# DOP and FCP in Generic Structures

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## 1 Context

We work throughout in a finite relational language $L$. This paper is built on [2] and [3]. We repeat some of the basic notions and results from these papers for the convenience of the reader but familiarity with the setup in the first few sections of [3] is needed to read this paper. Spencer and Shelah [6] constructed for each irrational $\alpha$ between 0 and 1 the theory $T^\alpha$ as the almost sure theory of random graphs with edge probability $n^{-\alpha}$. In [2] we proved that this was the same theory as the theory $T_\alpha$ built by constructing a generic model in [3]. In this paper we explore some of the more subtle model theoretic properties of this theory. We show that $T^\alpha$ has the dimensional order property and does not have the finite cover property.

We work in the framework of [3] so probability theory is not needed in this paper. This choice allows us to consider a wider class of theories than

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just the $T_\alpha$. The basic facts cited from [3] were due to Hrushovski [4]; a full bibliography is in [3]. For general background in stability theory see [1] or [5].

We work at three levels of generality. The first is given by an axiomatic framework in Context 1.10. Section 2 is carried out in this generality. The main family of examples for this context is described Examples 1.3. Sections 3 and 4 depend on a function $\delta$ assigning a real number to each finite $L$-structure as in these examples. Some of the constructions in Section 3 (labeled at the time) use heavily the restriction of the class of examples to graphs. The first author acknowledges useful discussions on this paper with Sergei Starchenko.

**1.1 Notation.** Let $K_0$ be a class of finite structures closed under substructure and isomorphism and containing the empty structure. Let $\overline{K}_0$ be the universal class determined by $K_0$.

**1.2 Notation.** Let $B \cap C = A$. The *free amalgam* of $B$ and $C$ over $A$, denoted $B \bigotimes_A C$, is the structure with universe $BC$ but no relations not in $B$ or $C$.

We write $A \subseteq_\omega B$ to mean $A$ is a finite subset of $B$. A structure $A$ is called *discrete* if there are no relations among the elements of $A$. Let $\delta : K_0 \mapsto \Re^+$ (the nonnegative reals) be an arbitrary function with $\delta(\emptyset) = 0$. Extend $\delta$ to $d : \overline{K}_0 \times K_0 \mapsto \Re^+$ by for each $N \in \overline{K}_0$,

$$d(N, A) = \inf\{\delta(B) : A \subseteq B \subseteq_\omega N\}.$$

We usually write $d(N, A)$ as $d_N(A)$. We only use this definition when $\delta$ is defined on every finite subset of $N$. We will omit the subscript $N$ if it is clear from context.

For $g = \delta$ or $d_N$ and finite $A, B$, we define *relative dimension* by $g(A/B) = g(AB) - g(B)$. For infinite $B$ and finite $A$, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_\omega B\}$. This definition is justified in e.g. Section 3 of [3]. For any finite sequence $\overline{a} \in N$, $d_N(\overline{a})$ is the same as $d_N(A)$ where $\overline{a}$ enumerates $A$.

Consider a finite structure $B$ for a finite relational language $L$. We assume that each relation of $L$ holds of a tuple $\overline{a}$ only if the elements $\overline{a}$ are distinct and if $R(\overline{a})$ holds, $R(\overline{a}')$ holds for any permutation $\overline{a}'$ of $\overline{a}$.

$R(B)$ denotes the collection of subsets $B_0 = \{b_1, \ldots b_n\}$ of $B$ such that for some (any) ordering $\overline{b}$ of $B_0$, $B \models R(\overline{b})$ for some relation symbol $R$ of

$L$; $e(B) = |R(B)|$. Let $A$, $B$, $C$ be disjoint sets. We write $R(A,B)$ for the collection of subsets from $AB$ that satisfy some relation of $L$ (counting with multiplicity if a set satisfies more than one relation) and contain at least one member of $A$ and one of $B$. Write $e(A,B)$ for $|R(A,B)|$. Similarly, we write $R(A,B,C)$ for the collection of subsets from $ABC$ that satisfy some relation of $L$ and contain at least one member of $A$ and one of $C$. Write $e(A,B,C)$ for $|R(A,B,C)|$.

**1.3 Example.** The most important examples arise by defining $\delta$ as follows. In the last section of [3] we enumerated several other examples to which this axiomatization applies. Let

$$\delta_{\beta,\alpha}(A) = \beta|A| - \alpha e(A).$$

We may write $\delta_\alpha$ for $\delta_{1,\alpha}$. The class $K_\alpha$ is the collection of finite $L$-structures $A$ such that for any $A' \subseteq A$, $\delta_\alpha(A') \geq 0$. We denote by $T_\alpha$ the theory of the generic model of $K_\alpha$.

**1.4 Axioms.** Let $N$ be in $\overline{K}_0$ and let $A$, $B$, $C \in K_0$ be substructures of $N$.

1. If $A$, $B$, and $C$ are disjoint then $\delta(C/A) \geq \delta(C/AB)$.
2. For every $n$ there is an $\epsilon_n > 0$ such that if $|A| < n$ and $\delta(A/B) < 0$ then $\delta(A/B) \leq -\epsilon_n$.
3. There is a real number $\epsilon$ independent of $N, A, B, C$ such that if $A, B, C$ are disjoint subsets of a model $N$ and $\delta(A/B) - \delta(A/BC) < \epsilon$ then $R(A,B,C) = \emptyset$ and $\delta(A/B) = \delta(A/BC)$.
4. For each $A \in K_0$, and each $A' \subseteq A$, $\delta(A') \geq 0$.

