STRONGLY MINIMAL STEINER SYSTEMS II: COORDINATIZATION AND QUASIGROUPS

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Abstract. We note that a strongly minimal k-Steiner system (M, R) from (Baldwin-Paolini 2020) can be 'coordinatized' in the sense of (Ganter-Werner 1975) by a quasigroup if k is a prime-power. But for the basic construction this coordinatization is never definable in (M, R). Nevertheless, by refining the construction, if k is a prime power there is a $(2, k)$-variety of quasigroups which is strongly minimal and definably coordinatizes a Steiner k-system.

A *linear space* is collection of points and lines that satisfy a minimal condition to call a structure a geometry: two points determine a line. A linear space is a Steiner k-system if every line (block) has cardinality k. Such mathematicians as Steiner, Bose, Skolem, and Bruck have established deep connections between the existence of a Steiner system with v points and blocks of size k and divisibility relations among k and v. Much of the history of Steiner systems interacts with the general study of non-associative algebraic systems such as quasigroups. A quasigroup is a structure with a single binary operation whose multiplication table is a Latin square (each row or column is a permutation of the universe). Stein [Ste56] made deep connections between Steiner k-systems and quasigroups. A line of work from the 1950-1980's including [Ste56, Grä63, GW75, Eva82] proved that Steiner k-systems are 'informally coordinatized' by varieties of quasigroups if and only if k is a prime power. We extend these results to certain Steiner system of every cardinality with two contrasting results. We build on Theorem 0.1.1 to prove items 2) and 3) in this paper.

{disp2}

Theorem 0.1 (Theorem 3.11). *(1) ([BP20]) For each $k \geqslant$ there are uncountably many μ such that T_μ is the theory of a Steiner k-systems (M, R).*
(2) If k is a prime power, there is a 'coordinatizing quasigroup' for (M, R).
(3) Unless $k = 3$, the 'coordinatizing quasigroup' is not definable in (M, R).

The key to the new, (2),(3) parts of Theorem 0.1 is the relationship of so-called $(2, k)$ varieties [Pad72, GW75, Qua92] to a two-transitive finite structure and thus eventually to the reconstruction of a finite quasigroup. Theorem 0.1.2 rests primarily on work of [GW75, Ste56, Ś61] and others who achieved an 'informal coordinatization'(Definition 1.1) of certain classes of Steiner systems by varieties of quasigroups. The non-definablity result is vastly strengthened in [BV21].

Strongly minimal sets are the building blocks of $\aleph_1$-categorical theories. A complete first order theory is strongly minimal if every definable set is finite or cofinite. Alternatively, in a strongly minimal theory T the model theoretic notion of

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algebraic closure[1] determines a combinatorial geometry (matroid). Zilber conjectured that these geometries were all disintegrated ($\mathrm{acl(A)} = \bigcup_{\mathrm{a}\in\mathrm{A}} \phi(\mathrm{x},\mathrm{a})$), locally modular (group-like) or field-like. The examples here are based on Hrushovki's construction refuting this conjecture [Hru93]. These counterexamples, with 'flat[2] geometries' [Hru93, Section 4.2], have generally been regarded as an incohate class of exotic structures. Indeed, a distinguishing characteristic is the inability to formally define an associative operation with infinite domain in any structure with a flat acl-geometry.

However, we showed in Section 2 of [BP20] that linear spaces can be naturally formulated in a one-sorted logic with single ternary 'collinearity' predicate. Performing the construction on linear spaces with a geometrically motivated predimension produces strongly minimal Steiner k-systems (Theorem 0.1.1) that are model complete and satisfy the usual properties of counterexamples to Zilber's trichotomy conjecture. Their acl-geometries are flat, but not disintegrated nor locally modular. Further work shows there is a wide diversity of such examples.

Despite Theorem 0.1.(3), when $k = q$ is a prime power we can find strongly minimal quasigroups that define strongly minimal Steiner k-systems.

{disp3}

Theorem 0.2. *(Theorem 4.3) For each q and each of the T_μ in Theorem 0.1 with line length $k = q = p^n$ (for prime p) and certain varieties of quasigroups V, there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that defines a strongly minimal Steiner q-system.*

In [Bal21] we investigate various combinatorial problems about the classes of quasigroups constructed here. In particular, we find strongly minimal Steiner triple systems of every infinite cardinality that are two-transitive, and then easily deduce with uniform cycle graphs [CW12], and further that are ∞-sparse in the sense of [CGGW10]. The current paper deals with connections to universal algebra and poses a number of questions about the associated varieties of quasigroups.

We discussed in the introduction and Remark 5.27 of [BP20] the connections of this work with [BC19, CK16, HP]. These works construct first order theories of Steiner systems or projective planes that are at the other end of the stability spectrum from those here. Evans [Eva04] uses the Hrushovski construction to address combinatorial issues about Steiner systems.

This paper depends heavily on the results and notation of [BP20, BV21]. Certain arguments will require consulting those papers. We acknowledge helpful discussions with Joel Berman, Omer Mermelstein, Gianluca Paolini, and Viktor Verbovskiy.

{coord}

1. Coordinatization

Descartes's coordinatizaton identified addition and multiplication of line segments in a Euclidean plane as algebraic operations on what became over the next 300 years a (sub)field (which field depends on the geometric axioms) of the real numbers. Hilbert, building on Von Staudt, and before the notions were formally defined, established a *bi-interpretation* between an ordered field F where every positive number has a square root and the Euclidean plane coordinatized by F.

[1] $a \in \mathrm{acl(B)}$ if for some $\phi(x, \mathbf{b})$ with $\mathbf{b} \in B$, $\phi(a, \mathbf{b})$ and $(\exists^{<k} x)\phi(x, \mathbf{b})$ for some k.

[2] The dimension of a closed subspace is determined from its own closed subspaces by the inclusion-exclusion principle [BP20, Definition 3.8]. We use this paper as a common reference for many earlier results and definitions that are scattered in the literature.

