

TOWARDS A FINER CLASSIFICATION OF STRONGLY MINIMAL SETS

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ABSTRACT. Let M be strongly minimal and constructed by a ‘Hrushovski construction’. If the Hrushovski algebraization function μ is in a certain class \mathcal{T} (μ triples) we show that for independent I with $|I| > 1$, $\text{dcl}^*(I) = \emptyset$ (* means not in dcl of a proper subset). This implies the only definable truly n -ary functions f (f ‘depends’ on each argument), occur when $n = 1$. We prove, indicating the dependence on μ , for Hrushovski’s original construction and including analogous results for the strongly minimal k -Steiner systems of Baldwin and Paolini that the symmetric definable closure, $\text{sdcl}^*(I) = \emptyset$, and thus the theory does not admit elimination of imaginaries. In particular, such strongly minimal Steiner systems with line-length at least 4 do not interpret a quasigroup, even though they admit a coordinatization if $k = p^n$. The proofs depend on our introduction for appropriate $G \subseteq \text{aut}(M)$ the notion of a G -normal substructure \mathfrak{A} of M and of a G -decomposition of \mathfrak{A} . These results lead to a finer classification of strongly minimal structures with flat geometry, according to what sorts of definable functions they admit.

Strongly minimal sets are the building blocks of well-behaved first order theories such as algebraically and differentially closed fields. A definable subset D of a model of a first order theory T is strongly minimal if in each $M \models T$ every definable subset of $D(M)$ is finite or cofinite. When D is $x = x$, we say the model (theory) is strongly minimal. In any such set algebraic closure¹ gives a dependence relation satisfying Van der Waerden’s axioms, which implies the existence of a basis and an isomorphism between two models of a strongly minimal T whose bases have the same cardinality. Zilber conjectured that all geometries of strongly minimal sets were a) disintegrated (discrete/trivial) (e.g., (\mathbf{Z}, S)); b) locally modular (vector space-like) e.g., $(\mathbf{Q}, +)$; or c) field-like (e.g., $(\mathbb{C}, +, \times)$). That is, strongly minimal sets are associated with canonical mathematical structures. Hrushovski [Hru93] refuted this conjecture by a subtle extension of the Fraïssé construction. His *ab initio* (built from finite structures) examples, with *flat* geometries, have largely been treated as an undifferentiated collection of exotic structures because they admit no associative function (with infinite domain). Is there some ‘classical’ structure to which they all relate? [BP20] shows that strongly minimal Steiner systems are an exemplar and [Bal21a] that strongly minimal quasigroups arise. In this paper

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¹Model theorists generalize the usual field theoretic notion by saying $a \in \text{acl}(B)$ if for some formula $\phi(x, \mathbf{b})$ with $\mathbf{b} \in B$, $\phi(a, \mathbf{b})$ is true and $\phi(x, \mathbf{b})$ has only n solutions for some $n < \omega$. If $n = 1$, $a \in \text{dcl}(B)$.

we dash any such hope of a canonical strongly minimal theory with a strictly flat geometry by showing that most of Hrushovski’s original family of examples and the Steiner system (with one ternary relation) admit no definable binary functions (See Theorem 4.1.10.). While, there was considerable attention to adapting the construction to expansions of algebraically closed fields [BMPZ07, BH00, Poi01], work for the *ab initio* case has concentrated on the associated geometries. For example, [EF11, EF12] show the geometries are locally isomorphic to those of an associated ω -stable theory.

[BP20] shows that for each q this construction applied to *finite linear spaces* [BB93] yields strongly minimal q -Steiner systems. (We think of a q -Steiner systems as a collection of points and a ternary collinearity relation R such that two points determine a line and all lines have the same length q .) And [Bal21a], building on [Ste56, GW75], shows that if q is a prime power the Steiner system can be ‘coordinatized’ by a quasigroup. A variant of the original construction shows in [Bal94] the existence of a non-desarguesian almost strongly minimal projective planes.

Section 2 analyzes *elimination of imaginaries* in strongly minimal sets with no dependence on the Hrushovski construction. This notion plays several roles in model theory. A structure M is defined in another N if there a subset $D^M \subseteq N^k$ for some k such that the formulas of N define a structure isomorphic to M with domain D^M . For example, $k \times k$ matrix groups over a field F are defined on F^{k^2} . Interpretation is a more liberal notion that allows mapping M to a definable quotient of a substructure of N . Shelah introduced the notion of imaginary elements to obtain certain important technical definability results (‘canonical bases’ for definable sets) and guarantee closure under definable quotients. This notion plays a central role in geometric stability theory by extending a theory T to T^{eq} in a (in general) larger vocabulary. The models of T^{eq} form a pre-topos completion of the models of T [Har11]. Poizat introduced the property that ‘ T has elimination of imaginaries’ if this extension is not necessary. In particular if T admits elimination of imaginaries, any theory interpretable in T is actually definable in T . As in Poizat’s use of symmetric polynomials to prove algebraically closed fields admit elimination of imaginaries, in general, elimination of imaginaries is closely connected to the existence of symmetric functions in n -variables for each n . We make this precise by introducing the *symmetric definable closure*.

The original motives for this paper were to show i) (first author) (Pure) strongly minimal Steiner systems with line length greater than 3 admit neither \emptyset -definable quasigroups nor more generally ‘non-trivial’ \emptyset -definable binary functions (Theorem 5.13, Theorem 5.18) and ii) (second author) the original Hrushovski construction does not admit elimination of imaginaries (Theorem 4.4.1). The question of elimination of imaginaries for *ab initio* strongly minimal sets has largely lain fallow for 30 years although B. Baizhanov [Bai96] asked whether any strongly minimal theory in a finite vocabulary that admits elimination of imaginaries must be an algebraically closed field. [Hru93, p 160] observed that flat geometries obeyed the weaker ‘geometric’ elimination of imaginaries and this was adequate for studying the geometry. But as Zilber, we are interested in the actual theories.

In order to state the main results, we introduce new notions: $a \in (s) \text{dcl}^*(X)$ (Section 2) which clarify the notion of elimination of imaginaries. The $*$ in $a \in (s) \text{dcl}^*(X)$ means every element of X is used to witness a is in the (symmetric)

definable closure of X . A theory T_μ *triples* if for any good pair C/B (Definition 1.1.5.(3)), $\delta(B) \geq 2$ implies $\mu(C/B) \geq 3$. This implies that every primitive extension of a ‘well-placed’ (Definition 3.5) base has at least 3 copies in the generic.

- Notation 0.1.**
- (1) \hat{T}_μ denotes a strongly minimal theory constructed with the same δ , same vocabulary of one ternary relation R (required to be a hypergraph), the same \mathbf{L}_0 , and an appropriate μ as in the main construction in [Hru93].
 - (2) T_μ^S denotes a strongly minimal theory of Steiner systems constructed with the δ and vocabulary $\{R\}$ as in [BP20]) (Definition 1.2.2).
 - (3) T_μ is used for a strongly minimal theory of either sort; in both cases the geometry is *flat* but not *disintegrated*.

\emptyset -definable abbreviates parameter-free definable.

Theorem 0.2 (Main Results). *Let T_μ be a strongly minimal theory as in Definition 0.1. Let $I = \{a_1, \dots, a_v\}$ be a tuple of independent points with $v \geq 2$.*

- (1) *If T_μ triples then $\text{dcl}^*(I) = \emptyset$ and every definable function is essentially unary (Definition 2.9).*
- (2) *In any case $\text{sdcl}^*(I) = \emptyset$ and there are no \emptyset -definable symmetric (value does not depend on order of the arguments) truly v -ary function.*

Consequently, in both cases T_μ does not admit elimination of imaginaries. Nevertheless the algebraic closure geometry is not disintegrated.

Section 2 introduces the notion of symmetric definable closure and its connection with elimination of imaginaries for any strongly minimal theory T . As shown in Lemma 2.12, ‘no definable truly binary function’ is equivalent to ‘no definable truly n -ary function for any n ’ and to ‘essentially unary’. For binary functions, symmetric becomes commutative in Theorem 0.2.2. Moreover, Corollary 4.1.10 shows that \emptyset -definable truly binary’ can be replaced by ‘definable truly binary’ (parameters don’t help). If T has weak elimination of imaginaries, any of these equivalents implies ‘no finite coding’ and so imaginaries cannot be eliminated.

The crucial tool for this result is a close study of two kinds of action of the automorphism group of a structure M on certain finite definable subsets. For a subgroup G of the automorphism group of a generic structure that fixes (pointwise or setwise) a finite independent subset I of M we define the notion of a finite G -normal set. While the field-theoretic notion of normality allows the use of the definable closure $\text{dcl}(X)$ of a set of parameters X to study the solutions of algebraic equations, we show that such a decomposition by definable closure is impossible for flat geometries. Then we provide a G -invariant tree decomposition of such a set. The pointwise stabilizer is sufficient for absence of truly binary functions. But the extension to show $\text{sdcl}^*(I) = \emptyset$ and thus failure of elimination of imaginaries relies on $G_{\{I\}}$ and is more complex.

The problem of classifying strongly minimal sets by the first order theory of their associated acl-geometry was begun in the 1970’s, by several people with Zilber the most successful. [Bal84] set out a vocabulary (of n -independence predicates) for this purpose. Since Zilber’s conditions are expressible in this language his tricotomy is a coarse division. \aleph_0 -categorical disintegrated theories clearly have many different associated geometries since the size of finite algebraic closures is visible to the geometry. Since finite dimensionality is expressible it is natural to restrict the classification to the geometry, $\text{acl}(M)$, where M is the countable saturated model.

Our results indicate a profound distinction at the level of definable functions rather than merely geometry between the ‘field-like’ strongly minimal sets and the known counterexamples to Zilber’s conjecture as well as within the counterexamples. Fine classification has the dual meaning common in stability theory. We propose refining the classification strongly minimal theories (with flat geometry) by the existence of definable truly n -ary functions. And this proceeds by giving a ‘ G -tree decomposition’ of the algebraic closure of a finite set. This decomposition is reminiscent of the decomposition of model for the main gap; but the dependence on the choice of the group G refines the conception.

We refine Zilber’s trichotomy by exploring strongly minimal sets with flat geometries. In cases 1) and 2) below we have distinct theories of the acl-geometry. Within 2) we find for dcl the same unary nature that distinguishes acl in the Zilber classification. The distinction concerns properties of dcl in M , not an associated acl-geometry. We find the following:

Remark 0.3 (Classes of Theories with flat acl-geometries).

- (1) **disintegrated geometry** For any A , $\text{acl}(A) = \bigcup_{a \in I} \text{acl}(a)$;
- (2) **strictly flat geometry** (Definition 1.1.3. 2) *acl is not disintegrated but:*
 - (a) M is dcl-disintegrated: $\text{dcl}(I) = \bigcup_{a \in I} \text{dcl}(a)$ for independent I (no \emptyset -definable truly n -ary functions);
 - (b) M is not dcl-disintegrated: For some n there are truly n -ary functions:
 - (i) M is sdcl-disintegrated: $\text{sdcl}(I) = \bigcup_{a \in I} \text{sdcl}(a)$ for independent I (no commutative \emptyset -definable truly n -ary functions);
 - (ii) \emptyset -definable binary functions with domain M^2 exist; e.g. quasi-groups [Bal21a]² and non-commutative counterexamples found here.
- (3) **Further examples:**
 - (a) ternary rings [Bal95] that coordinatize a non-desarguesian plane³. There are parameter definable binary functions; but the ternary ring is not a composition of the ‘addition’ and ‘multiplication’ functions.
 - (b) 2-transitive strongly minimal sets [Hru93, Proposition 18], [Bal21b].

No theory in class 1) or 2) (and finite vocabulary) admits elimination of imaginaries. However, Verbovskiy [Ver06] has an example with elimination of imaginaries in an infinite vocabulary. Unlike those with locally modular geometries, strongly minimal theories with strictly flat geometries, like field type, have continuum many automorphism of each countable model (Lemma 3.12).

By [EF12] and⁴ [Mer20] the geometries of the countable saturated models of theories in 2a) and 2b) are elementarily equivalent (indeed have isomorphic localizations). For class 3) the geometries have not been investigated.

This diversity is obtained by realizing that the ‘Hrushovski construction’ actually has 5 parameters: $(\sigma, \mathbf{L}_0^*, \epsilon, L_0, \mathbf{U})$. \mathbf{L}_0^* is a collection of finite structures in a vocabulary σ , not necessarily closed under substructure. ϵ is a predimension as

²The vocabulary in the construction has two ternary relations.

³Because of the *ad hoc* nature of this construction, the methods of the current paper do not apply. The geometry has not been analyzed.

⁴In a private communication Mermelstein showed the infinite rank case of Steiner has the same geometry as the original example.

in Hrushovski with the requirement that it be flat. L_0 is a subset of L_0^* defined using ϵ . From such an ϵ , one defines notions of \leq , primitive extension, and good (minimally simply algebraic) pair. The function μ counts the number of allowed realizations of a good pair. Hrushovski gave the technical admissibility condition on μ that ensured the theory is strongly minimal rather than ω -stable of rank ω . Fixing a class \mathbf{U} of functions μ satisfying variants of this condition provides examples satisfying a wide range of combinatorial and algebraic conditions.

We show the elimination of imaginaries fails when L_0^* is the collection of finite linear spaces. In particular, B. Baizhanov asked whether any strongly minimal theory in a finite vocabulary, that admits elimination of imaginaries defines a field. We answer this question positively for the most evident counterexamples. But, the question of whether this can be extended when the class L_0^* is expanded to arbitrary $\forall\exists$ classes of finite structures seems wide open.