We call a function $d = d_N$ derived from $\delta$ satisfying Axioms 1.4 a *dimension function.*

***1.5 Lemma.*** *If $\delta$ is a dimension function satisfying the properties of Axiom 1.4 and $\leq_s$ ( read* strong submodel *) is defined by $A \leq_s N$ if $d_N(A) = d_A(A)$, then $\leq_s$ satisfies the following propositions. Let $M, N, N' \in \overline{K}_0$.*

***A1***. *$M \leq_s M$.*
***A2***. *If $M \leq_s N$ then $M \subseteq N$.*
***A3***. *$M \leq_s N' \leq_s N$ implies $M \leq_s N'$.*
***A4***. *If $M \leq_s N$, $N' \subseteq N$ then $M \bigcap N' \leq_s N$.*
***A5.*** *For all $M \in \overline{K}_0$, $\emptyset \leq_s M$.*

We need to analyze extensions which are far from being strong.

**1.6 Definition.** For $A, B \in S(K_0)$, $A \leq_i B$ if $A \subseteq B$ but there is no $A'$ properly contained in $B$ with $A \subseteq A' \leq_s B$. If $A \leq_i B$, we say $B$ is an *intrinsic* extension of $A$.

**1.7 Definition.** The intrinsic closure of $A$ in $M$, $icl_M(A)$ is the union of $B$ with $A \subseteq B \subseteq M$ and $A \leq_i B$. When $M$ is clear from context, we write $\overline{A}$ for $icl_M(A)$. The intrinsic closure can be more finely analyzed as follows.

1. For any $M \in K$, any $m \in \omega$, and any $A \subseteq M$,
$$icl_M^m(A) = \cup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$
2. $$icl_M(A) = \cup_m icl_M^m(A).$$
3. $M$ has *finite closures* if for each finite $A \subseteq M$, $icl(A)$ is finite.
4. $K$ has *finite closures* if each $M \in K$ has finite closures.

Using **A4**, note that the intrinsic closure of $A$ in $M$ is the intersection of the strong substructures of $M$ which contain $A$. Thus, when finite, $icl_M(A) \in K_0$ and is a strong substructure of $M$. Moreover, a countable $M$ has finite closures if and only if $M$ can be written as an increasing union of finite strong substructures.

**1.8 Definition.** The countable model $M \in \overline{K}_0$ is $(K_0, \leq_s)$-*generic* if

1. If $A \leq_s M, A \leq_s B \in K_0$, then there exists $B' \leq_s M$ such that $B \cong_A B'$,
2. $M$ has finite closures.

**1.9 Fact.** If $(K_0, \leq_s)$ satisfies the properties of Lemma 1.5 and the amalgamation property with respect to $\leq_s$ then there is a countable $K_0$-generic model.

**1.10 Context.** Henceforth, $(K_0, \leq_s)$ is class of finite structures closed under isomorphism and substructure with $\leq_s$ induced by a function $\delta$ obeying Axioms 1.4. Moreover, we assume $(K_0, \leq_s)$ satisfies the amalgamation property and $K$ is the class of models of the theory of the generic model $M$ of $(K_0, \leq_s)$. $\mathcal{M}$ is a large saturated model of $T = Th(M)$. In the absence of other specification, the dimension function $d$ is the function induced on $\mathcal{M}$ by $\delta$ and we work with substructures of $\mathcal{M}$.

# 2 Independence and Orthogonality

As indicated in Context 1.10, the following definitions take place in a suitably saturated model elementarily equivalent to the generic. We work in that context throughout this section.

**2.1 Definition.** We say the finite sets $A$ and $B$ are $d$-independent over $C$ and write

1. $A \downarrow^d{}_C B$ if
   (a) $d(A/C) = d(A/CB)$.
   (b) $\overline{AC} \cap \overline{BC} \subseteq \overline{C}$.
2. We say the (arbitrary) sets $A$ and $B$ are $d$-independent over $C$ and write $A \downarrow^d{}_C B$ if for every finite $A' \subseteq A$ and $B' \subseteq B$, $A' \downarrow^d{}_C B'$

The compatibility of the two definitions is shown, e.g., in Section 3 of [3]. The following is well known (cf. 3.31 of [3]).

***2.2 Lemma.*** *Suppose $A, B$ and $C = A \cap B$ are closed and $A \downarrow^d{}_C B$. Then $AB$ is closed, i.e. $\overline{AB} = \overline{A} \cup \overline{B}$.*

The equivalence of $d$-independence and stability theoretic independence was first proved in this generality in [3] but the basic setup comes from [4].

**2.3 Fact.** Suppose $T$ satisfies Context 1.10. If $C$ is intrinsically closed then for any $A$ and $B$, $A \downarrow_C B$ if and only if $A \downarrow^d{}_C B$.

We give a different proof that is not as involved with the intricacies of amalgamation in the case without finite closures as the one in [3].