Makowsky [Mak19] pointed out the subtleties of the coordinatization notion as a property connecting theories. In particular he emphasized the necessity of establishing the *definability* of the interpretation in *both directions* and verifying the composition is the identity. In the case at hand, there may be an informal coordinatization, but the Descartes direction is *never definable* in the geometric language containing only a ternary relation (for line length $\geqslant 4$). We contrast two notions of 'coordinatize'.

{coordef}

Definition 1.1. (1) [GW75, GW80] *A class of structures (specifically geometries (M, R)) is (informally)* coordinatizable *if there is* 1-1 *correspondence between it and a well-behaved class of algebras (specifically quasigroups $(M, *)$). If so, we say algebra $(M, *)$ coordinatizes the structure (M, R)*

(2) *The coordinatization is* definable *if there is a first order $\{R\}$-formula that defines $*$ that defines $*$ in (M, R).*

Ganter and Werner identify those classes of Steiner systems which are coordinatizable as in Definition 1.1.1 by certain varieties of quasigroups. But, as they point out, this identification is not unique; the same Steiner-system can be coordinatized using the same method by different algebras (that are not even in the same variety). Thus the theory of the Steiner system (M, R) does not even predict the equational theory of the informally coordinatizing algebra and certainly does not control the first order theory.

2. Background

{bg}

We first give a quick survey of a set of notions from combinatorics and universal algebra and then a short introduction to the Hrushovski construction. Hrushovski's signal achievement is to modify the Fraïssé generalization of the Cantor/Hausdorff development of 'homogeneous-universal' models to license constructions of various model theoretic complexities.

A Steiner (t, k, v)-system is a pair (P, B) such that $|P| = v$, B is a collection of k element subsets of P and every t element subset of P is contained in exactly one block. We showed in [BP20, §2.3] the bi-interpretability of the two sorted Steiner systems (P, B) with one-sorted structures (M, R) with a single ternary relation R. Since we are primarily interested in interested infinite structures, we omit the v unless it is crucial and since we deal with linear spaces $t = 2$. So, by Steiner k-system we mean a Steiner $(2, k)$ system of arbitrary cardinality. A *groupoid*[3] (also called *magma*) is a structure $(A, *)$ with one binary function $*$.

The salient characteristic of the equational theories below is that each defining equation involves only two variables. In particular, none of the varieties are associative.

Most of the current section[4] is a series of definitions needed to apply the Hrushovski construction as modified in [BP20]. The basic ideas of the Hrushovski construction are i) to modify the Fraïssé construction of countable homogeneous-universal structures by replacing substructure by a notion of strong substructure, defined

[3]In category theory the term groupoid means all morphisms are invertible. Thus, it is more analogous to our 'quasigroup'. But most of the references for this paper use groupoid with no explanation to mean binary function.

[4]The two page Section 2.1 of the online version of [BP20] summarises the role of strongly minimal sets in model theory and how strongly minimal Steiner systems arise.

using a pre-dimension δ (Definition 2.5) so that independence with respect to the dimension induced by δ is a combinatorial geometry[5] and ii) to employ a function μ to bound the number 0-primitive extensions of each finite structure so that closure in the geometry on the generic model for the resulting class of finite structures is algebraic closure.

We phrase our work in the generalization of the Fraïssé and Hrushovski constructions laid out in [KL92].

{smcl}

Notation 2.1. (1) *A* smooth class $(\boldsymbol{L}_0, \leqslant)$ *is a countable collection of finite structures with a transitive relation on* $\boldsymbol{L}_0$, strong substructure $(\leqslant)$, *refining substructure such that* $B \leqslant C$ *implies* $B \leqslant C'$ *if* $B \subseteq C' \subseteq C$. *However,* $\boldsymbol{L}_0$ *need not be closed under substructure. If a smooth class satisfies amalgamation and joint embedding there is a countable generic model* M, *i.e. if finite* A, B *are each strong in* M, *they are automorphic in* M. *The theory of the generic structure,* $\mathcal{G}_\mu$, *is the desired strongly minimal* T_μ.

(2) *Given a class of finite structures* L_0, $\hat{L}_0$ *denotes the collection of structures of direct limits of members of* L_0.

The next notation outlines the parameters for the variations on the construction that appear below.

{hruclass}

Notation 2.2. A *Hrushovski sm-class* is determined by a quintuple $(\sigma, \boldsymbol{L}_0^*, \epsilon, L_0, \mathbf{U})$. $\boldsymbol{L}_0^*$ is a collection of finite structures in a vocabulary σ, not necessarily closed under substructure. ϵ is a function from members of $\boldsymbol{L}_0^*$ to integers satisfying the conditions imposed on δ in Definition 2.5. $\boldsymbol{L}_0$ is a subset of $\boldsymbol{L}_0^*$ defined using ϵ. From such an ϵ, one defines notions of $\leqslant$, primitive extension, and good pair. Hrushovski gave one technical condition on the function μ counting the number of realizations of a good pair that ensured the theory is strongly minimal rather than ω-stable of rank ω. Fixing a class $\mathbf{U}$ of functions μ satisfying that condition in the base case and others for special purposes provides a way to index a rich group of distinct constructions. At various times in this paper $\mathbf{U}$ is instantiated as $\mathcal{U}$, $\mathcal{C}$, $\mathcal{F}$ or $\mathcal{T}$.

When we move from the general axiomatic framework to specific developments, which depend on the definition of δ, L becomes K, ϵ becomes δ and a various script letters are substituted for U to describe the counting function μ.

{not1}

Notation 2.3. *We work in a vocabulary* τ *with one ternary relation* R, *and assume always that* R *can hold only of three distinct elements and then in any order (i.e., a* 3*-hypergraph). We say a* maximal R*-clique in a structure* M *is a line (block) and sometimes write (partial) line for a clique that is not maximal.*

{lines}

Definition 2.4. (1) $\boldsymbol{K}_{-1}$ *is the collection of* linear spaces, τ*-structures such that* 2*-points determine a unique line (maximal clique); we interpret* R *as collinearity. By convention two unrelated elements constitute a* trivial line.

