1953], it seems to be fully established in [Tarski and Vaught, 1956] as the fundamental notion of *elementary extension* requires such a stipulation.

In particular, Goodstein's [Goodstein, 1953] review of Robinson's paper, which provides the first known publication of the completeness theorem[28] in the modern (*) form, does not regard this reformulation of the theorem as a foundational issue.

> ...the Metamathematics of algebra is a book for algebraists not logicians; its claim to a place in the new series of studies in logic and the foundations of mathematics is very slender.
>
> Only the first fifth of the book, which is devoted to an extension of G6del's completeness theorem to non- denumerable systems of statements, has any bearing on the foundations of mathematics, and the remaining four-fifths may be read without reference to this first part which could with advantage have been omitted.

McKinsey (also noting the uncountable application) and Heyting give straightforward accounts in Mathematical Reviews of the result of Henkin's papers on first order and theory of types respectively with no comments on the significance of the result. Still more striking, Ackermann's review [Ackerman, 1950] of Henkin's proof gives a routine summary of the new argument and concludes[29]. with, 'The reviewer can not follow the author when he speaks of an extension to an uncountable set of relation symbols, since such a system of notations can not exist.'

As we promised in the introduction, we see that the significance of an explanation depends on the audience; the depth of an argument is often not apparent to the author. Thus, we return to the original remark

[28] See footnote 6.

[29] This is my very rough translation. The sentence reads, 'Ref. kann dem Verf. aber nicht folgen, wenn er von der Möglichkeit einer mehr als abzählbaren Menge von primitiven Symbolen spricht, da es ein derartiges Bezeichnungssystem doch nicht geben kann.

of Mancosu, explanations that lead to a major recasting of an entire discipline. Henkin's argument was a major component in the turn from model theory as a (mathematical) attempt to understand mathematical reasoning to model theory as tool in many areas of mathematics. This change in paradigm bloomed with Shelah's classification theory [Baldwin, 2017]. An essential step was to move from studying a logic which encompassed relations of all orders to the study of theories about particular areas of mathematics in their native vocabularies. Henkin's proof enabled this view and he was one of the pioneers.

However, these ultimately revolutionary features were invisible at the time. This slow reinterpretation of basic notions is a fundamental feature of mathematical development[30]

References

[Ackerman, 1950] Ackerman, W. (1950). Reviewed: The completeness of the first-order functional calculus by Leon Henkin. *Journal of Symbolic Logic*, 15:68. German.

[Addison et al., 1965] Addison, Henkin, and Tarski, editors (1965). *The Theory of Models*. North-Holland, Amsterdam.

[Avigad and Morris, 2014] Avigad, J. and Morris, R. (2014). The concept of 'character' in dirichlet's theorem on primes in an arithmetic progression. *Archive for History of Exact Sciences*, 68:265–326.

[Baldwin, 2016a] Baldwin, J. (2016a). Foundations of mathematics: Reliability and clarity, the explanatory role of mathematical induction. In Väänänen, J., Hirvonen, Å., and de Queiroz, R., editors, *Logic, Language, Information, and Computation, 23rd international workshop, Wollics 2016, Puebla Mexico, August 16-19, 2016 Proceedings*, pages 68–82.

[Baldwin, 2017] Baldwin, J. (2017). Formalization without foundationalism; model theory and the philosophy of mathematics practice.

[Baldwin and Lachlan, 1971] Baldwin, J. and Lachlan, A. (1971). On strongly minimal sets. *Journal of Symbolic Logic*, 36:79–96.

[Baldwin and Laskowski, 2017] Baldwin, J. and Laskowski, C. (2017). Henkin constructions of models with size continuum. in preparation.

[Baldwin, 2016b] Baldwin, J. T. (2016b). Axiomatizing changing conceptions of the geometric continuum I: Euclid and Hilbert. 29 pages, submitted, `http://homepages.math.uic.edu/~jbaldwin/pub/axconIsub.pdf`.

[Barwise and Feferman, 1985] Barwise, J. and Feferman, S., editors (1985). *Model-Theoretic Logics*. Springer-Verlag.