There are a number of applications of these various construction methods in universal algebra and combinatorics. Early on, [Bal95] used the general method to construct an \aleph_1 -categorical projective plane at the very bottom of the Lenz-Barlotti hierarchy. [Bal21a] shows two results. 1) if the line length k is a prime power then a strongly minimal Steiner system admits a ‘coordinatization’ by a quasigroup. As shown here as a by-product of our main construction, this coordinatization is not definable in the vocabulary $\{R\}$. 2) Nevertheless, if k is a prime power, there are strongly minimal quasigroups⁵ $(Q, *)$ (created in the vocabulary $\{H, R\}$) with L_0^* $\forall\exists$ axiomatizable) which induce Steiner systems. 2) In [Bal21b] we generalize the notion of cycle decomposition [CW12] from Steiner 3-systems to that of path graph for Steiner k -systems and introduce a uniform method of proof for the following results. For each of the following conditions, there are 2^{\aleph_0} families of elementarily equivalent Steiner systems, each with \aleph_0 countable models and one model of each uncountable cardinal satisfying the following condition: i) (extending [CGGW10]) each Steiner triple system is ∞ -sparse and has a uniform path graph; ii) (extending [CW12] each Steiner k -system (for $k = p^n$) is 2-transitive and has a uniform path graph (infinite cycles only) iii) extending [Fuj06] each is anti-Pasch (anti-mitre); iv) Items ii) and iii) have definable quasi-group structure. Moreover, by varying L_0^* classes we can demand all models are 2-transitive. Unlike most combinatorial constructions, each example presents a *class of models* with the desired property.

Section 1 outline the general framework of *ab initio* Hrushovski strongly minimal sets and then describes the variant for Steiner systems. In Section 2, we introduce subgroups G_I and $G_{\{I\}}$ of $\text{aut}(M)$ (fixing I setwise and pointwise respectively) and explain the connections of definable closure and symmetric definable closure with the elimination of imaginaries. The key tool of G -tree-decomposition appears in Section 3 along with the basic properties. There are now two steps in the proof. Section 4.1 proves the non- \emptyset -definability of truly n -ary functions for triplable theories, using dcl^* and G_I . Section 4.2 shows the necessity of the tripling hypothesis. Section 4.3 describes the distinction between flowers and bouquets which is essential for Section 4.4, the proof that $\text{sdcl}^*(I) = \emptyset$ for finite independent sets in a standard Hrushovski theory using $G_{\{I\}}$. In Section 5 we adapt that proof to Steiner systems and prove that examples with line length at least 4 from [BP20] do not define quasigroups. We raise some further questions in Section 6.

⁵As in [BC19], H is the graph of the quasigroup operation and R is collinearity.

1. TWO CONTEXTS

In Subsection 1.1 we give an axiomatic description of the properties of functions δ and μ for Hrushovski constructions of ω -stable and strongly minimal sets. In Subsection 1.2 we describe the specific definitions of δ for the two main cases considered here: the original Hrushovski context and Steiner systems.

In Section 1.1 we denote our classes as (\mathbf{L}, ϵ) with various decorations; these conditions provide a general framework for the study of the family of flat strongly minimal sets announced in the previous paragraph. From Section 1.2 on, we write (\mathbf{K}, δ) to emphasize the restriction to a single ternary relation, particular choices of δ and to prepare for the variations in [Bal21b].

1.1. General Context

The notions in this section are well-known, under various names. We both fix the notation used here and give some of the definitions in a greater generality needed here but not in e.g., [Hru93]). Fix a countable relational vocabulary τ . We write \mathbf{L}^* for the collection of all τ -structures and \mathbf{L}_0^* for the finite τ -structures.

All constructions studied here satisfy the properties of 3.4–3.7 of [BS96] and flatness, which follow from flatness of the underlying ‘preimension’ function ϵ [BP20, 3.8, 3.10].

Axiom 1.1.1. *Let \mathbf{L}_0 be a countable subset of \mathbf{L}_0^* that is closed under isomorphism⁶. Let $\hat{\mathbf{L}}_0$ be the collection containing any union of members of \mathbf{L}_0 . Further ϵ is a map from \mathbf{L}_0^* into \mathbf{Z} . We require that $\mathbf{L}_0, \hat{\mathbf{L}}_0, \epsilon$ satisfy the following requirements. Let $N \in \hat{\mathbf{L}}_0$ and $A, B, C \in \mathbf{L}_0$ be substructures of N .*

- (1) $\epsilon(\emptyset) = 0$
- (2) If $B \in \mathbf{L}_0$ and $A \subseteq B$ then $\epsilon(A) \geq 0$.
- (3) If A, B , and C are disjoint then $\epsilon(C/A) \geq \epsilon(C/AB)$.
- (4) If A, B, C are disjoint subsets of $N \in \hat{\mathbf{L}}_0$ and $\epsilon(A/B) - \epsilon(A/BC) = 0$ then $r(A, ABC, C) = \emptyset$.⁷
- (5) ϵ is flat (Definition 1.1.3)[Hru93, BP20]).
- (6) **Canonical Amalgamation** If $A \cap B = C$, $C \leq A$ and $A, B, C \in \mathbf{L}_0$ there is a direct sum $G = A \oplus_C B$ such that $G \in \mathbf{L}_0$. Moreover, $\epsilon(A \oplus_C B) = \epsilon(A) + \epsilon(B) - \epsilon(C)$ and any D with $C \subseteq D \subseteq A \oplus_C B$ is also free. Thus, $B \leq G$.

Disjoint union is the canonical amalgamation for the basic Hrushovski construction and Definition 3.14 of [BP20] gives the appropriate notion satisfying Axiom 1.1.1.5 for linear spaces. Axiom 1.1.1.2 can be rephrased as: $B \subseteq C$ and $A \cap C = \emptyset$ implies $\epsilon(A/B) \geq \epsilon(A/C)$; so we can make the following definition.

Definition 1.1.2. *Extend ϵ to $d : \hat{\mathbf{L}}_0 \times \mathbf{L}_0 \rightarrow N$ by for each $N \in \hat{\mathbf{L}}_0$, $d(N, A) = \inf\{\epsilon(B) : A \subseteq B \subseteq_\omega N\}$, $d_N(A/B) = d_M(A \cup B) - d_M(B)$. We usually write $d(N, A)$ as $d_N(A)$ and omit the subscript N when clear.*

⁶In this paper \mathbf{L}_0 is closed under substructure. But this condition is relaxed to construct flat strongly minimal quasigroups in [Bal21a].

⁷See Definition 1.2.1 for details concerning r . In the general case, we count each of a sequence of predicates R_i separately. We make appropriate modification for linear spaces when they are considered.

Hrushovski defined a crucial property of the algebraic closure (pre) geometry⁸: *flat*. [BP20] generalized the notion of flatness for a pregeometry to a general predimension function.

Definition 1.1.3. (1) Consider a class (\mathbf{L}_0, ϵ) , $N \in \mathbf{L}_0$ and a sequence F_1, \dots, F_s of subsets of N . For $\emptyset \subsetneq T \subseteq \{1, \dots, s\} = I$, we let $F_T = \bigcap_{i \in T} F_i$ and $F_\emptyset = \bigcup_{1 \leq i \leq s} F_i$. We say that ϵ is flat if for all such F_1, \dots, F_s we have:

$$(*) \quad \epsilon\left(\bigcup_{1 \leq i \leq s} F_i\right) \leq \sum_{\emptyset \neq T} (-1)^{|T|+1} \epsilon(F_T).$$

- (2) Suppose (A, cl) is a pregeometry on a structure M with dimension function d and F_1, \dots, F_s are finite-dimensional d -closed subsets of A . Then (A, cl) is flat if d satisfies equation (*).
- (3) (A, cl) is strictly flat if it is flat but not distintegrated ($\text{acl}(ab) \neq \text{acl}(a) \cup \text{acl}(b)$).

What Hrushovski called *self-sufficient* closure is in the background.

Definition 1.1.4. (1) We say $A \subseteq N$ is strong in N and write $A \leq N$ if $\epsilon(A) \leq \epsilon(C)$ for any C with $A \subseteq C \subseteq_\omega N$, where \subseteq_ω stands for ‘be a finite subset’.

(2) For any $A \subseteq B \in \mathbf{L}_0$, the intrinsic (self-sufficient) closure of A , denoted $\text{icl}_B(A)$, is the smallest superset of A that is strong in B .

Note that $A \leq N$ if and only if $d_N(A) = \epsilon(A)$. It is well-known that since ϵ is integer valued then $\text{icl}(A)$ is finite if A is. The following definitions describes the pairs $B \subseteq A$ such that in the generic model M , A will be contained in the algebraic closure of B .

Definition 1.1.5. (1) A is a primitive extension⁹ of B if $B \leq A \cup B$, $A \cap B = \emptyset$, and there is no A_0 with $\emptyset \subsetneq A_0 \subsetneq A$ such that $B \leq B \cup A_0 \leq B \cup A$. A is a k -primitive extension if, in addition, $\epsilon(A/B) = k$. We stress that in this definition, while B may be empty, A cannot be.

Sometimes primitive is used with $B \subsetneq A$ and the primitive as $A - B$. In that case we sometimes write \hat{A} for $A - B$ when the disjointness is essential.

(2) We say that the 0-primitive pair A/B is good¹⁰ if every element of B is in some relation with an element of A .

(3) If A is 0-primitive over D and $B \subseteq A$ is such that A/B is good, then we say that B is a base for A over D if B is a minimal subset of D such that A/B is 0-primitive.

Remark 1.1.6. In the Hrushovski case, the definition of δ makes clear that the base defined by minimality in Definition 1.1.5.3 is also the maximal subset of D that is related to an element of A . This fails in the Steiner case; see Lemma 5.4.

Definition 1.1.7. Good pairs were defined in Definition 1.1.5.

(1) **Adequacy condition:** Fix a function μ assigning to every isomorphism type β of a good pair C/B in \mathbf{L}_0 a number $\mu(\beta) = \mu(B, C) = \mu(C/B) \geq \delta(B)$.

⁸A pregeometry/matroid becomes a geometry by modding out $\text{cl}(\emptyset)$.

⁹In [Hru93], 0-primitive is called *simply algebraic* and good is called *minimally simply algebraic*.

¹⁰In the Hrushovski case, it is equivalent to say if there is no $B' \subsetneq B$ such that (A/B') is 0-primitive. But for linear spaces these conditions are no longer equivalent.

- (2) For any good pair (A/B) with $B \subseteq M$ and $M \in \hat{\mathbf{L}}_0$, $\chi_M(A/B)$ denotes the maximal number of disjoint copies of A over B in M . A priori, $\chi_M(A/B)$ may be 0.
- (3) Let \mathbf{L}_μ be the class of structures M in \mathbf{L}_0 such that if (B, C) is a good pair then $\chi_M(B, C) \leq \mu(B, C)$.
- (4) $\hat{\mathbf{L}}_\mu$ is the universal class generated by \mathbf{L}_μ .
- (5) [*d*-closed] For $M \in \hat{\mathbf{L}}_\mu$ and $X \subseteq M$, X is *d*-closed in M if $d(a/X) = 0$ implies $a \in X$ (equivalently, for all $Y \subseteq_\omega M - X$, $d(Y/X) > 0$).
- (6) Let \mathbf{L}_d^μ consist of those $M \in \hat{\mathbf{L}}_\mu$ such that $M \leq N$ and $N \in \hat{\mathbf{L}}_\mu$ imply M is *d*-closed in N .

The restriction on μ in Definition 1.1.7.1 comes from [Hru93]. It appeared as a useful condition to guarantee the amalgamation. Surprisingly, we find the following slight strengthening plays a central role in preventing the definability of *any* truly binary functions.

Definition 1.1.8. We say T_μ triples if $\delta(B) \geq 2$ implies $\mu(C/B) \geq 3$ for any good pair C/B with $|C| > 1$.

We can show that any element of $\hat{\mathbf{L}}_\mu$ (not just \mathbf{L}_μ) can be amalgamated (possibly with identifications) over a (necessarily finite) *strong* substructure D of F with a strong extension of D to a member E of \mathbf{L}_μ . This yields the following conclusions; they are largely the same as [Hru93]; in order to treat line length 3, [BP20] make the adequacy requirement Definition 1.1.7.1 apply only when $|B| \geq 3$ and add $\mu(\beta) \geq 1$, if $\beta = \alpha$ (Definition 1.2.6).

Recall that a *generic model* for a class (\mathbf{L}_0, \leq) is an M such that if $A \leq M$, $A \in \mathbf{L}_0$ and if $A, A' \leq M$ are isomorphic, the isomorphism extends to an automorphism of M .

Conclusion 1.1.9. Suppose \mathbf{L}_μ satisfies the properties described in Axiom 1.1.1 and Definition 1.1.7:

If $D \leq F \in \hat{\mathbf{L}}_\mu$ and $D \leq E \in \mathbf{L}_\mu$ then there is $G \in \hat{\mathbf{L}}_\mu$ that embeds (possibly with identifications) both F and E over D . Moreover, if $F \in \mathbf{L}_d^\mu$, then $F = G$. In particular, (\mathbf{L}_μ, \leq) has the amalgamation property, and there is a generic structure $\mathcal{G}_\mu \in \hat{\mathbf{L}}_\mu$ for (\mathbf{L}_μ, \leq) .

The more refined conclusion of model completeness is shown in [Hol99] and for the linear space case in [BP20].

Conclusion 1.1.10. Under the hypotheses of Conclusion 1.1.9 there is a collection Σ_μ of π_2 sentences which

- (1) axiomatize the complete theory T_μ of the class \mathbf{L}_d^μ , *d*-closed models in $\hat{\mathbf{L}}_0$.
- (2) T_μ is model complete and strongly minimal.
- (3) The acl-geometry of T_μ is flat (e.g. 1.1.3. 2)

1.2. 3-hypergraphs and Linear Spaces

We now describe the main examples for this paper of the context axiomatized in Subsection 1.1. We replace \mathbf{L} and ϵ by \mathbf{K} and δ to indicate that properties here may depend on the specific definition of the class.

Definition 1.2.1 (Context). (1) The vocabulary τ contains a single ternary relation R . We require that R is a predicate of 3-elements sets (distinct in any order).

- (2) Let A, B, C each be a subset of $D \in \mathbf{K}^*$. We write $R(A, B, C)$ for the collection of tuples \mathbf{x} such that $D \models R(\mathbf{x})$ and \mathbf{x} intersects each of A, B, C . (The letters may be repeated to indicate only two sets are represented.) We write $R(A, B)$ for $R(A, A \cup B, B)$. We write $r(A, B, C)$ for the number¹¹ of tuples (up to permutation) in $R(A, B, C)$. Finally for $A \in \mathbf{K}_0^*$, $r(A) = r(A, A, A)$.

We restrict to integer valued δ which is essential (but not sufficient) to guarantee ω -stability. The crucial distinction between [Hru93] and linear spaces is restricting the class of finite structures by more than the assertion that R is a ternary predicate of sets (3-hypergraph).