(2) *For* $\ell \subseteq A$, *we denote the cardinality of a line* ℓ *by* $|\ell|$, *and, for* $B \subseteq A$, *we denote by* $|\ell|_B$ *the cardinality of* $\ell \cap B$.

(3) *We say that a non-trivial line* ℓ *contained in* A *is* based in $B \subseteq A$ *if* $|\ell \cap B| \geqslant 2$, *in this case we write* $\ell \in L(B)$.

[5]The requirement that the range of this function is well-ordered is essential to get a geometry rather than just a dependence notion; using rational or real coefficients yields a stable theory and the dependence relation of forking [BS96].

{nullity}

(4) *The* nullity of a line ℓ *contained in a structure* $A \in \boldsymbol{K}_{-1}$ *is:*

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

Now we define our geometrically based pre-dimension function [Pao21].

{defdelrank}

Definition 2.5. (1) *Recall* $\boldsymbol{K}_{-1}$ *is the collection of finite linear spaces. For* $A \in \boldsymbol{K}_{-1}$ *let:*

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

(2) *Let:* {k0def}

$$\boldsymbol{K}_0 = \{A \in \boldsymbol{K}_{-1} \text{ such that for any } A' \subseteq A, \delta(A') \geqslant 0\},$$

and by define $(\boldsymbol{K}_0, \leqslant)$ *by* $A \leqslant B$ *if and only if:*

$$A \subseteq B \wedge \forall X(A \subseteq X \subseteq B \Rightarrow \delta(X) \geqslant \delta(A)).$$

(3) *In any* $N \in \hat{\boldsymbol{K}}_0$, $d_N(A/B) = \min\{\delta(B/A) : A \subseteq B \subseteq N\}$.
(4) *We write* $A < B$ *to mean that* $A \leqslant B$ *and* A *is a proper subset of* B.
(5) *We say that* B *is a* primitive extension *of* A *if* $A \leqslant B$ *and there is no* B_0 *with* $A \subsetneq B_0 \subsetneq B$ *such that* $A \leqslant B_0 \leqslant B$.

The class $\boldsymbol{K}_0$ satisfies amalgamation with the following construction.

Definition 2.6. [BP20, Lemma 3.14] *Let* $A \cap B = C$ *with* $A, B, C \in \boldsymbol{K}_0$. *We define* $D := A \oplus_C B$ *as follows:* {defcanam}

(1) *the domain of* D *is* $A \cup B$;
(2) *a pair of points* $a \in A - C$ *and* $b \in B - C$ *are on a non-trivial line* ℓ' *in* D *if and only if there is line* ℓ *based in* C *such that* $a \in \ell$ *(in* A*) and* $b \in \ell$ *(in* B*). Thus* $\ell' = \ell$ *(in* D*).*

[BP20] demonstrates the class $(\boldsymbol{K}_0, \delta)$ satisfies the basic properties (including flatness) of a δ function in a Hrushovski construction and of the associated algebraic closure geometry. For $M \in T_\mu$, $a \in \mathrm{acl(B)}$ if and only $d_M(a/B) = 0$. The flatness implies that no model of T_μ (Fact 0.1) admits a definable binary associative function with infinite domain.

The following definitions describe the pairs $B \subseteq A$ such that in the generic model A will be contained in the algebraic closure of B.

{prealgebraic}

Definition 2.7. *Let* $A, B \in \boldsymbol{K}_0$ *with* $A \cap B = \emptyset$ *and* $A \neq \emptyset$.

(1) B *is a* primitive extension *of* A *if* $A \leqslant B$ *and there is no* $A \subsetneq B_0 \subsetneq B$ *such that* $A \leqslant B_0 \leqslant B$.
   B *is a* k-primitive extension *if, in addition,* $\delta(B/A) = k$.
   We stress that in this definition, while B *may be empty,* A *cannot be.*
(2) *We say that the* 0*-primitive pair* A/B *is* good *if there is no* $B' \subsetneq B$ *such that* (A/B') *is* 0*-primitive. (Hrushovski called this a minimal simply algebraic or m.s.a. extension.)*
(3) *If* A *is* 0*-primitive over* B *and* $B' \subseteq B$ *is such that we have that* A/B' *is good, then we say that* B' *is a* base *for* A *(or sometimes for* AB*).*
(4) *If the pair* A/B *is good, then we also write* (B, A) *is a* good pair.

The following notation singles out the effect of the fact that our rank depends on line length rather than the number of occurrences of a relation. We isolate the isomorphism type of good pair that provides the invariant for the Steiner systems. {linelength}

Notation 2.8 (Line length)**.** *We write $\boldsymbol{\alpha}$ for the isomorphism type the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$. By Lemma 5.18 of* [BP20]*, lines in models of T_μ have length k if and only if $\mu(\boldsymbol{\alpha}) = k - 2$.*

{Kmu}

{itemKmu} **Definition 2.9.** *Good pairs were defined in Definition 2.7.*

(1) Let $\mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:
   (i) a number $\mu(\beta) = \mu(C/B) \geqslant \delta(B)$, if $|C - B| \geqslant 2$;
   (ii) a number $\mu(\beta) \geqslant 1$, if $\beta = \boldsymbol{\alpha}$.

(2) For any good pair (B, C) with $B \subseteq M$ and $M \in \hat{\boldsymbol{K}}_0$, $\chi_M(C/B)$ denotes the {Kmuitem} *number of disjoint copies of C over B in M. Of course, $\chi_M(C/B)$ may be 0.*

(3) Let $\boldsymbol{K}_\mu$ be the class of structures M in $\boldsymbol{K}_0$ such that if (C/B) is a good pair {Kmuhatitem} *$\chi_M(C/B) \leqslant \mu((C/B))$.*

(4) $\hat{\boldsymbol{K}}_\mu$ is the class of direct limits of structures in $\boldsymbol{K}_\mu$.

{assqg}

3. Associating Strongly Minimal Steiner systems with Quasigroups

We explore here the distinction between informal and definable coordinitazation (Definition 3.9). First we summarise the substantial literature on informal coordinatization of k-Steiner systems. Then we give a short proof that the 'natural' coordinatizing quasigroup is not definable in the strongly minimal k-Steiner system (M, R) when $k \geqslant 3$. [BV21] contains the vastly more complicated argument that no quasigroup (indeed, in many case, no binary function with domain M^2) is definable in (M, R).