[Beth, 1959] Beth, E. (1959). *The Foundations of Mathematics*. North Holland Publishing Co.

[Birkhoff, 1935] Birkhoff, G. (1935). On the structure of abstract algebras. *Proc. Cam. Phil. Soc.*, 31:433–454.

[Chang and Keisler, 1973] Chang, C. and Keisler, H. (1973). *Model theory*. North-Holland. 3rd edition 1990.

[30] See, for example, [Avigad and Morris, 2014], [Baldwin, 2016b], [Lakatos, 1976], [Werndl, 2009].

[Cintual and Noguera, 2015] Cintual, P. and Noguera, C. (2015). A Henkin-style proof of completeness for first-order algebraizable logics. *Journal of Symbolic Logic*, 80:341–358.

[Dawson, 1993] Dawson, J. W. (1993). The compactness of first-order logic: from Gödel to Lindstrom. *History and Philosophy of Logic*, 14:1537.

[Detlefsen, 2014] Detlefsen, M. (2014). Completeness and the ends of axiomatization. In Kennedy, J., editor, *Interpreting Gödel*. Cambridge University Press.

[Dickmann, 1985] Dickmann, M. (1985). Larger infinitary logics. In Barwise, J. and Feferman, S., editors, *Model-Theoretic Logics*, pages 317–363. Springer-Verlag.

[Farah et al., 2016] Farah, I., Hart, B., Lupini, M., Robert, L., Tikuisis, A., A., V., and Winter, A. (2016). Model theory of c*-algebras. math arXiv:1602.08072v3.

[Franks, 2010] Franks, C. (2010). Cut as consequence. *History and Philosophy of Logic*, 31:349–379.

[Franks, 2013] Franks, C. (2013). Logical completeness, form, and content: an archaeology. In Kennedy, J., editor, *Interpreting Gödel: Critical Essays*. Cambridge University Press., Cambridge. forthcoming.

[Gentzen, 1932] Gentzen, G. (1932). Üuber die existenz unabhängiger axiomensysteme zu unendlichen satzsystemen. *Mathematische Annalen*, 107:329–350. translated as *On the existence of independent axioms systems for infinite sentence systems* in Szabo 1969, 29-52.

[Gödel, 1929] Gödel, K. (1929). On the completeness of the calculus of logic. In S. Feferman, e. a., editor, *Kurt Gödel, Collected Works, Vol. Publications 1929-36*, volume I, pages 61–101. Oxford University Press, New York. thesis, published under the auspices of Association for Symbolic Logic: Providence, Rhode Island.

[Gödel, 1930] Gödel, K. (1930). The completeness of the axioms of the functional calculus of logic. In S.Feferman and et al., editors, *Kurt Gödel, Collected Works, Vol. I*, volume I, pages 103–123. Oxford University Press, New York. under the auspices of Association for Symbolic Logic: Providence, Rhode Island;1930c in collected works; first appeared:Monatshefte für Mathematik und Physik.

[Goodstein, 1953] Goodstein, R. L. (1953). On the metamathematics of algebra by a. robinson. *The Mathematical Gazette*, 37(321):224–26.

[Hafner and Mancosu, 2005] Hafner, J. and Mancosu, P. (2005). The varieties of mathematical explanation. In Mancosu, P., Jørgensen, K., and Pedersen, S., editors, *Visualization, Explanation, and Reasoning Styles in Mathematics*, pages 251–249. Springer.

[Hafner and Mancosu, 2008] Hafner, J. and Mancosu, P. (2008). Beyond unification. In Mancosu, P., editor, *The Philosophy of Mathematical Practice*, pages 151–178. Oxford University Press.

[Henkin, 1949] Henkin, L. (1949). The completeness of the first-order functional calculus. *J. Symbolic Logic*, 14:159166.

[Henkin, 1953] Henkin, L. (1953). Some interconnections between modern algebra and mathematical logic. *Trans. American Mathematical Society*, 74:410–427.

[Henkin, 1996] Henkin, L. (1996). The discovery of my completeness proofs. *The Bulletin of Symbolic Logic*, 2:127–158.