Suppose for contradiction that $R(A, C, B) \neq \emptyset$. Then for $\epsilon$ chosen according to Axiom 1.4, $\delta(A/B) - \delta(A/BC) > \epsilon$. Now, construct a nonforking sequence $\langle A_i, B_i \rangle$ in $tp(AB/C)$. Since $A$ is not in the algebraic closure of $BC$, no $A_j$ is in the algebraic closure of the union of $B_i$ for $i < j$. We will use this fact to show that the types $p_i = tp(A_i/CB_i)$ are $n$-contradictory for some $n$. If not, for each $n$ there is an $A^*$ which is common solution for, say $p_1, \ldots, p_n$. Fix $n$ such that $n \cdot \epsilon > \delta(A/C)$. But $\delta(A^*/B_1, \ldots B_n) \leq \delta(A/C) - n \cdot \alpha$ so this implies $A \subseteq acl(CB_1 \ldots, B_n)$ and this contradiction yields the result. The extension property for nonforking types and uniqueness suffice to deduce the converse from $d$-dependence implies forking dependence so we finish as in Lemma 3.35 of [3].

We extend our notion of dimension to a global real-valued rank on types.

**2.4 Definition.** Let $p \in S(A)$. Define $d(p)$ as $d(\overline{a}/A)$ for some (any) $\overline{a}$ realizing $p$.

**2.5 Definition.** Let $p_1, p_2 \in S(A)$.

1. $p_1$ and $p_2$ are disjoint if for any $\overline{a}_1$, $\overline{a}_2$ realizing $p_1$, $p_2$, $icl(A\overline{a}_1) \cap icl(A\overline{a}_2) \subseteq icl(A)$.

2. $p_1 \in S(A)$ and $p_2 \in S(B)$ are disjoint if any pair of nonforking extensions of $p_1$ and $p_2$ to $AB$ are disjoint.

***2.6 Lemma.*** *Let $A \subset B$, $p \in S(B)$ and $p|A = q$ and suppose $A$ is intrinsically closed.*

1. *If $d(p) < d(q)$ then $p$ forks over $A$.*

2. *$q$ is stationary.*

Proof. 1) follows immediately from Fact 2.3; 2) is also proved in [3] (Lemma 3.38).

***2.7 Lemma.*** *Let $A$ be intrinsically closed, $p_1, p_2 \in S(A)$. If $p_1$ and $p_2$ are disjoint and $d(p_1) = 0$ then $p_1$ and $p_2$ are orthogonal.*

Proof. If not, there exist sequences $a_1 \ldots a_k$ and $b_1 \ldots b_m$ of realizations of $p_1$ and $p_2$ respectively, which are independent over $A$, such that $\overline{a} \not\downarrow_A \overline{b}$. Since $d(p_1) = 0$, $d(\overline{a}/A) = 0$ and $icl(A\overline{a}) \cap icl(A\overline{b}) \not\subseteq A$. By Lemma 2.2, intrinsic closure is a trivial dependence relation. Since the $a_i$ and the $b_j$ are independent, this implies that for some $i, j$, $icl(Aa_i) \cap icl(Ab_j) \not\subseteq A$. But this contradicts the disjointness of $p_1$ and $p_2$ and we finish.

The *dimensional order property* (DOP) and *dimensional discontinuity property* DIDIP are defined in [5]. Either of these conditions implies $T$ has many models in uncountable powers. $T$ has the eventually non-isolated dimensional order property (eni-dop) if some type witnessing the dimension order property is not isolated. This condition implies that $T$ has the maximal number of countable models. Since $T_\alpha$ is not small for irrational $\alpha$, this is not new information. However, the eni-dop seems to be a much more intrinsic feature of the construction than the smallness. (For precise definition see e.g. [1].)

***2.8 Theorem.*** *Let $K_0$ be a class satisfying Context 1.10. Let $T$ be the theory of the generic model for $(K_0, \leq_s)$. Suppose further that there is a pair of independent points $B = \{x, y\}$ and a nonalgebraic type $p$ with $d(p/B) = 0$ but $d(p/x) > 0$ and $d(p/y) > 0$.*

1. *The theory $T$ has the dimensional order property.*
2. *If $p$ is not isolated the theory $T$ has the eni dimensional order property.*
3. *The theory $T$ has the dimensional discontinuity property.*

Proof. i) Let $A = \{a, b\}$ where $a$ and $b$ are independent over the empty set. It suffices to show that there is a type $p \in S(A)$ with $d(p) = 0$ and such that if $\overline{c}$ realizes $p$, $\overline{c} \not\downarrow_a b$ and $\overline{c} \not\downarrow_b a$. For then we can construct an independent sequence of points $a_i$ and disjoint copies $p_{i,j}$ over $\{a_i, a_j\}$ which will be pairwise orthogonal by Lemma 2.7. The required type is constructed in Theorem 3.6. ii) follows by the same argument if $p$ is not isolated.