{quasigroups} **Definition 3.1** ([Smi07])**.** *A quasigroup*[6] *$(Q, *)$ is a groupoid $(A, *)$ such that for $a, b \in Q$, there exist unique elements $x, y \in Q$ such that both*

$$a * x = b, y * a = b.$$

The general notion is a universal Horn class, not a variety. See Definition 3.6 and Remark 3.7.

We will discuss in detail three (families of) varieties (equational classes) of quasigroups corresponding to $k = 2, 3, p^n$.

{defrelvar} **Definition 3.2.** [Smi07]

(1) A Steiner quasigroup is a groupoid which satisfies the equations: $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

(2) A Stein quasigroup is a groupoid which satisfies the equations: $x * x = x$, $(x * y) * y = y * x$, $(y * x) * y = x$.

(3) Given a (near)-field[7] *$(F, +, \cdot, -, 0, 1)$ of cardinality q and a primitive element $a \in F$, define a multiplication $*$ on F by $x * y = y + (x - y)a$. An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a* block algebra [GW80] *over $(F, *)$. Note $(F, *)$ is idempotent (i.e., $x * x = x$).*

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[6] Alias: multiplicative quasigroup [MMT87], combinatorial quasigroup [Smi07].

[7] A near-field is an algebraic structure satisfying the axioms for a division ring, except that it has only one of the two distributive laws. They were introduced by Dickson in 1905; we focus on the field case. We have seen the term 'block algebra' only in [GW80], but it seems appropriate.

While every group is an quasigroup, the Stein and Steiner quasigroup are rather special quasigroups since they are idempotent. Thus, a Stein or Steiner quasigroup $(Q, *)$ cannot be a group unless it has only one element.

Steiner triple systems and Steiner quasigroups are actually *interdefinable.*

{defsq}

Fact 3.3. *Each Steiner* triple *system is interdefinable with a Steiner quasigroup (Definition 3.2).*

Proof. Given the algebra, the blocks are the 2-generated subalgebras; given a Steiner triple system, let $x \circ y$ be the third element of the block if $x \neq y$ and $x \circ x = x$. Since all blocks are isomorphic to the unique 3 element Steiner quasigroup, the resulting algebra is a Steiner quasigroup. ■

Corollary 3.4. *There are $2^{\aleph_0}$ strongly minimal theories T_μ of Steiner quasigroups and so non-isomorphic Steiner triple systems of cardinality $\aleph_0$.*

Proof. We have an explicit 1-dimensional (the domain and range of the interpretation is the universe) bi-interpretation between Steiner triple systems and the Steiner 3-systems that were given by Theorem 0.1. ■

While these algebras are in the variety of Steiner quasigroups, for each μ we have selected a single algebra in each uncountable cardinality. So we are distinguishing first order theories not varieties. The following example illustrates the issue addressed in more generality and more detail in Fact 3.11. Can the theory of the Steiner system distinguish the two Steiner quasigroups?

Fact 3.5 (Stein quasigroups)**.** [GW80, page 5] *Each $(2,4)$-Steiner system (P, B) is coordinatized by a Stein quasigroup, $(Q, *)$* {Steinq}

Proof. Sketch: One direction is obvious; the blocks are the 2-generated subalgebras of the quasigroup. For the other direction, the universe of the algebra is $Q = P$. For each block $b \in B$, enumerate B as 0 to 3. We consider two possibilities for a four element Stein quasigroup on B: A_1 requires $0 * 1 = 2$ while A_2 requires $0 * 1 = 3$. Regardless of the choice of A_i, the entire structure is a Stein quasigroup. It clearly satisfies the three equations of Definition 3.2.2 because they involve elements only within a single block and also the requirement that each equation $ax = b$ $(ya = b)$ has a unique solution, as again the solution is within the block determined by a, b. ■

Following [GW75, Pad72], we say

{rkdef}

Definition 3.6. *(1) The variety V is an (r, k)-variety if every r-generated subalgebra of any $A \in V$ is isomorphic to the free V-algebra on r, $F_r(V)$, and $|F_r(V)| = k$.*

(2) A Mikado *variety* [GW75, 128] *is $(2, q)$-variety with all fundamental operations binary and with an equational base of 2-variable equations.*

This is one of 5 equivalent characterizations of an (r, k) variety in [Pad72]. Obviously, the collection of r-generated subalgebras $A \in V$ form an Steiner (r, k)-system; we need a third: the automorphism group of any r generated algebra is strictly (i.e. sharply) r-transitive.

{vocab}

Remark 3.7. In general a quotient of a quasigroup $(Q, *)$ need not be a quasigroup. But if $V(Q)$ (i.e. HSP(Q)) is an (r, k) variety, then every algebra in $V(Q)$ is a quasigroup [Qua92, Theorem 3]. So in this paper[8] we can regard quasigroups as structures with one binary operation.

We rely heavily on a 'classical' observation of Trevor Evans. It requires no proof, but a little thought. Evans [Eva82] calls a variety V *binary* if both all function symbols of V are binary and the defining equations involve only 2 variables.

{2propagates}

Fact 3.8 ([Ste64, Eva76])**.** *If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.*

In this context, Definition 1.1 becomes:

{defcoord}

Definition 3.9. *A variety (equational class) or more generally a first order theory of algebras V* coordinatizes *a class $\mathcal{S}$ of $(2, k)$-Steiner systems if:*

The universe M of each member (M, R) of $\mathcal{S}$ is the domain of an algebra $(M, *)$ in V and the lines are the 2-generated $*$-subalgebras.

If $*$ is definable from R, this is a definable coordinatization*.*

To clarify the situation, we sketch the proof of the relevant parts of the classification of (r, k) varieties.