[Hilbert and Ackermann, 1938] Hilbert, D. and Ackermann, W. (1938). *Grundzuge der Theoretischen Logik, 2nd edn*. Springer, Berlin. first edition, 1928.

[Hintikka, 1955] Hintikka, J. (1955). Form and content in quantification theory,. *Acta Philosophica Fennica*, pages 7–55.

[Hodges, 1985] Hodges, W. (1985). *Building Models by Games*. London Math Soc., Student Texts. Cambridge University Press, Cambridge.

[Karp, 1959] Karp, C. (1959). *Languages with Expressions of Infinite Length*. PhD thesis, University of Southern California. Advisor, Leon Henkin: `http://cdm15799.contentdm.oclc.org/cdm/ref/collection/p15799coll17/id/140603`.

[Keisler, 1971] Keisler, H. (1971). *Model theory for Infinitary Logic*. North-Holland.

[Kreisel and Krivine, 1967] Kreisel, G. and Krivine, J. (1967). *Elements of Mathematical Logic (Model Theory)*. North Holland, Amsterdam.

[Lakatos, 1976] Lakatos, D. (1976). *Proofs and Refutations*. Cambridge University Press, Cambridge.

[Makkai, 1969] Makkai, M. (1969). On the model theory of denumerably long formulas with finite strings of quantifiers. *J. Symbolic Logic*, 34:437–459.

[Mancosu, 2008] Mancosu, P. (2008). Mathematical explanation: Why it matters. In Mancosu, P., editor, *The Philosophy of Mathematical Practice*, pages 134–150. Oxford University Press.

[Marker, 2002] Marker, D. (2002). *Model Theory: An Introduction*. Springer-Verlag.

[Resnik and Kushner, 1987] Resnik, M. and Kushner, D. (1987). Explanation, independence, and realism in mathematics. *British J. Philos. Sci.*, 38:141–158.

[Robinson, 1951] Robinson, A. (1951). *Introduction to Model Theory and to the Metamathematics of Algebra*. Studies in Logic and the Foundations of Mathematics. North Holland, Amsterdam. 1st edition 1951; 2nd edition 1965.

[Robinson, 1952] Robinson, A. (1952). On the application of symbolic logic to algebra. In *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30-September 6,1950*, volume 1, pages 686–694. American Mathematical Society, Providence, Rhode Island.

[Robinson, 1970] Robinson, A. (1970). Forcing in model theory. In *Actes, Congrs intern, math., 1970.*, volume 1, pages 245–250. American Mathematical Society, Providence, Rhode Island.

[Skolem, 1967] Skolem, T. (1967). Some remarks on axiomatic set theory. In Van Heijenoort, J., editor, *From Frege to Godel: A Sourcebook in Mathematical Logic, 1879-1931*, pages 290–301. Harvard University Press. German original published in 1923; address delivered in 1922.

[Smullyan, 1963] Smullyan, R. M. (1963). A unifying principle in quantification theory. *Proceedings of the National Academy of Science*, 49:828–832.

[Smullyan, 1966] Smullyan, R. M. (1966). Trees and nest structures. *Journal of Symbolic Logic*, 31:828–832.

[Steiner, 1978] Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, 34:135–151.

[Tarski, 1935] Tarski, A. (1935). Der Wahrheitsbegriff in den formalisierten Sprachen. *Studia Philosophica*, 1:261–405. translated as: The concept of truth in formalized languages in Logic, [Tarski, 1956].

[Tarski, 1946] Tarski, A. (1946). A remark on functionally free algebras. *Ann. of Math*, 47:163–165.

[Tarski, 1954] Tarski, A. (1954). Contributions to the theory of models, I and II. *Indag. Math.*, 16:572,582.

[Tarski, 1956] Tarski, A. (1956). *Logic, semantics and metamathematics: Papers from 1923-38*. Oxford: Clarendon Press. translated by J.H. Woodger.

[Tarski and Vaught, 1956] Tarski, A. and Vaught, R. (1956). Arithmetical extensions of relational systems. *Compositio Mathematica*, 13:81–102.

[Werndl, 2009] Werndl, C. (2009). Justifying definitions in mathematics going beyond Lakatos. *Philosophia Mathematica*, pages 313–340.