For iii) it suffices to find an independent sequence of sets $B_n$ for $n < \omega$ and $p \in S(B)$ where $B = \cup B_n$ such that $p \dashv \cup_{n<j} B_n$ for each $j$. Choose $B_n$ and $C_n$ as described at the beginning of the proof of Theorem 3.6. Let $B$ be the union for $n < \omega$ of $B_n = \{x_n, y_n\}$ with no relations on $B$. For each $n$, let $f_n$ map $c_n$ to $c$, $x$ to $x_n$ and $y$ to $y_n$. Then $B \cup \{c\}$ is as required. That is, $d(t(c/B)) = 0$ but $d(t(c/\cup_{n<m} B_n)) > 0$.

## 3 Constructing types of $d$-rank 0

We construct a nonalgebraic type $p$ over a two element set with $d(p) = 0$.

**3.1 Context.** We work with a class $K_0$ of finite structures as in Example 1.3. Thus, $(K_0, \leq_s)$ witnesses Contex 1.10. Recall that $K$ is the class of models of the theory of the generic $M$, $\mathcal{M}$ is a saturated model of this theory, and $S(K)$ is the universal class it determines.

Finally, the $\alpha$ parameterizing the dimension function may be rational or irrational. This distinction affects only the question of whether the type with rank 0 is isolated and we discuss that when it arises.

**3.2 Definition.** $(K_0, \leq_s)$ has the *full amalgamation property* if $B \cap C = A$ and $A \leq_s B$ imply $B \bigotimes_A C \in K_0$ and $C \leq_s B \bigotimes_A C$.

It is easy to check (Section 4 of [3])that if $(K_0, \leq_s)$ is closed under free amalgamation then it has full amalgamation.

**3.3 Assumption.** $(K_0, \leq_s)$ has the *full amalgamation property.*

**3.4 Examples.** Each of the following classes is closed under free amalgamation.

1. The class $(K_\alpha, \leq_s)$ of all finite $L$-structures $A$ with $\delta_{1,\alpha}(A)$ hereditarily positive. The resulting theory is $\omega$-stable if $\alpha$ is rational and stable if $\alpha$ is irrational.

2. The class yielding the stable $\aleph_0$-categorical pseudoplane of [4].

The main aim of this section is to establish the following result which leads easily by Theorem 2.8 to showing the theory of the generic model $\mathcal{M}$ has DOP and DIDIP.

**3.5 Definition.** We say $C$ is a *primitive* extension of $B$ if $B \leq_s C$ but there is no $B'$ properly between $B$ and $C$ with $B' \leq_s C$.

***3.6 Theorem.*** *There exists a triple $\{x, y, c\} \in \mathcal{M}$ such that $B = \{x, y\}$ is an independent pair over $\emptyset$ and $d(c/xy) = 0$ but $d(c/x) > 0$, $d(c/y) > 0$ and $c \notin acl(x, y)$.*

Proof. Fix a discrete structure $B$ with universe $\{x, y\}$. We will construct a family $\langle (C_n, x_n, y_n, c_n) : n < \omega \rangle$ of structures in $K_0$ which satisfy the following conditions. Let $B_n = \{x_n, y_n\}$. The inequalities in the following discussion automatically become strict inequalities if $\alpha$ is irrational.

1. $0 \leq \delta(C_n/B_n) < 1/n$.
2. $(x_n, y_n, c_n)$ is a discrete substructure of $C_n$.
3. $C_n$ is a primitive extension of $B_n$.

Now map each $B_n$ to $B$ and amalgamate the images of the $C_n$ disjointly over $B$. Then identify all the $c_n$ as $c$ to form a structure $A$. Without loss of generality we can assume $A$ is strongly embedded in $\mathcal{M}$. Thus, $icl_{\mathcal{M}}(cB) = A$. Then $d(c/B) = 0$ but $d(c/x)$ and $d(c/y)$ are both at least one. Thus $c \not\downarrow_x xy$ and $c \not\downarrow_y xy$. Since $\delta(C_n/B_n) \geq 0$, for every $n$, $c \notin acl(B)$.

**3.7 Remark.** If $\alpha$ is irrational, all the $C_n$ are necessary and $tp(c/xy)$ is nonprincipal. If $\alpha$ is rational, for some $n$, $\delta(C_n/B_n) = 0$. (We expand on this remark after Observation 3.9.) The type is principal but still not algebraic since in this context there are infinitely many copies (in a generic) of a primitive extension with relative dimension 0.

The construction of the $C_n$ follows a rather tortured path. We first need to consider structures with negative dimension over $B$.

**3.8 Definition.** Let $\mathcal{A} = \mathcal{A}_\alpha$ be the class of structures of the form $(A, a, b, e)$ which satisfy the following conditions. Let $B$ be the structure with universe $\{a, b\}$ and no relations.

1. $A \in K_0$.
2. $\{a, b, e\}$ is the universe of a discrete substructure of $A$.
3. For each $A'$ with $B \subseteq A'$ and $A'$ properly contained in $A$, $\delta(A') > \delta(A)$.
4. $-1 < \delta(A/B) \leq 0$.