{uaoracle}

Fact 3.10.
(1) [Ś61] *The only (r, k) varieties are those where $r = 0$, $k = 0$; $r = k$; $r = 2$, $k = q = p^n$, for a prime p and a natural number n; $r = 3$, $k = 4$.*
(2) [GW75, GW80] *For each q, the class of q-Steiner systems is coordinatized by a $(2, q)$-variety V of block algebras (Definition 3.2).*

Proof sketch: Only Steiner $(2, q)$-systems with $q = p^n$ for some prime p, and $n \geqslant 1$ are relevant here. It is easy to check that the block algebras defined in Definition 3.2 are $(2, k)$ algebras. But, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out, an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Given the Steiner q-system, we assign to each line a copy of the unique q element algebra $F_2(V)$. This gives an algebra in V by Fact 3.8 $\blacksquare_{3.10}$

We easily see 1) of Theorem 3.11 from Facts 3.8 and 3.10.

{coordthm}

Theorem 3.11. *If T_μ is a strongly minimal Steiner k-system (from Fact 0.1) and V is a Mikado $(2, k)$ variety of quasigroups, then*
(1) Each $(M, R) \models T_\mu$ is informally coordinatizable *by an algebra $(Q_M, *)$ in V.*
(2) $R(x, y, z)$ is definable in $(Q_M, *)$.
(3) There is an (incomplete) first order theory $\check{T}_\mu$ in the vocabulary $\{*\}$ such that each model of T_μ is coordinatized by a model of $\check{T}_\mu$.
(4) If $\mu(\boldsymbol{\alpha}) = k > 1$ this coordinatization is not definable in (M, R).

[8] In general the variety generated by a quasigroup contains groupoids that are not quasigroups. See [Qua92], [MMT87, page 126], [SR99, Example 2.2]. In the general situation, to obtain a variety rather than a quasivariety, the requirement that the binary relation has inverses must be enforced by binary left and right division operators.

Proof. 1) is immediate from Fact 3.10. For 2) let $\theta_F(x, y, z)$ be the disjunction of the terms $z = f_i(x, y)$ where the $f_i(x, y)$ list the terms generating $F = F_2(V)$. Thus, $R(x, y, z) \leftrightarrow \theta_F(x, y, z)$ defines R in $(Q, *)$.

3) Let $\Delta_F(x, y, f_1(x, y), \ldots f_k(x, y))$ denote the quantifier-free diagram of F. The two-transitivity of F guarantees the particular choice of the two elements x, y does not matter. Letting R/θ denote the substitution of θ_F for R, $\check{T}_\mu$ is axiomatized by

$$Eq(V) \cup \{(\forall x, y)\Delta_F(x, y, f_1(x, y), \ldots f_k(x, y))\} \cup \{\phi\restriction(R/\theta_F) : \phi \in T_\mu\}.$$

4) Without loss of generality, let (M, R) be the countable generic and suppose it is coordinatized by $(Q_M, *)$. Let $\{a, b\}$ be a strong substructure of (M, R) (i.e. $d(\{a, b\} = 2)$ and let $c_1, \ldots c_k$ fill out the line through a, b to a structure A. Since V is a Mikado variety, by genericity there is a strong embedding of A into M.

Then all triples a, b, c_i realize the same quantifier free R-type and $A \leqslant M$ implies for any permutation ν of k fixing $0, 1$, for $2 \leqslant i < k$, there is an automorphism of (M, R) fixing a, b and taking c_i to $c_{\nu(i)}$. Thus, $a{*}b$ cannot be definable in (M, R). ■

Despite Theorem 3.11.4, which shows there is no reason to think $\mathrm{Th}(\mathrm{Q_M}, *)$ is strongly minimal, we find strongly minimal theories of quasigroups in Section 4.

{uncoord}

QUESTION 3.12. We have found a coordinatizing algebra Q_M for each model M of T_μ. The construction depends on M and a particular free algebra F on two generators. The choice of the block algebra variety in 1) is not unique. Ganter and Werner [GW80, page 7] describe two different varieties of block algebras (one commutative and one not) over F_5, depending on the choice of the primitive element a of F_5 (Definition 3.2). Thus $\check{T}_\mu$ is not complete. Our constructions (Theorem 4.3) show there are continuum many first order theories of strongly minimal block algebras.

(1) *Are all the $(Q_M, *)$ (for the same F) elementarily equivalent? in the same (equationally complete?) variety?*
(2) *Do they represent continuum many distinct varieties? I.e, are the classes $HSP(\mathcal{G}_\mu)$ distinct for (sufficiently) distinct μ?*

Since $\check{T}_\mu$ is not complete, it can't be interpreted in the complete theory T_μ. But there is a much stronger reason for the failure to define $*$ in (M, R). For this, we need some further hypotheses on μ.

{tripdef}

Definition 3.13. *Let $\boldsymbol{K}_{-1}$ be the class of finite linear spaces as in Definition 2.3. Recall from* [BP20] *that we allowed $\mu(\boldsymbol{\alpha}) = 1$ to accommodate Steiner triple systems. We say that the function μ from good pairs into ω satisfies*

(1) *$\mu \in \mathcal{U}$ if the standard Hrushovski condition is met: $\mu(A/B) \geqslant \delta(B)$.*
(2) *μ* triples *($\mu \in \mathcal{T}$) if for $\mu(A/B) \geqslant 3$ unless $\delta(B) = 1$ or B is an independent pair.*

If μ is in the class $\mathcal{T}$ of triplable μ-functions, [BV21] ensures that there are no definable truly binary (indeed n-ary) functions.

{essunary}

Definition 3.14 (Essentially Unary functions)**.** *Let T be a strongly minimal theory. An $\emptyset$-definable function $f(x_0 \ldots x_{n-1})$ is called* essentially unary *if there is an $\emptyset$-definable function $g(u)$ such that for some i, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.*

With Verbovskiy we introduced the notion of a decomposition of finite G-normal subsets of Hrushovski strongly minimal sets with respect to automorphism groups to prove:

{bvnobin}

Fact 3.15. *[*[BV21]*] For any strongly minimal Steiner system* (M, R)

(1) If $\mu \in \mathcal{T}$ (μ-triples), every definable function in a model of T_μ is essentially unary and so T_μ does not have elimination of imaginaries.
(2) If $\mu \in \mathcal{U}$ T_μ does not admit elimination of imaginaries. In particular, there is no commutative definable binary function.