Definition 1.2.2. *The choices here for δ*

- (3-hypergraph) For a finite τ -structure A , $\delta(A) = |A| - r(A)$.
- (linear space)
 - (1) A linear space is a τ -structure such that 2-points determine a unique line. We interpret R as collinearity. By convention two unrelated elements constitute a trivial line.
 - (2) For B, ℓ subsets of A , we say $\ell \in L(B)$ (is supported by B) if ℓ is a maximal R -clique contained in A and $|\ell \cap B| \geq 2$.
 - (3) Let

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

- (4) Then \mathbf{K}_0 is the collection of finite linear spaces A such that for any $A' \subseteq A$, $\delta(A') \geq 0$.

Note, that as opposed to Section 1.1, we have restricted \mathbf{K}_0 both by (a different) δ and by the linear space axiom.

Notation 1.2.3. Let $\delta(B/A)$ denote $\delta(A \cup B) - \delta(A)$. Suppose $A \cap B = C$ and $A, B, C \in \mathbf{L}_0$:

- (1) We say A and B are δ -independent if $\delta(A \cup B) = \delta(A) + \delta(B) - \delta(C)$.
- (2) We say A and B are fully independent over C if there are no relations involving elements from each of $A - C$ and $B - C$ and possibly C , i.e. $R(A - C, A \cup B, B - C) = \emptyset$.

Just rewriting the definition, we have $\delta(A \cup B/C) = \delta(A/B \cup C) + \delta(B/C)$.

Remark 1.2.4. Note that when δ just counts relations as in [Hru93], δ -independence implies full independence. The situation is more complicated for linear spaces; see Definition 5.6 and Remark 5.12.

The following useful tool is easy from the definition ([BP20, 4.7]).

Lemma 1.2.5. *If C is 0-primitive over \mathfrak{D} with base $B \subseteq \mathfrak{D}$ and $|C - B| \geq 2$, then every point $b \in B$ satisfies $R(c_1, c_2, b)$ for some $c_1, c_2 \in C$ and every point in C satisfies at least two instances of R .*

We single out an isomorphism type of a good pair (a 1-element extension, α) that plays a special role in the general proof that elimination of imaginaries fails. T_μ is a Steiner k -system if $\mu(\alpha) = k - 2$ [BP20].

¹¹For the Steiner system case count lines and compute δ as in Section 5.

Notation 1.2.6 (Line length). We write α for the isomorphism type of the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$.

By Lemma 5.18 of [BP20], lines in models of the theory T_μ of a Steiner system have length k if and only if $\mu(\alpha) = k - 2$.

2. ELIMINATION OF IMAGINARIES AND (SYMMETRIC) DEFINABLE CLOSURE

In this section we work in the context of Section 1.2 and lay out the connections among elimination of imaginaries, definable closure and a new notion *symmetric definable closure*. Recall that [She78, III.6] introduced extensions of structures by *imaginary elements* and [Poi83] discovered the importance of a theory not needing to add them.

Definition 2.1. *Elimination of Imaginaries:*

- (1) [Poi83] A theory T admits elimination of imaginaries if for every model M of T , for every formula $\varphi(\mathbf{x}, \mathbf{y})$ and for every $\mathbf{a} \in M^n$ there exists $\mathbf{b} \in M^m$ such that

$$\{f \in \text{aut}(M) \mid f|\mathbf{b} = \text{id}_{\mathbf{b}}\} = \{f \in \text{aut}(M) \mid f(\varphi(M, \mathbf{a})) = \varphi(M, \mathbf{a})\}$$

- (2) [Poi85, Theorem 16.15] A theory T admits weak elimination of imaginaries if and only if for every formula $\phi(\mathbf{x}; \mathbf{a})$ there exists a formula $\psi_{\mathbf{a}}(\mathbf{x}; \mathbf{y})$ such that there are only finitely many parameters $\mathbf{b}_1, \dots, \mathbf{b}_n$ such that each of $\psi_{\mathbf{a}}(\mathbf{x}; \mathbf{b}_1), \dots, \psi_{\mathbf{a}}(\mathbf{x}; \mathbf{b}_n)$ is equivalent to $\phi(\mathbf{x}; \mathbf{a})$.

The next notion arose in [Poi85]; we use the precise version of [Tsu93, Def. 2.3].

Definition 2.2. A finite set $F = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \dots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \text{id}_S \quad \text{for any } \sigma \in \text{aut}(M/A).$$

We say $T = \text{Th}(M)$ has the *finite set property* if every finite set of tuples F is coded by some set S over \emptyset .

Part 1) of the next result appears in [Pil99].

Fact 2.3. (1) Every strongly minimal theory such that $\text{acl}(\emptyset)$ is infinite has weak elimination of imaginaries.

- (2) If T admits weak elimination of imaginaries then T satisfies the finite set property if and only if T admits elimination of imaginaries.

Immediately from Fact 2.3, since in almost all¹² of the examples studied here $\text{acl}(\emptyset)$ is infinite [BP20, Fact 5.26], if T admits elimination of imaginaries there is an \emptyset -definable truly binary function (given by the coding of a pair of independent points). Section 4.2 exhibits a strongly minimal theory with a truly binary function that still fails to (by Section 4.1) eliminate imaginaries, as it is not commutative.

Below X denotes an arbitrary subset of a structure M and I denotes an v -element independent set $\{a_1, \dots, a_v\}$ with $I \leq M$.

We work with two groups of automorphisms; Section 3 treats properties that hold of both of them so the group is denoted G . Section 4.1 uses G_I and $G_{\{I\}}$ is needed in Section 4.4.

¹²[Bal21b] shows $\text{acl}(\emptyset)$ is infinite or empty.

Notation 2.4. Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

We introduce the (minimal) definable closure dcl^* of a set X to distinguish points which depend on *all* elements of X . Recall that for any first order theory T , if $X \subseteq M \models T$, then $c \in \text{dcl}(X)$ means c is the unique solution of a formula with parameters in X . This implies the orbit of c under $\text{aut}_X(M)$ consists of just c and the converse holds in any ω -homogenous model. All the models considered here are ω -homogeneous (since \aleph_1 -categorical [BL71]).

Notation 2.5. By $b \in \text{dcl}^*(X)$ we mean $b \in \text{dcl}(X)$, but $b \notin \text{dcl}(U)$ for any proper subset U of X (and analogously for acl^*). Note that $\text{dcl}^*(X)$ consists of the subset of $\text{dcl}(X)$ of elements not fixed by G_T for any $T \subsetneq X$.

Definition 2.6. For G either G_I or $G_{\{I\}}$, D is said to be G -invariant if D contains the G orbits of each of its elements, equivalently, $g(D) = D$ whenever $g \in G$.

The notion of symmetric definable closure, $\text{sdcl}(I)$, captures one direction of finite coding.

Notation 2.7. The symmetric definable closure of X , $\text{sdcl}(X)$, is those a that are fixed by every $g \in G_{\{X\}}$. $b \in \text{sdcl}^*(X)$ means $b \in \text{sdcl}(X)$ but $b \notin \text{sdcl}(U)$ for any proper subset U of X .

Note $\text{sdcl}^*(X) \subseteq \text{dcl}^*(X) \subseteq \text{dcl}(X)$. However, $\text{sdcl}(X)$ may not be contained in $\text{dcl}^*(X)$.¹³

Remark 2.8. We will give in Example 4.2.1 a theory \hat{T}_μ where for some B with $\delta(B) = 2$, there is a good pair A/B with $\mu(A/B) = 2$, and \hat{T}_μ admits an independent set $I = \{a, b\} \subseteq M \models \hat{T}_\mu$ with $\text{dcl}^*(I) \neq \emptyset$. Given a v -element independent set I with $v \geq 2$, we will show in Section 4.1 assuming $\mu(A/B) \geq 3$ (whenever $\delta(B) = 2$), that $\text{dcl}^*(I) = \emptyset$ and show in Section 4.4 that, even if some $\mu(A/B)$ might be 2, $\text{sdcl}^*(I)$ is empty.

Definition 2.9 (non-trivial functions). Let T be a strongly minimal theory.

1 (Essentially unary) An \emptyset -definable function $f(x_0 \dots x_{n-1})$ is called essentially unary if there is an \emptyset -definable function $g(u)$ such that for some i , for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \dots b_{i-1}, c, b_i \dots b_{n-1}) = g(c)$.

2 (truly n -ary)

(a) Let $\mathbf{x} = \langle x_0 \dots x_{n-1} \rangle$: a function $f(\mathbf{x})$ truly depends on x_i if for any independent sequence \mathbf{a} and some (hence any) independent¹⁴ \mathbf{a}' which disagrees with \mathbf{a}' only in the i th place $f(\mathbf{a}) \neq f(\mathbf{a}')$.

(b) f is truly n -ary if f truly depends on all its arguments and $f(\mathbf{a})$ is not a component of \mathbf{a} for all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$.

Lemma 2.10. For a strongly minimal T the following conditions are equivalent:

(1) for any $n > 1$ and any independent set $I = \{a_1, a_2, \dots, a_n\}$, $\text{dcl}^*(I) = \emptyset$;

¹³For a simple example, consider the theory of $(\mathbf{Z}, S, 0)$. Then $\text{dcl}^*(a, b) = \emptyset$ for a, b in distinct \mathbf{Z} -chains but $0 \in \text{sdcl}(\emptyset) \subseteq \text{sdcl}(X)$.

¹⁴This definition is more restrictive than the standard (e.g. [Grä79, p. 35] as for our definition, in a ring the polynomial $xy + z$ does not depend on y while usually one is allowed to substitute 0 to witness dependence.

- (2) every \emptyset -definable n -ary function ($n > 0$) is essentially unary;
(3) for each $n > 1$ there is no \emptyset -definable truly n -ary function in any $M \models T$.

Proof. 1) implies 2). Fix I as in the statement and let f be an \emptyset -definable n -ary function. Then 1) implies that for some i , say, 1, there is an \emptyset -definable function $g(u)$, with $g(a_1) = f(\mathbf{a}) = d$. Let p^1 denote the generic type over \emptyset realized by each a_i and p^n the type of the n -tuple. There are parameter-free formulas $\psi(u, y)$ and $\chi(u, \mathbf{v}, w)$ ($\lg(\mathbf{v}) = n - 1$) such that $p^1(u)$ entails $\psi(u, y)$ defines g (i.e. $y = g(u)$) and p^n implies χ defines f (i.e. $w = f(u, \mathbf{v})$).

Now we have $\phi(a_1, d) \wedge \chi(\mathbf{a}, d)$. But since a_1 is independent from (a_2, \dots, a_n) , for cofinitely many $c \in M$, we have $\exists y(\phi(c, y) \wedge \chi(c, a_2 \dots a_n, y))$. For each k with $1 \leq k < n$ let

$$A_{k+1}^c = \{x : (\exists^\infty x_1 \dots \exists^\infty x_k) g(x) = f(x_1, x_2 \dots x_k, x, a_{k+2} \dots a_n)\}.$$

By induction, since $a_{k+1} \in A_{k+1}^c$ and is independent from the parameters defining A_{k+1}^c , each A_{k+1}^c is cofinite, so, its complement A_{k+1} is finite. Thus, the subset on which $f(\mathbf{x}) \neq g(x_1)$ is contained in $\bigcup_{1 \leq i \leq n} A_i \times M^{n-1}$, which has Morley rank at most $n - 1$. Thus, f is essentially unary witnessed by g .

2) implies 3): Suppose f is such a definable truly n -ary function, let \mathbf{a} enumerate an independent set I . By 2) there are i, g with $g(a_i) = f(\mathbf{a})$ and this holds on any independent sequence. For some $j \neq i$, let \mathbf{a}' be obtained from \mathbf{a} by replacing a_j by an a'_j such that \mathbf{a}' is independent. Then $f(\mathbf{a}) = g(a_i) = f(\mathbf{a}')$ so f is not truly n -ary since it doesn't depend on x_i .

3) implies 1): Suppose 1) fails. Fix the least $n \geq 2$ such that $\text{dcl}^*(I) \neq \emptyset$ for some independent set $I = \{a_1, a_2, \dots, a_n\}$. Let $d \in \text{dcl}^*(I)$. By the definition of dcl , there exists $\varphi(x, \mathbf{y})$ such that $\models \exists! x \varphi(x, \mathbf{a}) \wedge \varphi(d, \mathbf{a})$, so φ defines some n -ary function f . We now show f is truly n -ary. If $f(\mathbf{a})$ is a component of \mathbf{a} , then $d = a_i$ for some i and $d \in \text{dcl}(a_i)$, contradicting $d \in \text{dcl}^*(I)$. If f is not truly n -ary, there exists i such that for any independent sequences \mathbf{b} and \mathbf{b}' which disagree in the i th place it holds that $f(\mathbf{b}) = f(\mathbf{b}')$. We choose a'_i so that $\mathbf{a}a'_i$ is independent, then $\mathbf{a}'_i = (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n)$ is independent, too. Then $f(\mathbf{a}) = f(\mathbf{a}'_i)$. Let $\psi(x', \mathbf{a})$ denote $f(\mathbf{a}) = f(a_1, \dots, a_{i-1}, x', a_{i+1}, \dots, a_n)$. Since $\mathbf{a}a'_i$ is independent and a'_i satisfies $\psi(x', \mathbf{a})$, this formula has cofinitely many solutions. Then the formula

$$\begin{aligned} \exists x(\exists^\infty x')(f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, x', a_{i+1}, \dots, a_n) \wedge \\ \wedge y = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)) \end{aligned}$$

defines d , so $d \in \text{dcl}^*(I - \{a_i\})$, for a contradiction. \blacksquare

Remark 2.11. We cannot isolate ‘non-triviality’ by simply saying there is no definable n -ary function, nor even none which depends on all its variables. The insight is that if $a \in \text{acl}(B)$ by a formula $\phi(v, \mathbf{b})$ which has k solutions, any solution is in the definable closure of B and the other $k - 1$. Steiner systems with line length k give a stark example. Consider the function of $k - 1$ variables which projects on the first entry unless the $k - 1$ arguments are a clique (partial line) and gives the last element of the line in that case. This function satisfies the ‘depends’ hypothesis but not the projection hypothesis. So, although \emptyset -definable, it is not a truly $(k - 1)$ -variable function.

We now begin the use of this general analysis to study the elimination of imaginaries for Hrushovski constructions.