**3.9 Observation.** 1. The choice of $\delta$ as $\delta_\alpha$ makes $\mathcal{A}$ depend on $\alpha$.

2. If the last three conditions are satisfied, the first is as well.

3. The last condition implies that $\delta(A/a) > 0$ and $\delta(A/b) > 0$.

We first show that the set

$$X = X_\alpha = \{\beta : \beta = \delta(A/\{a,b\}) \text{ for some } (A,a,b,e) \in \mathcal{A}\}$$

is not bounded away from zero. If $\alpha$ is irrational, $0 \notin X$ so $X$ is infinite. If $\alpha = p/q$ is rational, every element of $X$ has the form $(mq - np)/q$ so there cannot be an infinite sequence of members of $X$ tending to 0. That is, there will be an $A$ with $\delta(A/B) = 0$. As indicated $X$ depends on $\alpha$ (through $\delta = \delta_\alpha$ and $\mathcal{A} = \mathcal{A}_\alpha$.) But the bulk of the proof is uniform in $\alpha$, so to enhance readability we keep track of $\alpha$ only for that part of the proof where the dependence is not uniform.

**3.10 Construction.** There are two elementary steps in the construction. It is easy to check that if the constituent models described here are in $K_0$, then so is the result.

1. If $\delta(A/B) = \beta$ and $\beta \in X$, and $A^*$ is the free amalgam over $B$ of $k$ copies of $A$, then $\delta(A^*/B) = k\beta$.

2. Let $(A_1, a_1, b_1, c_1)$ and $(A_2, a_2, b_2, c_2)$ be in $\mathcal{A}$. Let $A^*$ be formed by identifying $b_1$ and $a_2$ and freely amalgamating over that point.

***3.11 Lemma.*** *If $\beta > -1/k$ and $\beta \in X$ then $k\beta \in X$.*

Proof. Use Construction 3.10 i).

It is straightforward to determine the following properties of the second construction.

***3.12 Lemma.*** *Suppose $\delta(A_1/\{a_1,b_1\}) = \beta_1$, $\delta(A_2/\{a_2,b_2\}) = \beta_2$ and $\beta_1, \beta_2 \in X$. Let $A^*$ be formed as in Construction 3.10 ii).*

*1. $\delta(A^*/\{a_1,b_2\}) = \beta_1 + \beta_2 + 1$.*

*2. If $-2 < \beta_1 + \beta_2 \leq -1$ then $\beta_1 + \beta_2 + 1 \in X$ and $\langle A^*, a_1, b_2, c_1\rangle \in \mathcal{A}$.*

3. *If* $-1 \leq \beta_1 + \beta_2 < -1 + 1/n$ *then*

   (a) $0 \leq \delta(A^*/\{a_1, b_2\}) < 1/n$.

   (b) $\delta(A^*/a_1) \geq 1$ *and* $\delta(A^*/b_2) \geq 1$.

*Proof.* The key observations for 1)and thus 2) and 3a) is that for any $B \subseteq A_1 \subseteq A^*$,

$$\delta(A'/\{a_1, b_2\}) = \delta(A' \cap A_1/\{a_1, b_1\}) + \delta(A' \cap A_2/\{a_2, b_2\}) + 1.$$

For 3b) we need the further remark:

$$\delta(A'/a_1) = \delta(A'/b_2) = \delta(A'/\{a_1, b_2\}) + 1.$$

***3.13 Lemma.*** *If $L$ contains a single binary relation and $K_0 = K_\alpha$, then $X$ is not empty.*

*Proof.* It suffices to show that each $\mathcal{A}_\alpha$ is nonempty for $0 < \alpha \leq 1$. The construction is somewhat ad hoc and proceeds by a number of cases depending on $\alpha$. Thus to establish Lemma 3.13 we will use the notations $\mathcal{A}_\alpha, \delta_\alpha$. These constructions are very specific to graphs. The second author has an alternative argument which avoids the dependence on $\alpha$. However, it passes through hypergraphs and has it own computational complexities.

**3.14 Case 1.** $3/4 < \alpha < 1$: Let $A_1$ be the structure obtained by adding to $\{a, b, e\}$ two points $b_1, b_2$ such that $b_1$ is connected to $a$ and $e$ while $b_2$ is connected to $b$ and $e$. Then

$$-1 < \delta_\alpha(A_1/B) = 3 - 4\alpha < 0$$

for the indicated $\alpha$ and $(A_1, a, b, e) \in \mathcal{A}_\alpha$.

**3.15 Case 2.** $2/3 \leq \alpha < 4/5$: Let $A_2$ be the structure obtained by adding to $\{a, b, e\}$ two points $b_1, b_2$ such that $b_1$ is connected to $a$, $b$, and $e$ while $b_2$ is connected to $b$ and $e$. Then

$$-1 < \delta_\alpha(A_2/B) = 3 - 5\alpha < 0$$

for the indicated $\alpha$ and $(A_2, a, b, e) \in \mathcal{A}_\alpha$.

**3.16 Case 3.** $0 < \alpha < 2/3$: Let $A_{n,k}$ be the structure obtained by adding to $\{a, b, e\}$ both $n$ points $a_1, \ldots, a_n$ such that each $a_i$ is connected to $a$, $b$, and $e$ and $k$ points $b_1, \ldots, b_k$ such that each $b_i$ is connected to all the $a_i$.