Elimination of imaginaries is an important concept in geometric stability theory. For our purposes here, we consider a more algebraic condition which implies imaginaries cannot be eliminated: every definable function f is symmetric; that is, the value of $f(\mathbf{x})$ does not depend on the order of the arguments. See [BV21, §2].

Fact 3.15 is proved for the basic Hrushovski construction and strongly minimal Steiner systems in [BV21]. The crucial distinction between 1) and 2) in Fact 3.15 is that 2) there may be definable binary functions but they cannot be commutative.

We now show the versatility of the method of construction by finding Steiner systems which both do and don't admit definable unary functions. We repeat a short argument from [BP20] since the method is highly relevant here.

{realudc}

Lemma 3.16. *If $\mu \in \mathcal{U}$ and $\mu(\boldsymbol{\alpha}) \geqslant 2$, i.e. lines have length at least 4, there is a 12-element structure A that is 0-primitive over a singleton a. If $\mu(A) = 1$, then T_μ admits a non-trivial definable unary function.*

Proof. Let $\boldsymbol{\eta}$ be the isomorphism type of the pair $(\{a\}, \{b, c\} \cup \{d_i : 1 \leqslant i \leqslant 9\})$ where R holds of a, b, c, $ad_{2i+1}d_{2i+2}$ (for $0 \leqslant i \leqslant 3$) $bd_{2i+2}d_{2i+3}$ (for $0 \leqslant i \leqslant 2$), b, d_8, d_1, and finally each triple from $\{c, d_8, d_9, d_4\}$. There are 12 points, nine 3-point line segments and one with 4 points so A denotes the entire structure $\delta(A) = 12 - (9 + 2) = 1$. By inspection, each proper substructure A' has $\delta(A') \geqslant 2$ so A is 1-primitive over $\{a\}$. But d_9 is the unique point that is in exactly one clique within A. Thus, if $\mu(\boldsymbol{\eta}) = 1$, the formula $(\exists x_1, x_2 y_1, \ldots y_8)\Delta(x_0, x_1, x_2 y_1, \ldots y_8, y_9)$ (where Δ is the quantifier free diagram of A) defines d_9 over a in any model of T_μ. ∎

While we have given only one example, one can extend the length of the cycle and get infinitely many examples. Note that the construction in Lemma 3.16 is iterable so the definable closure *may not be locally finite*.

Suppose, however that $\mu(\boldsymbol{\eta}) \geqslant 2$. Then if $\{a\} \leqslant M$, by [BP20, Corollary 5.16], there are 2 realizations of A over $\{a\}$ and so $d_9 \in \mathrm{acl}(\{a\} - \mathrm{dcl}(\{a\}$. Moreover, if $\{a\} \nleqslant M$ then $d_9 \in \mathrm{acl}(\emptyset)$.

Recall $\mathcal{U}$ is the set of μ such that for any good pair B/A, $\mu(B/A) \geqslant \delta(B)$. While the construction with admissible $\mathcal{U}$ does not imply trivial unary closure, we can obtain triviality by changing the class U of admissible μ.

{defCscr}

Definition 3.17. *Define $\mathcal{C}$ by restricting $\mathcal{U}$ by requiring that if $|A| = 1$ and some point in B is determined by A (such as $\boldsymbol{\gamma}$ in Lemma 3.16), then $\mu(B/A) = 0$.*

We used the cycle graphs of [CW12] to prove in [BP20, 4.11] that there are $2^{\aleph_0}$ distinct strongly minimal Steiner systems T_μ; this proof remains valid if μ is restricted to $\mathcal{C}$ or even $\mathcal{C} \cap \mathcal{T}$. Clearly amalgamation can not introduce unary functions so we have:

{nounary}

Proposition 3.18. *If $\mu \in \mathcal{C} \cap \mathcal{T}$, and M is the generic for $\boldsymbol{K}_\mu$, for any $a \in M$,* $\mathrm{dcl(a)} = \{\mathrm{a}\}$.

Proof. Clearly amalgamation can not introduce unary functions so we have a generic M with no unary functions.

4. Constructing strongly minimal quasigroups

{findqg}

We have shown that in general the strongly minimal Steiner k-systems in the vocabulary $\tau = \{R\}$ for $k > 3$ do not define a quasigroup. We will jointly construct a Steiner system (M, R) and a multiplication $*$, requiring that the $*$-algebra be in a given $(2, q)$-variety V (Definition 3.6) that coordinatizes (M, R). Then taking the reduct of $(M, R, *)$ to the vocabulary containing only $*$ we have a strongly minimal flat quasigroup.

{defK'}

Definition 4.1. *Fix $\mu \in \mathcal{U}$ with $\mu(\boldsymbol{\alpha}) = q-2$ and a $(2, q)$-variety V of quasigroups.*

(1) Let the class $\boldsymbol{K}^q_\mu$ be the finite τ-structures A such that each maximal clique has q-elements. This is expressed by $\forall\exists$ τ-sentence.
(2) Expand $\tau = \{R\}$ to τ' by adding a ternary relation symbol H. (This will be graph of $*$.) Let $\boldsymbol{K}'_\mu$ $(= \boldsymbol{K}'_{\mu',V})$ be the finite τ'- structures A' such that $A' \restriction \tau \in \boldsymbol{K}^q_{\mu'}$ and $A' \restriction H$ is the graph of $F_2(V)$ on each line.
(3) Let (A/B) be a good pair for $\boldsymbol{K}^q_\mu$ with isomorphism type $\boldsymbol{\gamma}$. Let $\boldsymbol{\gamma_i}$ be a list of the possible ways to assign the quasi-group structure on the domain A' described in the previous paragraph[9].
(4) Note $\mu'(\boldsymbol{\alpha'})$ must be 1, when the τ-reduct is a line of length k, over a two-point base, since two points determine a line and since V is a $(2, k)$ all quasigroups on the line are isomorphic. For any other isomorphism type of a good pair with $\boldsymbol{\gamma} = \boldsymbol{\gamma'} \restriction \tau$, let $\mu'(\boldsymbol{\gamma'}) = \mu(\boldsymbol{\gamma})$.