Lemma 2.12. *Let $I = \{a_0, a_1, \dots, a_{v-1}\}$ be an independent set with $I \leq M$ and M be a model of a strongly minimal theory constructed as in Section 1.1.*

- (1) *For any finite X , if $\text{sdcl}(X) = \emptyset$ then X is not finitely coded.*
- (2) *If $\text{sdcl}^*(I) = \emptyset$ then I is not finitely coded.*
- (3) *If $\text{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter-free definable truly n -ary function for $n = |I|$.*

Proof. 1) is immediate from Definition 2.2 and 3) follows immediately from 2) since $\text{sdcl}^*(I) \subseteq \text{dcl}^*(I)$. 2) requires some effort. Suppose $t \in T$ where T is a finite code for I . If $t \in \text{sdcl}(I) - \text{sdcl}^*(I)$, either $t \in \text{dcl}(\emptyset)$ or $t \in \text{dcl}(\{a_i : i \in J\})$ for some $J \subset I$. In the first alternative, if T is a finite code for I so is $T - \{t\}$. And since I is independent, it cannot be coded by the empty set. So we must consider the second case. But if $t \in \text{dcl}(I) - \text{dcl}^*(I)$ is in, say, $\text{dcl}^*(J)$, a permutation switching a_i and a_j for some $j \in J$, $i \in I - J$ and fixing the other a_k takes t to some $t' \neq t$. Thus $t \notin \text{sdcl}(I)$. ■

We use G_I to prove hypotheses (1) and $G_{\{I\}}$ for (2) of Theorem 2.13. The ‘proof’ just below just indicates the organization of the argument that follows.

Theorem 2.13. *Let T_μ be a Hrushovski construction as in Theorem 4.1.2 or a strongly minimal Steiner system as in Theorem 5.2.*

- (1) *If T_μ triples ($\delta(B) = 2$ and $|C| > 1$ imply $\mu(C/B) \geq 3$), then $\text{dcl}^*(I) = \emptyset$.*
- (2) *In any case $\text{sdcl}^*(I) = \emptyset$.*

Consequently, T_μ does not admit elimination of imaginaries.

Proof. 1) By Lemma 2.12 and Theorem 4.1.2, I is not finitely coded. So by Fact 2.3, T does not admit elimination of imaginaries. And by Lemma 2.10, $\text{dcl}^*(I) = \emptyset$ implies there is no \emptyset -definable truly n -ary function. 2) Theorem 4.4.1 (Theorem 5.19 for the Steiner case) provides $\text{sdcl}^*(I) = \emptyset$ without the extra (triplable) hypothesis. ■

Importantly, if the A in a good pair A/B is fixed setwise by G then so is B .

Observation 2.14. *Assume \mathbf{L}_0 consists of all structures A in \mathbf{L}_0^* such $\emptyset \leq A$. Let $\mathfrak{A}, B, C, I \subseteq M$, $M \models T_\mu$ and $G = G_{\{I\}}$ or G_I . Suppose C is 0-primitive over \mathfrak{A} and based on $B \subseteq \mathfrak{A}$. If the automorphism $f \in G$ fixes \mathfrak{A} setwise, and fixes C setwise, then it fixes B setwise.*

Note that the first assumption in Observation 2.14 *fails* for the Steiner system case (and also Proposition 18 of [Hru93]). We find a suitable substitute in Lemmas 5.4 and 5.11.

Proof. By Definition 1.1.5.4, B is uniquely determined by C and \mathfrak{A} . Thus, if $f \in G$ satisfies $B \neq f(B)$, we have a contradiction since $f(C) = C$ and so some element of C is R -related to an element not in B . ■

3. G -DECOMPOSITION

We continue with the hypotheses of Sections 1.2. Our original goal was to show $\text{dcl}^*(I) = \emptyset$ for $I = \{a_1, \dots, a_v\}$ with $d(I) = v \geq 2$. For this we introduced the notion of a G_I -decomposition to analyze the algebraic closure of a finite set. However

we needed an additional hypothesis on μ to show $\text{dcl}^*(I) = \emptyset$. In order to eliminate that hypothesis, we consider decompositions with respect to two subgroups of $\text{aut}(M)$: $G = G_I$ or $G = G_{\{I\}}$ that fix I pointwise or setwise, respectively. Using the associated decomposition we inductively show the appropriate definable closure is empty. We give a joint account of the decomposition but by changing the group prove Theorem 0.2 1) (for dcl^*) or 2) (for sdcl^*).

Definition 3.1. *Let M be the generic model of T_μ , $I = \{a_1, \dots, a_v\}$ be independent with $I \leq M$, and let $G \in \{G_I, G_{\{I\}}\}$. A subset \mathfrak{A} is G -normal if it is finite, contains I , G -invariant (G fixes \mathfrak{A} setwise), and is strong in M .*

We need the following easy observation to prove Lemma 3.3; finite G -normal sets exist. The forward implication in Observation 3.2 holds for any first order theory. As in Notation 2.5 the conditions are equivalent here since all models are ω -homogeneous.

Observation 3.2. *Let $A \subseteq M$. 1) implies 2) and 3); all are equivalent in an ω -homogenous model.*

- (1) $c \in \text{acl}(A)$
- (2) *The orbit of c under $\text{aut}_A(M)$ (A fixed pointwise) is finite.*
- (3) *If A is finite then the orbit of c under $\text{aut}_{\{A\}}(M)$ (A fixed setwise) is finite.*

Lemma 3.3. *Each finite $U \subseteq \text{acl}(I) \subseteq M$ is contained in a finite G -normal set. If U is a finite G -normal set, then $U \subseteq \text{acl}(I)$.*

Proof. Given any finite set U , let $G(U) = \{g(u) : u \in U, g \in G\}$, and $\mathfrak{A}_U^G = \text{icl}(I \cup G(U))$. Then \mathfrak{A}_U^G is G -normal. For this, note $G(U)$ is finite by Lemma 3.2. The intrinsic closure of a set is unique, so \mathfrak{A}_U is fixed setwise for either G . The second part of the lemma is immediate since $\text{icl}(X) \subseteq \text{acl}(X)$. ■

Note that for given I and U in M , both the set \mathfrak{A}_U^G and the height (Definition 3.8) of the G -decomposition depend on the choice of G .

We need the following result from [Ver06, 4.2] to carry out the decomposition.

Lemma 3.4. *Suppose $A_1 \subset A_2 \subset A_3$ are such that $C_i = A_{i+1} \setminus A_i$ is 0-primitive over A_i , for $i = 1, 2$. If C_2 is 0-primitive over A_1 , then C_1 is 0-primitive over $A_1 \cup C_2$.*

Proof. Let $D \subseteq C_1$. Then

$$\begin{aligned} \delta(D/A_1 \cup C_2) &= \delta(D \cup C_2/A_1) - \delta(C_2/A_1) = \delta(D \cup C_2/A_1) \\ &= \delta(D/A_1) + \delta(C_2/A_1 \cup D) = \delta(D/A_1) \end{aligned}$$

The first three equalities follow easily from the definition of $\delta(X/Y)$ and the conditions of the lemma. The last equality follows from: $0 = \delta(C_2/A_2) \leq \delta(C_2/A_1 \cup D) \leq \delta(C_2/A_1) = 0$. So C_1 is 0-primitive over $A_1 \cup C_2$. ■

The next definitions and theorems provide the tools for the decompositions. Roughly speaking, capital Roman letters (A, B, C) denote specific components of the decomposition; script letters $\mathfrak{A}, \mathfrak{D}$ range over initial segments of the decomposition. In particular, this means that each of $\mathfrak{A}, \mathfrak{D}$ contains I and is closed in M .

Definition 3.5. (1) *We call a good pair A/B well-placed by \mathfrak{D} , if $B \subseteq \mathfrak{D} \leq M$ and A is 0-primitive over \mathfrak{D} , and*

- (2) A/B is well-realized in Y if $\chi_Y(A/B) = \mu(Y/B)$. If $Y = M$ we omit it and write simply A/B is well-realized.

Lemma 3.7 is crucial for the general decomposition Construction 3.8.

Definition 3.6. Let A and C be subsets of \mathfrak{D} . We say A splits over \mathfrak{D} if both $A \cap \mathfrak{D}$ and $A - \mathfrak{D}$ are non-empty.

Lemma 3.7. Let M be the countable generic model for T_μ . Suppose A/B well-placed by \mathfrak{D} and \mathfrak{D} is G -invariant.

- (1) Then $\chi_M(A/B) = \mu(A/B)$.
 (2) For each $i < \mu(A/B)$ there is partial isomorphism h_i fixing B pointwise with domain $B \cup A$ either i) $h_i(A) \cap \mathfrak{D} = \emptyset$ or ii) $h_i(A) \subsetneq \mathfrak{D}$. Moreover, by Lemma 1.1.7.3 the $h_i(A)$ are disjoint over B . In case i) there is a $g \in G$ that fixes IB (and indeed \mathfrak{D}) pointwise and takes A to $h_i(A)$. That is, G acts transitively on the copies of A that are disjoint from \mathfrak{D} .

While the proof uses that M is generic, the conclusion passes to any model of T_μ , because models are algebraically closed.

Proof. 1) Part 1 is the translation to this notation of a result proved for the Hrushovski case in [Ver02, 2.25]; for the Steiner case, it is [BP20, 5.14]. Since the article [Ver02] is difficult to access, we repeat the proof here.

Let $\mathfrak{D} \leq M$ and A be a 0-primitive extension of \mathfrak{D} with the base B . If $\chi_{\mathfrak{D}}(A/B) = \mu(A/B)$ we are done, so we assume that $\chi_{\mathfrak{D}}(A/B) < \mu(A/B)$. Let A_1 be an isomorphic copy of A over \mathfrak{D} and let E be the canonical amalgam of $\mathfrak{D} \cup A$ and $\mathfrak{D} \cup A_1$ over \mathfrak{D} . By [Hru93, Lemma 3], $E \in \mathbf{L}_\mu$ and there is an embedding $g : E \rightarrow M$ such that $g \upharpoonright \mathfrak{D} \cup A = id_{\mathfrak{D} \cup A}$ and $g(E) \leq M$. Then $\chi_E(A/B) = \chi_{\mathfrak{D}}(A/B) + 1$ and we proceed by induction.

2) By 1) there are partial isomorphisms h_i for $i < \mu(A/B)$, fixing B , but not necessarily I , giving structures $A_i = h_i(A)$ isomorphic to A over B . Note that A_i can not split over \mathfrak{D} and since I is independent $I \leq M$. Moreover, we have $BA \approx BA_i$, $\mathfrak{D} \leq M$, and there are no relations between A and $\mathfrak{D} - B$. So, if $A_i \cap \mathfrak{D} = \emptyset$ there must be no relations between A_i and $\mathfrak{D} - B$. Else, $\delta(\mathfrak{D}A_i) < \delta(A)$. As $\mathfrak{D}A \approx \mathfrak{D}A_i$, there is an automorphism of M taking A to A_i and fixing \mathfrak{D} and in particular I . ■

The following definition and description of the decomposition of a G -normal set is intended to be evident (modulo the references). The next diagram gives an overall view; Example 3.13 gives a closer view.

Construction 3.8. Let \mathfrak{A} be G -normal. We can linearly decompose \mathfrak{A} as the union of X_n , $n \leq r$, where $X_0 = I$ and X_{n+1} is 0-primitive over X_n and good over $Y_{n+1} \subseteq X_n$ for $n < r$. This is a cumulative decomposition: $X_n \subseteq X_{n+1}$.

Since we aim to prove that $\text{dcl}^*(I) \cap X_r = \emptyset$ ($\text{sdcl}^*(I) \cap X_r = \emptyset$) by induction on n , it would be convenient to assume that X_n is G -invariant for each $n < r$. But it is not true. In order to reach an induction on G -invariant sets, we create, by grouping the images of various partial isomorphisms of the $\hat{X}_{n+1} = (X_{n+1} - X_n)$ over Y_{n+1} , G -invariant strata \mathfrak{A}^{m+1} of components that are independent over \mathfrak{A}^m . The new tree decomposition creates strata $\langle \mathfrak{A}^m : m < m_0 \rangle$; m_0 is called the *height* of the decomposition.

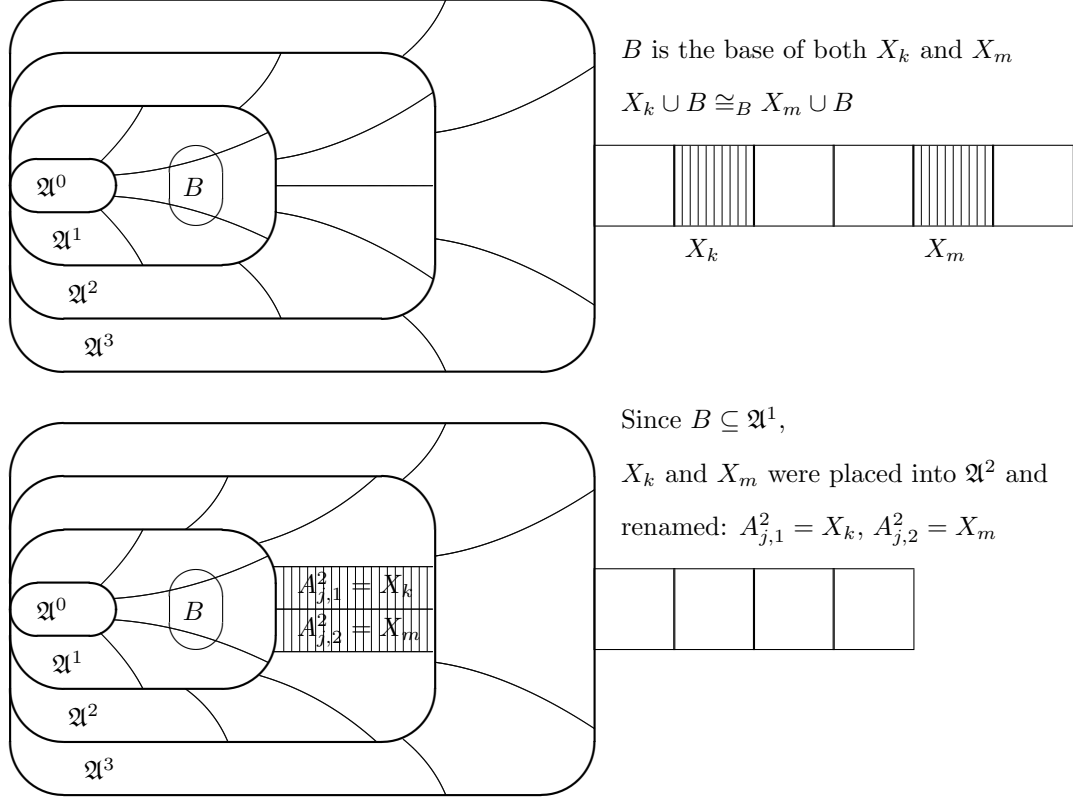


FIGURE 1. From a linear to a tree-decomposition: One Step

We define the new decomposition of \mathfrak{A} into strata \mathfrak{A}^m by inductively assigning to each \hat{X}_{n+1} an integer $S(\hat{X}_{n+1}, Y_{n+1})$, the *strata* of \hat{X}_{n+1} , the least $m+1 \leq n$ such that $Y_{n+1} \subseteq \mathfrak{A}^m$ and renaming $\hat{X}_{n+1} = (X_{n+1} - X_n)$ as an $A_{x,y}^{m+1}$ for an appropriate x, y (more detail below). The Y_{n+1} may be omitted when clear from context. By fiat, $S(X_0, \emptyset) = 0$.

i) \mathfrak{A}^0 :

(a) Let $D_0 = \mathfrak{A} \cap \text{acl}(\emptyset)$.