Then $\delta_\alpha(A_{n,k}/B) = n + k + 1 - (nk + 3n)\alpha$. We say $\alpha$ is *acceptable* for $n$ and $k$ if the following inequality is satisfied.

$$\ell_{n,k} = \frac{n+k+1}{nk+3n} < \alpha < \frac{n+k+2}{nk+3n} = u_{n,k}.$$

To show that if $\alpha$ is acceptable for $n$ and $k$, then $(A_{n,k}, a, b, e) \in \mathcal{A}_\alpha$ we need several claims.

**3.17 Claim 1.** For each $k$,

1. $u_{n+1,k} > \ell_{n,k}$,
2. $\ell_{n+1,k} < \ell_{n,k}$,
3. $\lim_{n\to\infty} \ell_{n,k} = 1/(k+3)$.

Claim 1 is established by routine computations.

**3.18 Claim 2.** For every $\alpha$ that is acceptable for $n$ and $k$, if $B \subseteq A' \subseteq A_{n,k}$, $\delta_\alpha(A'/B) \geq \delta_\alpha(A_{n,k}/B)$.

To see this, note that any such $A'$, for some $m \leq n$ and $\ell \leq k$, either $A'$ has the form $A_{m,\ell}$ or the form $B_{m,\ell}$, where $B_{m,\ell}$ is the structure obtained by omitting the element $e$ from $A_{m,\ell}$. Now note that if $\delta_\alpha(B_{m,\ell}/B) < 0$ then $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m+1,\ell}/B)$ and $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m,\ell+1}/B)$. The same assertion holds when $A_{m,\ell}$ is substituted for $B_{m,\ell}$. Finally, $\delta_\alpha(B_{n,k}/B) \geq \delta_\alpha(A_{n,k}/B)$. These three observations yield the second claim.

From these two claims we see that for each $\alpha$, there is a pair $n, k$ with $A_{n,k} \in \mathcal{A}_\alpha$. The remainder of the argument does not depend on $\alpha$ so we return to the use of the notation $X$ and $\mathcal{A}$.

***3.19 Lemma.*** *For every $n$ there is an element $\beta$ of $X$ with $\beta > -1/n$.*

Proof. If not, fix the least $n$ such that all elements of $X$ are at most $-1/(n+1)$ and fix $\beta_0 \in X$ with $-1/n < \beta_0 \leq -1/(n+1)$. (If $\beta_0 = -1/(n+1)$, $\beta_1 = 0$ and we finish.) Define by induction $\beta_{\ell+1} = (n+1)\beta_\ell + 1$. Combining the two elementary steps we see that each $\beta_\ell \in X$. Let $\beta'_\ell$ be the distance between $-1/n$ and $\beta_\ell$. That is, $\beta'_\ell = |-1/n - \beta_\ell| = 1/n + \beta_\ell$. Now $\beta_\ell \leq -1/(n+1)$ if and only if $\beta'_\ell \leq 1/(n)(n+1)$.

But

$$\beta'_{\ell+1} = 1/n + (n+1)\beta_\ell + 1 = (n+1)\beta'_\ell.$$

So

$$\beta'_\ell = (n+1)^\ell \beta'_0.$$

As $\beta'_0 > 0$, for sufficiently large $\ell$, $\beta'_\ell > 1/(n)(n+1)$ so $\beta_\ell > -1/(n+1)$ as required.

With a few more applications of our fundamental constructions, we can find the $C_n$ needed for Theorem 3.6.

By applying Construction 3.10 i) and Lemma 3.19 for any $n$, and $i = 1, 2$ we can find $(A^n_1, x^n_1, y^n_1, c^n_1)$ and $(A^n_2, x^n_2, y^n_n, c^n_2)$ containing $B^n_i = \{x^n_i, y^n_i\}$ such that $\{x^n_i, y^n_i, c^n_i\}$ is discrete and $\delta(A^n_i/B^n_i) = \beta^n_i$ with $-1 < \beta^n_1 + \beta^n_2 < -1 + 1/n$.

To construct $A^n_1$, choose using Lemma 3.19 a $(D^n, x^n_1, y^n_1, c^n_1) \in \mathcal{A}$ with $-1/n < \delta(D^n/B^n_1) \leq 0$. Take an appropriate number, $k$, of copies of $D^n$ over $B^n_1$ and apply Construction 3.10 i) to form $A^n_1$ with

$$-1 < k\delta(D^n/B^n_1) = \delta(A^n_1/B^n_1) = \beta^n_1 < -1 + 1/n$$

and choose $c^n_1 \in A^n_1$ so that $(x^n_1, y^n_1, c^n_1)$ is discrete. By Lemma 3.19 again choose $(A^n_2, x^n_2, y^n_2, c^n_2) \in \mathcal{A}$ with

$$-(\beta^n_1 + 1)/2 < \delta(A^n_2/B^n_2) = \beta^n_2 < 0.$$

Now apply Construction 3.10 ii) to $(A^n_1, x^n_1, y^n_1, c^n_1)$ and $(A^n_2, x^n_2, y^n_n, c^n_2)$ to form $(C_n, x_n, y_n, c_n)$ where $x_n = x^n_1$, $y_n = y^n_2$, and $c_n = c^n_1$. Denote $\{x_n, y_n\}$ by $B_n$. Then $0 < \delta(C_n/B_n) = 1 + \beta^n_1 + \beta^n_2 < 1/n$. Each $C_n$ contains a discrete set $\{x_n, y_n, c_n\}$ and the third property of the $C_n$ follows using the second part of Lemma 3.12. This completes the construction of the type of $d$-rank 0.