For each prime power $q \geqslant 4$, we show that the class of finite linear spaces such that each maximal non-trivial (at least three elements) clique has size q and with $F_2(V)$ as the quasigroup on each such clique has the amalgamation property.

{constexp}

Construction 4.2. For $q > 3$ a prime power, fix μ with $\mu(\boldsymbol{\alpha}) = q - 2$ and V as Definition 4.1. We transform an $A \in \boldsymbol{K}^q_\mu$ to an $A' \in \boldsymbol{K}'_\mu$. First, there is a canonical extension of A to $\check{A} \in \boldsymbol{K}^q_\mu$. Namely extend each clique of length at least 3 in A to have length q; but with no new intersections. Now expand $\check{A}$ to a τ'-structure by imposing on each line ℓ a copy of $F_2(V)$ with graph $H \restriction \ell$. Call this expansion A'. We have in fact defined a finite family of possible expansions of A (depending on the interaction of H and R). The set of possible expansions of each $A \in \boldsymbol{K}_\mu$ as A varies through $\boldsymbol{K}^q_\mu$ is denoted $\boldsymbol{K}'_{\mu,V}$.

For any such A', let $\delta'(A') = \delta(A' \restriction \tau) = \delta(A)$. Note that each non-trivial line in A' has q elements. We denote by $\boldsymbol{\alpha}'$ the τ'-isomorphism type $(a_1, a_2, b_1 \ldots b_{k-2})$ of full line over two points; it is good with respect to $\boldsymbol{K}'_{\mu,V}$.

Note that, except for $\boldsymbol{\alpha'}$, each good pair $\boldsymbol{\gamma} = A/B$ in $\boldsymbol{K}_\mu$ has generated a finite number of distinct good pairs in $\boldsymbol{K}'_\mu$. As, the various copies have the same reduct

[9]This τ' structure on A' will be a substructure of a structure in $\boldsymbol{K}'_\mu$ but rarely itself a member of $\boldsymbol{K}'_\mu$.

to τ but differ in their quasigroup structure. With this framework in hand we can complete the proof of Theorem 4.3. We show how to vary the proofs of the crucial results 5.11 and 5.15 from [BP20] for this result.

{getsmquasigrp}

Theorem 4.3. *For each q and each $\mu \in \mathcal{U}$ each of the T_μ in Theorem 0.1 with line length $k = q = p^n$ (for some n) and any Mikado $(2, q)$ variety (block algebras) of quasigroups V (i.e. block algebras), there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that defines a strongly minimal q-Steiner system. The quasigroup is not commutative.*

Proof. We can construct a generic, provided we prove amalgamation for $\boldsymbol{K}'_{\mu,V}$. We now show that the amalgamation for the τ-class, as in Lemma 5.11 and Lemma 5.15 of [BP20] yields an amalgamation for τ'. Consider a triple D', E', F' in $\boldsymbol{K}'_{\mu,V}$ as in Lemma 5.15 of[BP20]. That is, $D' \subseteq F$ and E' is 0-primitive over D'. Since E' is primitive over D', although there may be a line contained in the disjoint amalgam G' with two points in each of D and $F - D$, each line that contains 2 points in $E - D$ can contain at most one from D. Thus, there is no issue with defining the relation H on the disjoint amalgamation. If μ' requires some identification for some (B', C'), just as in the original, it is because the (relational) τ'-structure $B'C'$ is $D'E'$ and (Note the 'further' in [BP20, Lemma 5.10].) there is a copy of E' over B' in F'. Now the strong minimality of the generic $\mathcal{G}'_{\mu,V}$ follows exactly as in Lemmas 5.21 and 5.23 of [BP20] and, letting $T_{\mu',V} = \mathrm{Th}(\mathcal{G}'_{\mu,\mathrm{V}})$, we have proved Theorem 4.3. The noncommutativity follows from blocking of symmetric functions that prevents elimination of imaginaries in Theorem 3.15. ∎

Notation 4.4. *For each $\mu \in \mathcal{U}$ with $\mu(\boldsymbol{\alpha}) = q-2$, and each $(2, q)$-variety V, we denote by $\mathcal{G}'_{\mu',V}$ the strongly minimal τ' generic structure constructed in Theorem 4.3. Its reduct to $\{H\}$ is a strongly minimal block algebra. The operation $*$ is quantifier free definable from H. The theory of that reduct is essentially $T_{\mu',V\restriction H}$ since R is definable in that reduct by $R(u, v, w)$ if and only $\bigvee_{\sigma(u,v)} H(u, v, \sigma(u, v))$, where is $\sigma(u, v)$ is an existential formula formed by translating the generating polynomials of $F_2(V)$ into relational formulas).*

{BarCas}

QUESTION 4.5. The use of the graph of the quasigroup in Construction 4.2 is similar to that in the study of model complete Steiner triple system of Barbina and Casanovas [BC19]. As noted in Remark 5.27 of [BP20], their generic structure M differs radically from ours: $\mathrm{acl_M(X)} = \mathrm{dcl_M(X)}$.

Is it possible to develop a theory of q-block algebras for arbitrary prime powers similar to that for Steiner quasigroups in their paper? That is, to find a model completion for each of the various varieties of block algebras discussed in Definition 3.2.3? Where do the resulting theories lie in the stability classification?

{incompleteness}

QUESTION 4.6. Necessarily in the construction given, a good pair (C/B) of τ-structures in the reduct of a model of $T_{\mu',V}$ will have many (but finitely) more copies of C over B than $\mu(C/B)$. Thus, $T_{\mu',V} \restriction \tau$ is not T_μ.