(b) For $i = 1, 2, \dots, v$, let $D_i = \mathfrak{A} \cap \text{acl}(a_i)$ and let $\mathfrak{A}^0 = D_1 \cup D_2 \cup \dots \cup D_v$.

Note that $D_i \cap D_j = D_0$ for any $1 \leq i < j \leq v$, \mathfrak{A}^0 is finite, $\mathfrak{A}^0 \leq M$, and so $\delta(\mathfrak{A}^0) = v$. Moreover, since $d(a_i) = 1$ the D_i are fully independent over D_0 .

As we continue the construction we will rearrange the components \hat{X}_n into a quasi-order by introducing sets \mathfrak{A}^m such that each component in \mathfrak{A}^m is based on a subset of \mathfrak{A}^{m-1} . At the n th stage of construction, considering (\hat{X}_{n+1}, Y_{n+1}) , \hat{X}_{n+1} is added to $\mathfrak{A}^{S(\hat{X}_{n+1}, Y_{n+1})}$ and given an appropriate name as described below. Each \mathfrak{A}^m is divided into q_m subsets \mathfrak{A}_i^m , where \mathfrak{A}_i^m consists of ℓ_i^m , disjoint off \mathfrak{A}^{m-1} , sets $A_{i,f}^m$ which are 0-primitive over \mathfrak{A}^{m-1} and pairwise isomorphic over \mathfrak{A}^{m-1} , and each $A_{i,f}^m$ is based on the same set $B_i^m \subseteq \mathfrak{A}^{m-1}$.

We call the $A_{i,f}^m$ *petals*. Lemma 3.7.(2) ensures that G acts 1-transitively on $\{A_{i,f}^m : f < \ell_i^m\}$. We describe further petals of $A_{i,1}^m/B_i^m$ in the next few paragraphs.

We now give a precise definition of \mathfrak{A}^{m+1} . We set $\mathfrak{A}^{-1} = \emptyset$ to allow uniform treatment for all $m \geq 0$. Note that new petals may be added to \mathfrak{A}^{m+1} at later stages in the construction.

- ii) \mathfrak{A}^{m+1} : Suppose $S(X_n) = m > 1$. We consider the good pair \hat{X}_{n+1}/Y_{n+1} with $Y_{n+1} \subseteq X_n$. If $Y_{n+1} = Y_{n'}$ for some $n' < n$ with $S(X_{n'}) = m$ then Y_{n+1} has already been denoted B_t^{m+1} for some $t \leq j$. If $\hat{X}_{n+1} \approx_{Y_{n'}} X_{n'}$, set \hat{X}_{n+1} as $\mathfrak{A}_{t,k}^{m+1} \in \mathfrak{A}_t^{m+1}$, where k is the least index not previously used with t .

If $X_n \not\approx_{Y_{n'}} X_{n'}$ and $Y_{n+1} \neq Y_{n'}$ for any $n' < n$ with $S(X_{n'}) = m$, set \hat{X}_{n+1} as $A_{u,1}^{m+1}$ and set Y_{n+1} as B_u^{m+1} for the next available u . Then $Y_{n+1} \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1}) \neq \emptyset$, $\mathfrak{A}^m \leq M$, $\mathfrak{A}^m \hat{X}_{n+1} \leq M$. It is possible that $(Y_{n+1}, \hat{X}_{n+1}) \approx (Y_{n'}, \hat{X}_{n'})$ for some smaller n' .

By Lemma 3.7.2, there are partial isomorphisms¹⁵ h_i for $i \leq \mu(\hat{X}_{n+1}/Y_{n+1})$ that fix Y_n and the $h_i(\hat{X}_{n+1})$ are independent (and so disjoint) over Y_n . Note that some of these h_i may not extend to automorphisms of M and if so by Lemma 3.7.2 they map X_{n+1} into \mathfrak{A}^m . Suppose that ℓ_{j+1}^{m+1} of these partial isomorphisms extend to automorphisms h_i of M that fix I and so $h_i(\hat{X}_n) \cap \mathfrak{A}^m = \emptyset$ for $1 \leq i \leq \ell_{j+1}^{m+1}$.

We have relabeled the $h_i(\hat{X}_{n+1})$ as $A_{j+1,f}^{m+1}$, for $1 \leq f \leq \ell_j^{m+1}$ and added them to \mathfrak{A}_j^{m+1} forming $\mathfrak{A}_{j+1}^{m+1} = \bigcup_{1 \leq f \leq \ell_{j+1}^{m+1}} \mathfrak{A}_{j+1,f}^{m+1}$, which is thus G -invariant.

Since \mathfrak{A} is G -normal, each of the $h_i(\hat{X}_{n+1})$ is an $\hat{X}_{n'}$ for some $n' \geq n+1$.

Let μ_{j+1}^{m+1} denote $\mu(\hat{X}_{n+1}/Y_{n+1}) = \mu(A_{j+1}^{m+1}/B_{j+1}^{m+1})$. The other $\mu_{j+1}^{m+1} - \ell_{j+1}^{m+1}$ images are subsets of \mathfrak{A}^m and are labeled as $C_{j+1,k}^{m+1}$ for $1 \leq k \leq \nu_{j+1}^{m+1} = \mu_{j+1}^{m+1} - \ell_{j+1}^{m+1}$.

Each of the $A_{i,f}^{m+1}$ for $f < \ell_{j+1}^{m+1}$ and the $C_{j,k}^{m+1}$ for $1 \leq k \leq \nu_{j+1}^{m+1}$ is a petal.

Note that $\hat{X}_{n+1} = X_{n+1} - X_n$ is based on Y_{n+1} , which we have designated¹⁶ as $B_{j+1}^{m+1} \subseteq \mathfrak{A}^m$; by the minimality of m , B_{j+1}^{m+1} intersects $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ non-trivially. Thus as we construct \mathfrak{A}_j^{m+1} , we are moving ℓ_{j+1}^{m+1} components down so they are directly above their base. This is possible by Lemma 3.4. We sometimes call the $A_{j,i}^{m+1}$ which have the same base B_j^{m+1} a *cluster* \mathfrak{A}_j^{m+1} .

At the conclusion of the construction for each $m < m_0$, for some $t_{m+1} < r$, there will be t_{m+1} ($t_{m+1} = \sum_{i < q_m} \ell_i^{m+1}$) distinct \hat{X}_{n+1} , labeled as $A_{j,f}^{m+1}$ with $S(X_n) = m$; the $A_{j,f}^{m+1}$ are independent over \mathfrak{A}^m . Then $\mathfrak{A}^{m+1} = \mathfrak{A}^m \cup \bigcup_{j < q_m} \mathfrak{A}_j^{m+1}$ and the union is a partition of $\mathfrak{A}^{m+1} - \mathfrak{A}^m$. While $\bigcup_{i \leq r} X_i$ is a chain, the \mathfrak{A}^m form a tree with the petals $A_{j,f}^{m+1}$ partitioning each level. More locally $B_j^{m+1} \cup \bigcup_{f=1}^{\ell_j^{m+1}} A_{j,f}^{m+1}$ looks like a flower with the base B_j^{m+1} and two collections of petals. $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ is a collection of petals $\bigcup_{j < q_m, f < \ell_j^{m+1}} A_{j,f}^{m+1}$ on the stem \mathfrak{A}^m . But for each j , the flower over B_j also contains the $C_{j,k}^{m+1} \subseteq \mathfrak{A}^m$ for $k < \nu_j^{m+1}$ for $j < q_m$. Further, $\mathfrak{A} = \bigcup_{m \leq m_0} \mathfrak{A}^m$.

¹⁵It is essential here that each (X_n/Y_n) is well-placed (Definition 3.5).

¹⁶We use a single subscript because, while we are considering several copies of the \hat{X}_n , there is a fixed base.

Note that any two petals on the same strata, say on \mathfrak{A}^{m+1} , are δ -independent over \mathfrak{A}^m and in the case of Hrushovski's construction are fully independent. For Steiner systems we obtain that if these petals do not belong to the same linear cluster (Definition 5.6) then they are fully independent.

Remark 3.9. *Note that a G -decomposition depends on, and is determined by, the original linear decomposition.*

The following observation is key to the proof of the ensuing Lemma 3.11 and Lemma 3.21.

Observation 3.10. [Ver02, Note 2.8] We say that a 3-hypergraph A is *disconnected over B* if there is a partition of A into $A_1 \cup A_2$ such that for every $a \in A_1$ and $b \in A_2$ there is no $d \in A \cup B$ such that $R(a, b, d)$. It is easy to see that if A is 0-primitive over B , then A is connected over B . As, $\delta(A_1/A_2B) = 0$, if A is disconnected over B ; but then A_2 is 0-primitive over B , contrary to the minimality of A .

Here are the basic properties of \mathfrak{A}^1 showing $\text{dcl}^*(I) \cap \mathfrak{A}^1 = \emptyset$; the situation is simpler than the $m > 1$ case as there are no maps of \hat{X}_n into \mathfrak{A}^m over Y_n when $m = 0$.

Lemma 3.11 (\mathfrak{A}^1). *Let \mathfrak{A} be G -normal and decomposed as $\langle \mathfrak{A}^n : n < m_0 \rangle$. Then for any i and $f < \ell_i^1$, $A_{i,f}^1$, and $B_i^1 \subseteq \mathfrak{A}^0$ the following hold: $d(B_i^1) \geq 2$, $\mathfrak{A}^0 \leq M$, $\ell_i^1 = \mu_i^1 = \mu(B_i^1, A_{i,1}^1)$, and each $A_{i,f}^1 \cap \mathfrak{A}^0 = \emptyset$. So, no $A_{i,f}^1$ is invariant under G .*

Proof. Note that for each i , $d(B_i^1) \geq 2$; otherwise there exists $b \in B_i^1$ such that $d(b) = d(B_i^1) = 1$. Since $b \in \mathfrak{A}^0$, $b \in \text{acl}(a_k)$ for some $k \in \{1, 2, \dots, v\}$; this implies that $B_i^1 \subseteq \text{acl}(a_k)$ and thus $A_{i,j}^1 \subseteq \text{acl}(a_k)$; the last assertion contradicts $A_{i,j}^1 \cap \mathfrak{A}^0 = \emptyset$. As noted in Construction 3.8.i), $I \subset \mathfrak{A}^0$ and $\delta(\mathfrak{A}^0) = v$ so $d(\mathfrak{A}^0) = v$ and $\mathfrak{A}^0 \leq M$.

We show there cannot be a copy, $C^1 = C_{i,x}^1$ for some i, x , of $A_{i,f}^1$ with base B_i^1 embedded in \mathfrak{A}^0 . Since $d(B_i^1) \geq 2$, B_i^1 intersects at least two $D_k - D_0$ and $D_j - D_0$ for some $k \neq j$. Note that $C^1 \subseteq \mathfrak{A}^0$ is not a singleton c , because otherwise $M \models R(c, d_k, d_j)$ for some $d_k \in B_i^1 \cap D_k - D_0$ and $d_j \in B_i^1 \cap D_j - D_0$, contradicting full independence of the D_i 's over D_0 . By Lemma 1.2.5 C^1 should intersect both $D_k - D_0$ and $D_j - D_0$. If not, there would be an $R(c_1, c_2, d)$ with $c_1, c_2 \in D_k$ and $d \in D_j \cap B_i^1$; this can't happen as the D_i for $i > 0$ are fully independent over D_0 (Construction 3.8 \mathfrak{A}^0 .b). But then C^1 is disconnected, contrary to Observation 3.10. Thus $\ell_i^1 = \mu_i^1 \geq \delta(B_i^1) \geq d(B_i^1) \geq 2$. ■

We pause to note a distinction between the flat geometries and the locally modular ones. [Bal73] showed that the rank dimension models of an \aleph_1 -categorical theory had either countably many or 2^{\aleph_0} automorphisms, with vector space-like strongly minimal sets on the first side and algebraically closed fields on the other. We now note:

Corollary 3.12. *If T_μ is constructed by a Hrushovski construction (including Steiner systems) with a flat geometry, each finite dimensional model M_n has 2^{\aleph_0} automorphisms.*

Proof. Suppose M is prime over the algebraically independent set X with n elements. There are countably many distinct good pairs (A_n/X) (Remark 4.1.11); each has multiplicity at least 2, and we can define automorphisms of M that fix or permute the realizations A_n at will to give 2^{\aleph_0} automorphisms. ■

The following example shows the situation gets much more complicated with the second strata.

Example 3.13. This example illustrates i) the shift from a chain to a strata decomposition, ii) $\text{acl}^*(\mathfrak{A}^0)$ may properly extend $\text{dcl}(\mathfrak{A}^0)$ and iii) that some $A_{1,i}^2$ may intersect \mathfrak{A}^0 . Let M be any model of \hat{T}_μ with $\mu(\alpha) = 2$. Suppose $I = \{a_1, a_2\}$ and let R hold of the triples $a_1 a_2 b_1, a_1 a_2 b_2, c_1 c_2 b_1, c_1 c_2 b_2$ and the entire six point diagram be strong in M .