Using the argument for constructing $A^n_1$, we easily show the following density result.

***3.20 Corollary.*** *For any $\gamma, \delta$ with $-1 \leq \gamma < \delta < 0$ there is a $(D, a, b, e) \in \mathcal{A}$ with $\gamma < \delta(D/\{a, b\}) < \delta$.*

The restriction to one-types in the following lemma is solely for ease of presentation.

***3.21 Lemma.*** *Suppose $A \subseteq M \models T_\alpha$ is intrinsically closed and $p_1, p_2 \in S_1(A)$ are disjoint. If $0 < d(p_i)$ for $i = 1, 2$ then $p_1 \not\perp p_2$.*

Proof. Clearly if $p_1$ and $p_2$ are not disjoint or if there is an edge between realizations of the two types, they are not orthogonal. Let $a_1, a_2$ realize $p_1, p_2$ and suppose for contradiction that $p_1$ and $p_2$ are orthogonal and $d(a_1a_2/A) = d(a_1/A) + d(a_2/A) = \beta > 0$. In particular, there is no edge linking $a_1$ and $a_2$. By Lemma 3.25 of [3] there are finite $A_1 \supseteq a_1a_2$ and $A_0 \subset A$ with $\beta \leq \gamma = \delta(A_1/A_0) < \beta + 1$. Lemma 3.20 allows us to choose a finite $B \supseteq \{a_1, a_2\}$ with

$$-1 < \delta(B/\{a_1, a_2\}) < \beta - \gamma < 0.$$

Then $Ba_1a_2$ is in $K_0$. By full amalgamation we can freely amalgamate $B$ with $AA_1$ over $\{a_1, a_2\}$ inside $\mathcal{M}$. Then $d(a_1a_2/A) \leq \delta(A_1B/A_0)$. Note $\delta(B/A_1A_0) = \delta(B/\{a_1, a_2\}) < \beta - \gamma$. So

$$\delta(A_1B/A_0) = \delta(B/A_1A_0) + \delta(A_1/A_0) < \beta.$$

This contradicts $d(a_1a_2/A) = \beta$ so we conclude $p_1 \not\perp p_2$.

Using the Lemmas 2.7 and 3.20 it is easy to see

***3.22 Corollary.*** *In $T_\alpha$,*

1. *For disjoint $p_1, p_2$, $p_1 \perp p_2$ if and only if $d(p_1) = 0$ or $d(p_2) = 0$.*

2. *Every regular type satisfies $d(p) = 0$.*

Our construction yields some further information.

**3.23 Definition.** The type $p \in S(A)$ is *minimal* if $p$ is not algebraic but for any formula $\phi(x, \overline{b})$ either $p \cup \{\phi(x, \overline{b})\}$ or $p \cup \{\neg\phi(x, \overline{b})\}$ is algebraic.

**3.24 Definition.** The type $p \in S(A)$ is $i$-minimal if for every $\overline{a}$ realizing $p$, if $c \in icl(A\overline{a})$, $icl(Ac) = icl(A\overline{a})$.

***3.25 Theorem.*** *If $p$ is constructed as in Lemma 3.6 then $p$ is minimal and trivial.*

*Proof.* If $d(p) = 0$ and $p$ is i-minimal then $p$ is minimal. We constructed $p$ so that $d(p) = 0$ but the fact that each $C_n$ is primitive over $B$ and $A$ is intrinsically closed guarantees that $p$ is $i$-minimal and we finish.

Clearly, $d(p) = 0$ does not imply $p$ is minimal. For, if $d(a/A) = d(b/A) = 0$ then $d(ab/A) = 0$ but if, for example, $a$ and $b$ are independent $tp(ab/A)$ is not minimal.

# 4 The Finite Cover Property

In this section we show that for classes as described in Example 1.3 with the full amalgamation property, and in particular for $(K_\alpha, \leq_s)$, the theory of the generic does not have the finite cover property. We rely on the following characterization due to Shelah [5, II.2.4].

**4.1 Fact.** If $T$ is a stable theory with the finite cover property then there is a formula $\phi(\overline{x}, \overline{y}, \overline{z})$ such that

1. For every $\overline{c}$, $\phi(\overline{c}, \overline{y}, \overline{z})$ defines an equivalence relation. We call this relation $\overline{c}$-equivalence.
2. For arbitrarily large $n$, there exists $\overline{c}_n$ such that the equivalence relation defined by $\phi(\overline{c}_n, \overline{y}, \overline{z})$ has exactly $n$ equivalence classes.

Here is some necessary notation.

**4.2 Definition.** Let $A, B$ be finite substructures of $M$ with $A \subseteq B$ then

1. $\chi_M(B/A)$ is the number of distinct copies of $B$ over $A$ in $M$.
2. $\chi^*_M(B/A)$ is the supremum of the cardinalities of maximal families of disjoint (over $A$) copies of $B$ over $A$ in $M$.