Is it possible to characterize those μ such that T_μ can be interpreted in a quasigroup? We guaranteed that each 2-generated subalgebra is $F_2(V)$ and V is a Mikado variety (in particular, determined by 2-variable equations), each quasi-group Q_M is in V. This is enough to show the full structure is a quasigroup. But different varieties of quasigroups may have the same free algebra on two generators. Construction 4.2 depends on both the original $\boldsymbol{K}^q_\mu$ and $F_2(V)$. *How many varieties can* *arise from the same $F_2(V)$? There are two variants on this question. One is, 'how many varieties of quasigroup can have the same free algebra on two generators?'. The second asks about only varieties that arise from a chosen μ and a variety V as in Construction 4.2.*

What varieties do the $\mathcal{G}_{\mu',V}$ generate? *Immediately from known results each such variety satisfies strong properties listed below.*

{uacor}

Corollary 4.7. *For each $T_{\mu',V}$ with prime power line length, any $M \models T_\mu$, the reduct to $*$ is in a variety (that is congruence permutable, regular and uniform* [Qua76, Theorem 3.1] *or* [GW75, Corollary 2.4] *but not residually small* [BM88, Corollary 8].

{uaques}

QUESTION 4.8. *Every finite algebra in a $(2, q)$-variety has a finite decomposition into directly irreducible algebras (*[GW75, Corollary 2.4]. Are there any similar results for infinite strongly minimal block algebras?

References

[Bal21] John T. Baldwin. Strongly minimal Steiner Systems III: Path Graphs and Sparse configurations. in preparation, 2021.

[BP20] John T. Baldwin and G. Paolini. Strongly Minimal Steiner Systems I. *Journal of Symbolic Logic*, 1-15, 2020. published online oct 22, 2020 `arXiv:1903.03541`.

[BS96] John T. Baldwin and Niandong Shi. Stable generic structures. *Annals of Pure and Applied Logic*, 79:1–35, 1996.

[BC19] Silvia Barbina and Enrique Casanovas. Model theory of Steiner triple systems. *Journal of Mathematical Logic*, 20: 2019. `https://doi.org/10.1142/S0219061320500105`.

[BV21] John T. Baldwin and V. Verbovskiy. Towards a finer classification of strongly minimal sets. in preparation, 2021.

[BM88] Clifford Bergman and Ralph McKenzie. On the relationship of AP, RS and CEP in congruence modular varieties. II. *Proc. Amer. Math. Soc.*, 103(2):335–343, 1988.

[Bur97] W. Burnside. *Groups of Finite Order.* Cambridge, 1897.

[CGGW10] K. M. Chicot, M. J. Grannell, T. S. Griggs, and B. S. Webb. On sparse countably infinite Steiner triple systems. *J. Combin. Des.*, 18(2):115–122, 2010.

[CK16] G. Conant and A. Kruckman. Independence in generic incidence structures. preprint, 2016.

[CW12] P. J. Cameron and B. S. Webb. Perfect countably infinite Steiner triple systems. *Australas. J. Combin.*, 54:273–278, 2012.

[Eva76] Trevor Evans. Universal Algebra and Euler's Officer Problem. *The American Mathematical Monthly,*, 86(6):466–473, 1976.

[Eva82] Trevor Evans. Finite representations of two-variable identities or why are finite fields important in combinatorics? In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 135–141. North-Holland, Amsterdam, 1982.

[Eva04] D. Evans. Block transitive Steiner systems with more than one point orbit. *Journal of Combinatorial Design*, 12:459–464, 2004.

[GW75] Bernhard Ganter and Heinrich Werner. Equational classes of Steiner systems. *Algebra Universalis*, 5:125–140, 1975.

[GW80] Bernhard Ganter and Heinrich Werner. Co-ordinatizing Steiner systems. In C.C. Lindner and A. Rosa, editors, *Topics on Steiner Systems*, pages 3–24. North Holland, 1980.

[Grä63] G. Grätzer. A theorem on two transitive permutation groups with application to universal algebras. *Fundamenta Mathematica*, 53, 1963.

[HP] Tapani Hyttinen and Gianluca Paolini. First order model theory of free projective planes: Part I. submitted.

[Hru93] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[KL92] D.W. Kueker and C. Laskowski. On generic structures. *Notre Dame Journal of Formal Logic*, 33:175–183, 1992.

[Mak19] J. A. Makowsky. Can one design a geometry engine? On the (un)decidability of affine Euclidean geometries. *Annals of Mathematics and Artificial Intelligence*, 85:259–291, 2019. online: `https://doi.org/10.1007/s10472-018-9610-1`.

[MMT87] Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor. *Algebras, lattices, varieties. Vol. I.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987.

[Pad72] R. Padmanabhan. Characterization of a class of groupoids. *Algebra Universalis*, 1:374–382, 1971/72.

[Pao21] Gianluca Paolini. New ω-stable planes. *Reports on Mathematical Logic*, 2021. to appear.

[Qua76] Robert W. Quackenbush. Varieties of Steiner loops and Steiner quasigroups. *Canad. J. Math.*, 28(6):1187–1198, 1976.

[Qua92] Robert W. Quackenbush. Quasigroups, universal algebra and combinatorics. In Jonathan D. H. Smith and Anna B. Romanowska, editors, *Universal Algebra and Quasigroup Theory*, pages 159–171. Helderman Verlag, Berlin, 1992.

[Rob82] D.J.S. Robinson. *A Course in the Theory of Groups*. Springer-Verlag, 1982.

[Ś61] S. Świerczkowski. Algebras which are independently generated by every n elements. *Fund. Math.*, 49:93–104, 1960/1961.

[Smi07] Jonathan D. H. Smith. Four lectures on quasigroup representations. *Quasigroups and Related Systems*, 15:109–140, 2007.

[SR99] Jonathan D. H. Smith and Anna B. Romanowska. *Post-modern algebra.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1999. A Wiley-Interscience Publication.

[Ste56] Sherman K. Stein. Foundations of quasigroups. *Proc. Nat. Acad. Sci.*, 42:545–546, 1956.

[Ste64] Sherman K Stein. Homogeneous quasigroups. *Pacific Journal of Mathematics*, 14:1091–1102, 1964.