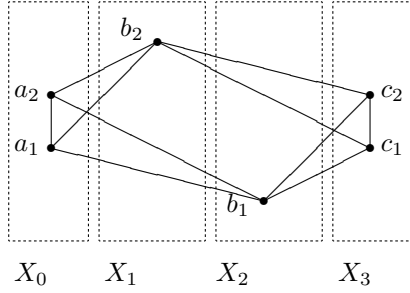


FIGURE 2. Chain

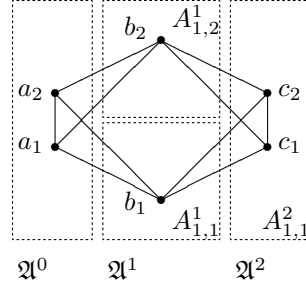


FIGURE 3. Decomposition

Figure 2 shows a chain decomposition; Figure 3 illustrates the downward embedding in a strata decomposition (as both $A_{1,1}^1$ and $A_{1,2}^1$ are based in \mathfrak{A}^0).

Now c_1, c_2 are both in $\text{acl}^*(\mathfrak{A}^0)$ but neither is in $\text{dcl}(\mathfrak{A}^0)$. Further $\text{sdcl}^*(\mathfrak{A}^0) = \text{dcl}^*(\mathfrak{A}^0) = \emptyset$. We will show in Section 4.2 that for an independent pair I , there may be elements in $\text{dcl}^*(I) - \text{sdcl}^*(I)$. X_1 and X_2 demonstrate that there may be components X_n and $X_{n'}$, both in strata m , such that $(Y_n, \hat{X}_n) \approx_I (Y_{n'}, \hat{X}_{n'})$; we provide the tool to study this situation in Definition 3.14.

Finally, in Figure 4 $A_{2,1}^2 \cap \mathfrak{A}^1 = \emptyset$ while $A_{2,2}^2 = \mathfrak{A}^0$ although both are based on and isomorphic over B_1^2 , as it shown in Figure 4.

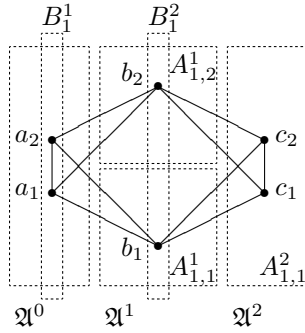


FIGURE 4. $A_{2,2}^2 = \mathfrak{A}^0$

Suppose further that $\mu(A_{1,1}^2/B_{1,1}^2) = 2$. Then this is a G -decomposition of $\mathfrak{A}^0 \cup \{c_2\}$ for either G . This shows that (in the presence of certain good pairs with $\mu(A/B) = 2$) we cannot avoid G -invariant petals.

Definition 3.14 (J_j^{m+1}). Let \mathfrak{A} be G -normal and decomposed by $\langle \mathfrak{A}^n : n \leq m_0 \rangle$. We let $J_{G,j}^{m+1}$ consist of all indices j' that $g(B_j^{m+1}) = B_{j'}^{m+1}$ for some $g \in G$. Thus we have an equivalence relation on the $j < q_m$ enumerating the bases B_j^{m+1} ; $j \sim j'$ if $B_{j'}^{m+1} = g(B_j^{m+1})$ for some $g \in G$.

If G is fixed we omit it in $J_{G,j}^{m+1}$ and write simply J_j^{m+1} . Note that $j \sim j'$ implies $J_j^{m+1} = J_{j'}^{m+1}$, $A_j^{m+1} \approx A_{j'}^{m+1}$, and $\mu_j^{m+1} = \mu_{j'}^{m+1}$.

Immediately,

Observation 3.15. Let everything be as in Definition 3.14. Then B_j^{m+1} is G -invariant if and only if $|J_{G,j}^{m+1}| = 1$.

Thus, \mathfrak{A}^m will consist of $\Sigma_j |J_{G,j}^{m+1}| \cdot \ell_j^m$ petals.

We summarise in Notation 3.16, which also depends on the choice of G . In Section 4.1 we are using G_I . In Section 4.4, we employ $G_{\{I\}}$.

Notation 3.16. The height of \mathfrak{A} is the maximal index, $m_0 \leq r$ of a non-empty strata.

q_m denotes the number of bases B_j^{m+1} that support elements of strata \mathfrak{A}^{m+1} .

And, for fixed G , (Definition 3.14), $|J_j^{m+1}|$ is the number of those $B_{j'}^{m+1}$ ($j' \in J_j^{m+1}$) that are isomorphic to B_j^{m+1} over I by some $g \in G$.

For each m, j , ℓ_j^{m+1} is the number of conjugates of A_j^{m+1} over $I \cup B_j^{m+1}$ under G . Thus, ℓ_j^{m+1} is the number of B_j^{m+1} -copies of $A_{j,1}^{m+1}$ that are not embedded in \mathfrak{A}^m .

We denote by ν_j^{m+1} the number of B_j^{m+1} -copies of $A_{j,1}^{m+1}$, labeled as $C_{j,q}^{m+1}$, that are embedded in \mathfrak{A}^m .

Finally, $\mathfrak{A}^m = \bigcup_{i \leq m} \mathfrak{A}^i$.

Lemma 3.17. Let \mathfrak{A} be G -normal and decomposed by $\langle \mathfrak{A}^n : n < m_0 \rangle$. For any positive $m \leq m_0$ and j it holds that $\ell_j^m + \nu_j^m = \mu(A_{j,1}^m/B_j^m)$.

Proof. Fact 3.7.(2) implies that $\chi_M(A_{j,1}^m/B_j^m) = \mu(A_{j,1}^m/B_j^m)$. Let C be a copy over B_j^m of $A_{j,1}^m$. Since $B_j^m \subseteq \mathfrak{A}^{m-1} \leq M$, the definition of a 0-primitive extension implies that either $C \subseteq \mathfrak{A}^{m-1}$ or $C \cap \mathfrak{A}^{m-1} = \emptyset$. ■

The following notion is central for analyzing the position of a G -invariant petal in \mathfrak{A} . As, a G_I -invariant singleton is in $\text{dcl}^*(I)$; our goal is to show this can't happen.

Definition 3.18. We say $A_{j,1}^{m+1}$ determines $A_{i,f}^m$ if $A_{i,f}^m$ is the unique petal based in \mathfrak{A}^{m-1} that intersects $B_j^{m+1} - \mathfrak{A}^{m-1}$.

Note that if $A_{i,f}^m$ is determined then $A_{i,f}^m$ is G -invariant, so we normally denote the determined petal by $A_{i,1}^m$. We now see that a G -invariant singleton determines a petal that contains B_j^{m+1} . The following lemmas show that when $|A_{j,1}^{m+1}| > 1$, under appropriate inductive hypothesis B_j^{m+1} is 'almost' contained in $A_{i,1}^m$ (Lemma 3.23.1).

Lemma 3.19. Let $m \geq 1$ and $B = B_j^{m+1}$ be the base of $A_{j,1}^{m+1}$ over \mathfrak{A}^m . If $|A_{j,1}^{m+1}| = 1$ and $A_{j,1}^{m+1}$ is G -invariant then

- (1) $A_{j,1}^{m+1}$ determines some $A_{i,f}^m$;

(2) and if B does not contain a G -invariant singleton, $B \subseteq A_{i,f}^m$.

Proof. (1) By Observation 2.14 B is G -invariant. Assume the contrary, that $A_{j,1}^{m+1} = \{c\}$, but B intersects at least two petals $A_{i,f}^m$ and $A_{i',f'}^m$. Observe that if the singleton c is primitive over \mathfrak{A}^m , then for some $b_1, b_2 \in \mathfrak{A}^m$, $((b_1, b_2), c)$ realizes the good pair α . So, $B = \{b_1, b_2\}$ and $M \models R(b_1, b_2, c)$. By construction $B \cap \mathfrak{A}^m \neq \emptyset$, so at least one of b_1, b_2 is in $A_{i,f}^m$ for some f and i , say b_1 .

Let $C^1 = \{c_1\}$ be an isomorphic over B copy of $A_{j,1}^{m+1}$ with $C^1 \subseteq \mathfrak{A}^m$. As there is no relation $R(b_1, b_2, c_1)$ with the b_i in $A_{i,f}^m$ and $A_{i',f'}^m$ (since they are fully independent over \mathfrak{A}^{m-1}), $B - \mathfrak{A}^{m-1} \subseteq A_{i,f}^m$. Since B is G -invariant, $A_{i,f}^m$ is G -invariant.

(2) Suppose for contradiction $b_2 \in \mathfrak{A}^{m-1}$. Then since B and \mathfrak{A}^{m-1} are each G -invariant both b_1 and b_2 are fixed by G violating the additional assumption for case (2). \blacksquare

Now we investigate the various images contained in \mathfrak{A}^m of $A_{j,1}^{m+1}$. To simplify notation we continue the special notations in Lemma 3.19 and add some more

Notation 3.20. We write (A/B) for the good pair $(A_{j,1}^{m+1}/B_j^{m+1})$ and μ_j^{m+1} for $\mu(A_{j,1}^{m+1}/B_j^{m+1})$. Let C^d , for $1 \leq d \leq \nu = \nu_j^{m+1} = \mu_j^{m+1} - 1$ (since $\ell_j^{m+1} = 1$) enumerate the automorphic images over $B = B_j^{m+1}$ of $A_{j,1}^{m+1}$ that lie in \mathfrak{A}^m . Let $C_+^d = C^d \cap \mathfrak{A}^{m-1}$, $C_-^d = C^d - \mathfrak{A}^{m-1}$, $B_+ = B \cap \mathfrak{A}^{m-1}$, and $B_- = B - \mathfrak{A}^{m-1}$.

With this notation we continue to set the stage; now, we assume both $|A_{j,1}^{m+1}| > 1$ and $|A_{i,f}^m| > 1$ for relevant i, f . The second assumption follows from the first, when $\mu_j^{m+1} \geq 3$, by Lemma 3.22 but will be an issue in Section 4.4. Recall Definition 1.2.1 of $R(X, Y)$ and $R(X, Y, Z)$.

Lemma 3.21. *Assume that $A_{j,1}^{m+1}$ is G -invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each i, f such that $A_{i,f}^m \cap B \neq \emptyset$. Then, for any d with $1 \leq d \leq \nu = \nu_j^{m+1}$:*

- (1) For any i, f such that $A_{i,f}^m \cap B \neq \emptyset$, $C^d \cap A_{i,f}^m \neq \emptyset$, i.e., $C_-^d \neq \emptyset$.
- (2) Using Notations 3.20 and 1.2.1, $R(B_-, C_+^d) = \emptyset$ and $R(B_-, C_+^d, B_+) = \emptyset$. Thus, $\delta(B_-/B_+ \cup \bigcup_{1 \leq d \leq \nu} C_+^d) = \delta(B_-/B_+)$.
- (3) If $C^d \cap \mathfrak{A}^{m-1} = \emptyset$, that is $C_+^d = \emptyset$, then there is a unique petal $A_{i,f}^m$ that contains both C^d and B_- . So, $A_{i,f}^m$ is G -invariant.

Proof. (1) Lemma 1.2.5 implies for any $b \in B_-$, there must be $c_1, c_2 \in C$ with $R(c_1, c_2, b)$. If a $c_i \in \mathfrak{A}^m - \mathfrak{A}^{m-1}$, it must be in $A_{i,f}^m$ since the petals are freely joined. If both are in \mathfrak{A}^{m-1} , $|A_{i,f}^m| = 1$, for a contradiction. So one c_i must be in $A_{i,f}^m$.

(2) Since $|A_{i,f}^m| > 1$, for any $b \in B_-$ there do not exist $x_1, x_2 \in \mathfrak{A}^{m-1}$ such that $M \models R(b, x_1, x_2)$. Hence $r(B_-, B_+ \cup \bigcup_{d=1}^{\nu} C_+^d) = r(B_-, B_+) + \sum_{d=1}^{\nu} r(B_-, C_+^d)$. The conditions $|A_{j,1}^{m+1}| > 1$ and $C^d \cong_B A_{j,1}^{m+1}$ imply for any $c \in C^d$ there are no $b_1, b_2 \in B$ such that $M \models R(b_1, b_2, c)$. Consequently, $r(B_-, C_+^d) = 0$.

(3) Assume that $C^d \cap \mathfrak{A}^{m-1} = \emptyset$. Assume also that $C^d \cap A_{s,t}^m \neq \emptyset$ for some s and t , but $C^d \not\subseteq A_{s,t}^m$. Since petals $A_{u,v}^m$ are free over \mathfrak{A}^{m-1} we obtain that C^d is disconnected over B , contradicting Observation 3.10. So, there is a unique petal

$A_{i,f}^m$ that contains C^d . Obviously, then $B_- \subseteq A_{i,f}^m$. The assumption $A_{j,1}^{m+1}$ is G -invariant implies (Observation 2.14) that B is G -invariant. So, B_- , and thus, $A_{i,f}^m$ are G -invariant. ■

We prove a consequence of: $\mu(A_{j,1}^{m+1}) \geq 3$ and $A_{j,1}^{m+1}$ is G -invariant.