**4.3 Definition.** $(A, B)$ is a *minimal pair* if $\delta(B/A) < 0$ and for every $B'$, with $A \subseteq B' \subseteq B$, $\delta(B/A) < \delta(B'/A)$.

The next result is proved in [3].

**4.4 Fact.** There is a function $t$ taking pairs of integers to integers such that if $A \leq_i B$ then for any $N \in K$ and any embedding $f$ of $A$ into $N$, $\chi_N(fB/fA) \leq t(|A|, |B|)$.

There is an easy partial converse to this result.

***4.5 Lemma.*** *For any $M \in K_0$, if $\chi^*_M(B/A) > t(|A|, |B|)$ then $A \leq_s B$.*

*Proof.* Suppose some $B'$ with $A \subseteq B$ satisfies $A \leq_i B'$. Then there are more than $t(|A|, |B|)$ disjoint copies of $B'$ over $A$ in $M$ contradicting Fact 4.4.

We also need the finer analysis of the intrinsic closure carried out in [2]. In fact, this argument depends on the slightly finer notion of a *semigeneric* which is defined in [2]. The crucial facts from [3] and [2] are the following.

**4.6 Fact.** If $(K_0, \leq_s)$ satisfies Context 1.10 and has the full amalgamation property then the theory of the generic $T$ satisfies

1. All models of $T$ are semigeneric.

2. $T$ is stable. For any formula $\phi(x_1 \ldots x_r)$ there is an integer $\ell = \ell_\phi$, such that for any semigeneric $M \in K$ and any $r$-tuples $\overline{a}$ and $\overline{a}'$ from $M$ if $icl_M^{\ell_\phi}(\overline{a}) \approx icl_M^{\ell_\phi}(\overline{a}')$ then $M \models \phi(\overline{a})$ if and only if $M \models \phi(\overline{a}')$.

***4.7 Theorem.*** *Let the language L contain only binary relation symbols. If $(K_0, \leq_s)$ satisfies Context 1.10 and has the full amalgamation property then the theory of the generic $T$ does not have the finite cover property.*

Proof. Suppose not. We know $T$ is stable so there is a formula $\phi$ satisfying the conditions of Fact 4.1. Each model of $T$ is semigeneric. Choose $\ell = \ell_\phi$ as in Fact 4.6 so that the isomorphism type of $icl_M^\ell(\overline{c}, \overline{a}, \overline{b})$ determines the truth of $\phi(\overline{c}, \overline{a}, \overline{b})$ for any triple of $\overline{c}, \overline{a}, \overline{b}$ of appropriate length. For any $n$ choose $m$ sufficiently large with respect to the maximal cardinality of $icl_M^\ell(\overline{c}, \overline{a}, \overline{b})$ and $n$ so that applying the pigeonhole principle and Ramsey's theorem we can choose $\overline{c}_m$ so that the $\langle \overline{a}_i : i < n \rangle$ are pairwise $\overline{c}_m$-inequivalent and for $i < n$ letting $A_i = icl_M^\ell(\overline{c}, \overline{a}_i)$ and $C = icl_M^\ell(\overline{c})$ the following property $P(C)$ holds.

1. for all $i, j$, $A_i \approx_C A_j$

2. for $i < j$, $A_0 A_1 \approx_C A_i A_j$.

If $n > t(k, |A_0|)!$ for $k < |A_0|$, applying the $\Delta$-system Lemma we can find $\hat{C}$ with $C \subseteq \hat{C} \subseteq A_0$ such that (without loss of generality) the $A_i$ are disjoint over $\hat{C}$. By appropriate choice of $n$, depending only on $|A_0|$, $|C|$, we may assume that $p(\hat{C})$ holds. By Fact 4.5, $\hat{C} \leq_s A_0$. We claim in fact that the structure imposed on $A_0 A_1$ is $A_0 \otimes_{\hat{C}} A_1$. If not, $R(A_0, \hat{C}, A_1)$ is nonempty. Let $E_i$ denote the substructure of M with universe $\bigcup_{j<i} A_j$. By Axiom 1.4 iii) for sufficiently large $k$, $\delta(A_k/E_k) < 0$. There is a minimal pair $(E_k', A_k')$ with $E_k' \subseteq E_k$ and $A_k' \subseteq A_k$. But then for each $j > k$ there is a copy $A_j'$ of $A_k'$, contained in $A_j$ and isomorphic to $A_k'$ over $E_k$ (since the language is binary). This contradicts the bound on the number of copies of a minimal pair, Fact 4.4. Thus we establish the claim. But now we have $E_{i+1} \approx E_i \otimes_C A_0$. Since $(K_0, \leq_s)$ has full amalgamation, this construction can be carried on indefinitely. But the definition of $\ell_\phi$ guarantees that the

$\overline{a}_i$ represent distinct $\overline{c}$-equivalence classes and this contradicts the hypothesis that there are only finitely many $\overline{c}$-equivalence classes.

**4.8 Conclusion.** The arguments in the paper are fully worked out only for languages with binary relation symbols. For Section 4, this is just a matter of easing notation; slight modifications of the argument work for any finite relational language. The combinatorial arguments in Section 3 are sufficiently complicated that the proof is the general case is less clear. But it would be quite surprising if the restriction to a binary language is actually necessary.

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