Lemma 3.22. *If $\mu(A_{j,1}^{m+1}) \geq 3$ and $A_{j,1}^{m+1}$ is G -invariant, then $\ell_j^{m+1} + 1 < \mu_j^{m+1}$ and $|A_{j,1}^{m+1}| > 1$ together imply $|A_{i,f}^m| > 1$ for any i, f such that $A_{i,f}^m \cap B \neq \emptyset$.*

Proof. Let $b \in B \cap A_{i,f}^m$. For $A_{i,f}^m$ there is a unique base¹⁷ B_i^m by Lemma 2.14. By the same observation as in Lemma 3.19.2, if $|A_{i,f}^m| = 1$, then $A_{i,f}^m = \{b\}$ and B_i^m is a pair $(c_1, c_2) \in \mathfrak{A}^{m-1}$ that satisfy $R(c_1, c_2, b)$. On the other hand, b satisfies $R(\alpha_1, \alpha_2, b)$ for some $\alpha_1, \alpha_2 \in A_{j,1}^{m+1}$ by Lemma 1.2.5. But, $|A_{i,f}^m| = 1$ implies there is no pair x, y from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ satisfying $R(x, y, b)$. Since $\ell_j^{m+1} + 1 < \mu_j^{m+1}$ and $\mu_j^{m+1} \geq 3$ there must be two disjoint embeddings of $A_{j,1}^{m+1}$ in \mathfrak{A}^m , this implies that some $d \in \mathfrak{A}^{m-1} - B_i^m$ is in relation with b ; this contradicts that $A_{i,f}^m$ is related only to elements of the doubleton B_i^m . ■

Lemma 3.23. *Let $m \geq 1$. Assume that $\delta(B) \geq 2$, $A_{j,1}^{m+1}$ is G -invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each i, f such that $A_{i,f}^m \cap B \neq \emptyset$ and $A_{i,f}^m$ is G -invariant. Further, assume that $C_+^d \neq \emptyset$ for each d . Then*

A) *If $\mu_j^{m+1} \geq 3$, then $A_{j,1}^{m+1}$ determines an $A_{i,f}^m$ and $\delta(B_+) = 0$.*

B) *If $\mu_j^{m+1} = 2$, then $2 \geq \sum_{i \in \mathcal{I}} \ell_i^m + \delta(B_+)$, where $\mathcal{I} = \{i : (\exists t) A_{i,t}^m \cap B \neq \emptyset\}$, and thus $\delta(B_+) \leq 1$.*

Proof. Most of the proof is the same for both A) and B); we split near the end. Let $\mathcal{I} = \{i : A_{i,t}^m \cap B \neq \emptyset \text{ for some } t\}$ and $\iota = |\mathcal{I}|$. Let $\nu = \mu(B_j^{m+1}, A_{j,1}^{m+1}) - 1$. The conditions $|A_{j,1}^{m+1}| > 1$ and $|A_{i,f}^m| > 1$ for each i and f such that $A_{i,f}^m \cap B_j^{m+1} \neq \emptyset$ imply that we may apply Lemma 3.21.2 below.

First, we show $\delta(B_-/B_+) \geq \nu \cdot \sum_{i \in \mathcal{I}} \ell_i^m$. For this, using notation 3.20, by Lemma 3.21.1 and the last hypothesis, both C_-^d and C_+^d are nonempty. So, invoking the definition of good pair we have $\delta(C_-^d \cap A_{i,f}^m/B \cup C_+^d) < 0$. Consequently, by submodularity of δ for any $i \in \mathcal{I}$ and $f = 1, \dots, \ell_i^m$, we can conclude:

$$(1) \quad \delta(C_-^d/B \cup C_+^d) = \sum_{i \in \mathcal{I}} \sum_{1 \leq f \leq \ell_i^m} \delta(C_-^d \cap A_{i,f}^m/B \cup C_+^d) \leq - \sum_{i \in \mathcal{I}} \ell_i^m.$$

Since $\mathfrak{A}^{m-1} \leq M$ and by monotonicity

$$(2) \quad 0 \leq \delta\left(\bigcup_{1 \leq d \leq \nu} C_-^d \cup B_-/\mathfrak{A}^{m-1}\right) \leq \delta\left(\bigcup_{1 \leq d \leq \nu} C_-^d \cup B_-/\bigcup_{1 \leq d \leq \nu} C_+^d \cup B_+\right).$$

But by the definition of $\delta(A/C)$, we can rewrite the last term to obtain

$$(3) \quad 0 \leq \delta\left(\bigcup_{1 \leq d \leq \nu} C_-^d/B \cup \bigcup_{1 \leq d \leq \nu} C_+^d\right) + \delta(B_-/(B_+ \cup \bigcup_{1 \leq d \leq \nu} C_+^d)).$$

¹⁷We will indicate a slight modification of the proof for the Steiner case in Lemma 5.14.

The last term of the right hand side of equation 3 equals $\delta(B_-/B_+)$ by Lemma 3.21.2. And, the first term satisfies

$$(4) \quad 0 \leq \delta\left(\bigcup_{1 \leq d \leq \nu} C_-^d/B \cup \bigcup_{1 \leq d \leq \nu} C_+^d\right) = \sum_{i \in \mathcal{I}} \sum_{\substack{1 \leq f \leq \ell_i^m \\ 1 \leq d \leq \nu}} \delta((C^d \cap A_{i,f}^m)/B \cup C_+^d) \leq -\nu \sum_{i \in \mathcal{I}} \ell_i^m$$

since each C^d contributes at most $-\sum_{i \in \mathcal{I}} \ell_i^m$ for each $d = 1, \dots, \nu$ by equation (1). Substituting our evaluations of the two terms on the right hand side of equation 3 (one from Lemma 3.21.2) and transposing, we have

$$(5) \quad \delta(B_-/B_+) \geq \nu \sum_{i \in \mathcal{I}} \ell_i^m.$$

Now, $B = B_+ \cup B_-$ implies $\delta(B) = \delta(B_-/B_+) + \delta(B_+)$. So

$$(6) \quad \delta(B) \geq \nu \sum_{i \in \mathcal{I}} \ell_i^m + \delta(B_+).$$

A) Assume that $\mu_j^{m+1} \geq 3$; then $\nu \geq 2$. If $\delta(B) = 2$, then $\nu = 2$, $\mathcal{I} = \{i\}$ for some i , $\ell_i^m = 1$, and $A_{i,1}^m$ is G -invariant.

If $\delta(B) \geq 3$, divide equation 6 by $\delta(B)$. Then, since $\nu \geq \delta(B) - 1$, substituting in equation 6, we obtain

$$(7) \quad 1 = \frac{\delta(B)}{\delta(B)} \geq \frac{(\delta(B) - 1) \sum_{i \in \mathcal{I}} \ell_i^m}{\delta(B)} + \frac{\delta(B_+)}{\delta(B)}$$

Since $\delta(B) \geq 3$, $\frac{\delta(B)-1}{\delta(B)} \geq \frac{2}{3}$ so

$$(8) \quad 1 \geq \frac{2 \sum_{i \in \mathcal{I}} \ell_i^m}{3} + \frac{\delta(B_+)}{\delta(B)}$$

Equation 8 implies $\mathcal{I} = \{i\}$ for some i , $\ell_i^m = 1$, and $A_{i,1}^m$ is G -invariant. Moreover, since the first term is not zero, $\delta(B_+) < 1$ and therefore $\delta(B_+) = 0$.

B) Now we assume that $\mu_j^{m+1} = 2$. Then, invoking the first hypothesis, $\delta(B) = 2$. In this case, since $\nu = 1$, equation 6 implies $2 \geq \sum_{i \in \mathcal{I}} \ell_i^m + \delta(B_+)$ and so $\delta(B_+) \leq 1$. \blacksquare

4. dcl IN HRUSHOVSKI'S FIRST EXAMPLE \hat{T}_μ

In this section we describe $\text{dcl}^*(I)$ in the main example \hat{T}_μ (Definition 0.1) in [Hru93]. We find that the values of μ for good pairs with $\delta(B) = 2$ distinguish whether $\text{dcl}^*(I)$ may be empty. Subsection 4.1 deals with dcl^* under a stronger hypothesis on μ to conclude $\text{dcl}^*(I) = \emptyset$. Subsection 4.2 provides an example that when $\mu(B, C) = 2$ for certain good pairs, there is a theory where $\text{dcl}^*(I) \neq \emptyset$. However, Subsection 4.4 sdcl^* shows that such examples still satisfy $\text{sdcl}^*(I) = \emptyset$ and fail elimination of imaginaries. That proof uses a deeper study of flowers and bouquets in Subsection 4.3.

The very raw idea is that if some petal A on the $(i+1)$ -th strata is G -invariant, and $\mu(A, B) \geq 3$ then it determines a petal on the i -th strata, which is also G -invariant, for each positive i ; but Lemma 3.11 implies that no petal on the 1st strata is G -invariant, for a contradiction. The hypothesis that $\mu(B, C) \geq 3$ for any good pair (B, C) with $\delta(B) = 2$ implies that this idea works and $\text{dcl}^*(I) = \emptyset$.

However in the general case $\text{dcl}^*(I)$ may not be empty. We consider in Section 4.4 a more complicated construction in order to prove that $\text{sdcl}^*(I) = \emptyset$.

Note, however that the family of theories described in Proposition 18 of [Hru93] (Steiner quasigroups) as well as the Steiner *triple* system of [BP20] have a truly binary function defined by R . The definition of \mathbf{K}_0 from δ in Hrushovski's Proposition 18 is non-standard; in linear space case, $\mu(\alpha) = 1$, for α the good pair of a line (Definition 1.2.6). Section 5 adapts our main results for Steiner systems.

4.1. G_I : No truly n -ary definable functions

We slightly vary Hrushovski's original example by adding a further adequacy requirement (Definition 1.1.7.1).

Definition 4.1.1. *We say that a function μ (or theory T_μ) bounding good pairs triples if for all good pairs (A/B) with $|A| > 1$, $\mu(A/B) \geq \delta(B)$:*

$$\delta(B) = 2 \Rightarrow \mu(A/B) \geq 3.$$

In this section with this triples condition, we G -decompose a finite set using G_I (fixing I pointwise) and show \hat{T}_μ is essentially unary and so eliminates imaginaries. We give a more refined argument using $G_{\{I\}}$ in Section 4.4 showing that even with truly n -ary function (i.e. dropping the 'triples' hypothesis), \hat{T}_μ must fail to eliminate imaginaries.

Theorem 4.1.2. *Assume that \hat{T}_μ triples. Let I be a finite independent set that contains at least 2 elements. Fix a G -normal $\mathfrak{A} \leq M \models \hat{T}_\mu$ with height m_0 . For every $m \leq m_0$, $\mathfrak{A}^m \cap \text{dcl}^*(I) = \emptyset$.*

Thus, $\text{dcl}^(I) \cap \mathfrak{A} = \emptyset$; so there is no truly n -ary \emptyset -definable function (Definition 2.9) for each $n \geq 2$ and \hat{T}_μ does not admit elimination of imaginaries. As a corollary, we obtain that $\text{dcl}(J) = \bigcup_{a \in J} \text{dcl}(a)$ for any independent set $J \subseteq M$.*

By the definition of \mathfrak{A}^0 , $\mathfrak{A}^0 \cap \text{dcl}^*(I) = \emptyset$. It suffices to show by induction on $m \geq 1$ that for each $e \in \mathfrak{A}^m$, $d(G_I(e)) \geq 2$. As, if $G_I(e) = \{e\}$ then $d(G_I(e)) = 1$. But we must begin with $m = 1$ since elements $e \in \mathfrak{A}^0$ may have $d(G_I(e)) = 1$. We obtain the conclusion by proving the following by double induction. Note that the truth of dim_m and moves_m each depend on the choice of G as G_I . Once we have this decomposition, satisfying moves for G_I -normal $\mathfrak{A} \leq M$ of any height, we can conclude $\text{dcl}_M^*(I) = \emptyset$.

Lemma 4.1.3. *Assume that \hat{T}_μ triples. For $m \geq 1$,*

- (1) dim_m : $d(E) \geq 2$ for any G_I -invariant set $E \subseteq \mathfrak{A}^m$, which is not a subset of \mathfrak{A}^0 .
- (2) moves_m : No $A_{f,k}^m$ is G_I -invariant.

The remainder of this section is devoted to the proof of Lemma 4.1.3. If m_0 is the height of \mathfrak{A} , then dim_{m_0} gives Theorem 4.1.2.

For each $m < m_0$, each \mathfrak{A}^{m+1} , and each j, k such that $A_{j,k}^{m+1} \subseteq \mathfrak{A}^{m+1}$, Observation 2.14 implies that if $A_{j,k}^{m+1}$ is G_I -invariant, then B_j^{m+1} is G_I -invariant. However the converse is false. The main part of the construction in Section 3 was to describe the family of G_I -conjugates $A_{j,k}^{m+1}$ over B_j^{m+1} of $A_{j,1}^{m+1}$. We now take into account that the B_j^{m+1} need not be G_I -invariant.

How do we use a joint induction? The $A_{j,i}^{m+1}$ are disjoint. If either $|J_j^{m+1}| > 1$ (Definition 3.14) or $\ell_j^m > 1$, $A_{j,i}^{m+1}$ moves and so no element of $A_{j,i}^{m+1}$ is definable over I . But, in Section 4.2, we show that when some $\mu_j^{m+1} = 2$, it may be that $|J_j^{m+1}| = 1$. So, in this section we add an hypothesis implying each relevant $\mu_j^{m+1} \geq 3$. In order to prove by induction on m that no $A_{j,i}^m$ is G_I -invariant (Conclusion 4.1.7), we need the dual hypothesis \dim_m . So, the simultaneous induction is organized as follows:

$$\begin{aligned} \dim_m \wedge \text{moves}_{m+1} &\Rightarrow \dim_{m+1} \text{ (Lemma 4.1.4)} \\ \dim_m \wedge \text{moves}_m &\Rightarrow \text{moves}_{m+1} \text{ (Conclusion 4.1.7)} \end{aligned}$$

In the following Corollary 4.1.5, we slightly modify the proof of Lemma 4.1.4 to ground the induction by showing \dim_1 and moves_1 .

We use without further notice the fact that for any $A \subseteq B$, $d(A) \leq d(B)$, e.g. $d(X) \leq d(G_I(X))$. Note that Construction 3.8 shows that any base B arising in the construction of \mathfrak{A}^{m+1} satisfies $1 \leq d(B) \leq v$, where $v = |I|$.

Lemma 4.1.4 is formulated for G ; it is applied for G_I in Section 4.1 while in Section 4.4 we use sdim_m instead of \dim_m^G with $G = G_{\{I\}}$. The ‘moves hypothesis’ in Lemma 4.1.4 will follow from the inductive hypothesis in the main proof.

Lemma 4.1.4. *Fix m with $1 \leq m \leq m_0$. If¹⁸ \dim_m^G and G moves $A_{j,k}^{m+1}$ then \dim_{m+1}^G . That is, for each $e \in A_{j,k}^{m+1}$, $d(G(e)) \geq 2$.*

Proof. Fix $m \leq m_0$, $j < q_m$, k and e with $e \in A_{j,k}^{m+1}$. We show $d(G(e)) \geq 2$. Let $E = G(e)$. Clearly $d(E) \leq v$, since $e \in \text{acl}(I)$. By Lemma 3.7.2, $E \cap A_{j,k}^{m+1} \neq \emptyset$ for each $k \leq \ell_j^{m+1}$. And for each $j' \in J_j^{m+1}$ the map taking B_j^{m+1} to $B_{j'}^{m+1}$ and Construction 3.8, guarantee each $E \cap A_{j',k'}^{m+1} \neq \emptyset$ for $k' \leq \ell_{j'}^{m+1} = \ell_j^{m+1}$.

Note that $\text{icl}(E) \subseteq \mathfrak{A}^{m+1}$, because $\mathfrak{A}^{m+1} \leq M$. The proof now breaks into three cases.

(1) If all the bases $B_{j'}^{m+1}$ are subsets of $\text{icl}(E)$ (for $j' \in J_j^{m+1}$), the hypothesis \dim_m and the monotonicity of d implies

$$2 \leq d\left(\bigcup_{g \in G_I} g(B_j^{m+1})\right) \leq d(\text{icl}(E)) \leq v$$

and we finish.

(2) Suppose no $B_{j'}^{m+1}$ (with $j' \in J_j^{m+1}$) is a subset of $\text{icl}(E)$. For each $j' \in J_j^{m+1}$ there is a t' such that $A_{j',t'}^{m+1} \cap \text{icl}(E) \neq \emptyset$. And since $B_{j'}^{m+1} \not\subseteq \text{icl}(E)$, Definition 1.1.5.5 of base¹⁹ implies $\delta(\text{icl}(E) \cap A_{j',t'}^{m+1} / \text{icl}(E) \cap \mathfrak{A}^m) \geq 1$. Thus,

$$\begin{aligned} v &\geq d(E) = \delta(\text{icl}(E)) = \delta((\text{icl}(E) - \mathfrak{A}^m) / \text{icl}(E) \cap \mathfrak{A}^m) + \delta(\text{icl}(E) \cap \mathfrak{A}^m) \geq \\ &\geq \delta((\text{icl}(E) - \mathfrak{A}^m) / \text{icl}(E) \cap \mathfrak{A}^m) \geq \sum_{j' \in J_j^{m+1}} \sum_{t=1}^{\ell_{j'}^{m+1}} \delta(\text{icl}(E) \cap A_{j',t}^{m+1} / \text{icl}(E) \cap \mathfrak{A}^m) \geq 2. \end{aligned}$$

¹⁸We do not use the adequacy hypothesis (Definition 4.1.1) in proving this lemma.

¹⁹That is, the base X of C/A is the least X such that $\delta(A'/X) = 0$ for every $A' \subseteq A$.

The double summation is at least 2 because either $|J_j^{m+1}| \geq 2$ and there are 2 non-zero outer summands or $J_j^{m+1} = \{j\}$ and since G_I moves $A_{j,1}^{m+1}$,

$$\sum_{t=1}^{\ell_j^{m+1}} \delta(\text{icl}(E) \cap A_{j,t}^{m+1} / \text{icl}(E) \cap \mathfrak{A}^m) \geq 2.$$

(3) Suppose one of the $B_{j'}^{m+1}$ (with $j' \in J_j^{m+1}$) is a subset of $\text{icl}(E)$ and another one $B_{j''}^{m+1}$ is not. Clearly,

$$d(E) = \delta(\text{icl}(E)) = \delta(\text{icl}(E) \cap \mathfrak{A}^m) + \delta(\text{icl}(E) - \mathfrak{A}^m / \text{icl}(E) \cap \mathfrak{A}^m).$$

Say $B_{j_1}^{m+1} \subseteq \text{icl}(E)$ and $B_{j_2}^{m+1} \not\subseteq \text{icl}(E)$. The first summand is at least 1, because $B_{j_1}^{m+1}$ is a subset of $\text{icl}(E) \cap \mathfrak{A}^m$, but is not a subset of $\text{acl}(\emptyset)$; as, otherwise $A_{j_1,1}^{m+1}$ is a subset of $\text{acl}(\emptyset)$. The second summand is also at least one, because

$$\delta(\text{icl}(E) - \mathfrak{A}^m / \text{icl}(E) \cap \mathfrak{A}^m) = \sum_{j' \in J_{j_2}^{m+1}} \sum_{t=1}^{\ell_{j'}^{m+1}} \delta(\text{icl}(E) \cap A_{j',t}^{m+1} / \text{icl}(E) \cap \mathfrak{A}^m)$$

and for some $t \leq \ell_{j_2}^{m+1}$, $\delta(\text{icl}(E) \cap A_{j_2,t}^{m+1} / \text{icl}(E) \cap \mathfrak{A}^m) \geq 1$, because $B_{j_2}^{m+1} \not\subseteq \text{icl}(E)$. \blacksquare

We establish the $m = 1$ step of Lemma 4.1.3 by emulating the proof of Lemma 4.1.4; we can't really apply the result as \dim_0 was not defined.

Corollary 4.1.5. *Both \dim_1 and moves_1 hold. More precisely, for any $e \in \mathfrak{A}^1 - \mathfrak{A}^0$, $d(G_I(e)) \geq 2$.*

Proof. By Lemma 3.11, we have moves_1 and for each j , $d(B_j^1) \geq 2$. Now follow the proof of Lemma 4.1.4, noting that it only uses that $d(B_j^1) \geq 2$. \blacksquare

Lemma 4.1.6. *If moves_m , \dim_m and $A_{j,1}^{m+1}$ is G_I -invariant, then*

- (1) For $B = B_j^{m+1}$, $d(B) \geq 2$.
- (2) $|A_{j,1}^{m+1}| > 1$.
- (3) If in addition, $\mu_j^{m+1} \geq 3$, then $|A_{i,f}^m| > 1$ for any i, f such that $A_{i,f}^m \cap B \neq \emptyset$.

Proof. By Lemma 4.1.5, we may assume $m \geq 1$. (1) By Observation 2.14, $A_{j,1}^{m+1}$ is G_I -invariant implies B_j^{m+1} is G_I -invariant. Then, since $B \not\subseteq \mathfrak{A}^0$, \dim_m implies that $d(B) \geq 2$.

(2) By 1) we may apply Lemma 3.19(2) to conclude that if $A_{j,1}^{m+1} = \{c\}$, then $A_{i,1}^m$ is G_I -invariant for some i , contradicting moves_m .

(3) Directly follows from Lemma 3.22. \blacksquare

With the next result we can complete the induction. The hypothesis that each $\mu_j^{m+1} \geq 3$ is essential for the induction.

Conclusion 4.1.7. *Under the hypotheses of Theorem 4.1.2, move_m and \dim_m imply move_{m+1} .*

Proof. Assume for contradiction that $A_{j,1}^{m+1}$ is G_I -invariant. The assumption \dim_m and Lemma 4.1.6 imply both $d(B) \geq 2$, so $\delta(B) \geq 2$, and that the hypotheses of Lemma 3.21 hold. And Lemma 3.21 gives the remaining hypotheses of Lemma 3.23. Indeed, if $C_+^d = \emptyset$ for some d , we obtain that $A_{i,f}^m$ is G_I -invariant for some i, f by

Lemma 3.21.3; that contradicts moves_m . Now, by Lemma 3.23 we obtain that $A_{i,1}^m$ is G_I -invariant for some i ; that contradicts moves_m . \blacksquare

Completing this induction gives Lemma 4.1.3 and so Theorem 4.1.2, asserting there are no \emptyset -definable truly n -ary functions. Now we generalize the result by allowing parameters (Corollary 4.1.10).

Lemma 4.1.8. *Let I be a finite independent set and $J \subset I$. Let a tuple $\mathbf{c} \in \text{acl}(J)$ and $d \in \text{acl}(I)$. Let \mathfrak{A}_I be the G_I -normal closure of $I \cup \{d, \mathbf{c}\}$, and \mathfrak{A}_J the G_J -normal closure of $J \cup \{\mathbf{c}\}$. Then there exists a tree decomposition \mathcal{T}_I of \mathfrak{A}_I such that the collection \mathcal{T}_J of petals $A_{f,j}^m$ that are based in \mathfrak{A}_J are a downwardly closed subset of \mathcal{T}_I , whose union is \mathfrak{A}_J . (By downwardly closed we mean that if $A_{f,j}^m \subseteq \mathfrak{A}_J$ then $B_f^j \subseteq \mathfrak{A}_J$.)*

Proof. We note $G_J(\mathbf{c}) = G_I(\mathbf{c})$ is the finite collection of automorphic images of \mathbf{c} over J , as for any automorphism ρ fixing J pointwise $\rho \upharpoonright \mathfrak{A}_J$ can be extended to one fixing I . By this equality and by monotonicity of the intrinsic closure we obtain

$$\mathfrak{A}_J = \text{icl}(J \cup G_J(\mathbf{c})) = \text{icl}(J \cup G_I(\mathbf{c})) \subseteq \text{icl}(I \cup G_I(d, \mathbf{c})) = \mathfrak{A}_I$$

Thus, $\mathfrak{A}_J \subseteq \mathfrak{A}_I$. Let $L_J = \langle X_i : i \leq r_J \rangle$, where $X_0 \leq X_1 \leq \dots \leq X_{r_J}$, be a linear decomposition of \mathfrak{A}_J ; that is, X_{i+1} is a 0-primitive extension of X_i for each $i < r_J$ and $X_{r_J} = \mathfrak{A}_J$. Since $I - J$ is independent over J , $I - J$ is independent over \mathfrak{A}_J , moreover $I - J$ and \mathfrak{A}_J are fully independent. Then

$$I = X'_0 \leq X_1 \cup (I - J) \leq \dots \leq X_{r_J} \cup (I - J)$$

is a linear chain of 0-primitive extensions and for each $i \leq r_J$ the base of X_{i+1} is a subset of \mathfrak{A}_J . Thus, it can be considered as an initial segment of a linear decomposition L_I of \mathfrak{A}_I by Lemma 3.4. Let \mathcal{T}_K be the tree decomposition of \mathfrak{A}_K determined by L_K for $K \in \{I, J\}$. We now show that if a petal in \mathfrak{A}_I intersects \mathfrak{A}_J , it is one of $X_{i+1} - X_i$ and a subset of \mathfrak{A}_J . Clearly, $\{\text{acl}(a_i) \cap \mathfrak{A}_I : a_i \in J\} \subseteq \{\text{acl}(a_i) \cap \mathfrak{A}_I : a_i \in I\}$ and $\{\text{acl}(a_i) \cap \mathfrak{A}_I : a_i \in J\} \subseteq \mathfrak{A}_J^0$ by Construction 3.8. So we only have to show the result for petals of \mathcal{T}_I of the form $A_{f,j}^{m+1}$. Note that any $A_{f,j}^{m+1}$ that nontrivially intersects \mathfrak{A}_J is contained in \mathfrak{A}_J , since $\mathfrak{A}_I^m \cup \mathfrak{A}_J \leq M$ and $A_{f,j}^{m+1}$ is 0-primitive over \mathfrak{A}_I^m .

We need to show $A_{f,j}^{m+1} \in \mathcal{T}_J$.

Level: 1 If $m = 0$ and $B_f^{m+1} \not\subseteq \bigcup_{a \in J} \text{acl}(a)$, then $A_{f,j}^{m+1} \not\subseteq \mathfrak{A}_J$ by Construction 3.8. So, $A_{f,j}^{m+1} \subseteq \mathfrak{A}_J^1$ implies $B_f^{m+1} \subseteq \mathfrak{A}_J$.

Level: $m + 1$ Since $A_{f,j}^{m+1} \subseteq \mathfrak{A}_J$ and is some \hat{X}_i of the given initial segment of the linear decomposition, $B_f^{m+1} \subseteq \mathfrak{A}_J$; so $A_{f,j}^{m+1} \in \mathcal{T}_J$.

By induction we have Lemma 4.1.8. Since \mathfrak{A}_J is G_J -normal, $\bigcup \mathcal{T}_J = \mathfrak{A}_J$. \blacksquare

We have the following immediate corollary. Let r be the height (i.e. the largest index k of an $A_{f,i}^k$ with $A_{f,i}^k \subseteq \mathfrak{A}_J$) of \mathfrak{A}_J .

Lemma 4.1.9. *For any $m \leq r$, and any $A_{p,q}^{m+s+1} \subseteq \mathfrak{A}_J - \mathfrak{A}_I^m$, and for any $A_{f,j}^{m+1} \subseteq \mathfrak{A}_I^{m+1} - \mathfrak{A}_J$, $r(A_{p,q}^{m+s+1}, A_{f,j}^{m+1}) = 0$.*

Proof. If $s = 0$ the result is clear since the petals over \mathfrak{A}_I^m are fully independent. If $0 < s < r - m$, we have shown in Lemma 4.1.8 for any $m \leq r$, any p, q with $A_{p,q}^{m+s+1} \subseteq \mathfrak{A}_J - \mathfrak{A}_I^m$, that $B_p^{m+s+1} \subseteq \mathfrak{A}_J^{m+s}$. So for any $s \geq 1$, $r(A_{p,q}^{m+s+1}, \mathfrak{A}_I^{m+s} -$

$\mathfrak{A}_J) = 0$. In particular, $r(A_{p,q}^n, A_{f,j}^{m+1}) = 0$ for any $A_{f,j}^{m+1} \subseteq \mathfrak{A}_I - \mathfrak{A}_J$, and any $A_{p,q}^n \subseteq \mathfrak{A}_J$ with $m+1 \leq n \leq r$. ■

Corollary 4.1.10. *Assume that \hat{T}_μ triples. Then, for $n > 1$, no truly n -ary function is definable in \hat{T}_μ even with parameters.*

Proof. Let $M \models \hat{T}_\mu$ and suppose $\phi(y, \mathbf{x}, \mathbf{c})$ defines a truly n -ary function $y = g(\mathbf{x})$ on M^n . Taking M saturated, we can choose a_1, a_2, \dots, a_n independent over \mathbf{c} . Fix \mathbf{c}' maximal independent inside \mathbf{c} . Then $I = \{a_1, \dots, a_n\} \cup \{x \in \mathbf{c}'\}$ is independent. Let $\mathfrak{A} = \mathfrak{A}_I$ be the G_I -normal closure of $I \cup \{d, \mathbf{c}\}$, where $d = g(\mathbf{a})$.

For each $q \in \{1, \dots, n\}$ we define $I_q = \{a_q\} \cup \{x \in \mathbf{c}'\}$. Clearly, I_q is independent as a subset of the independent set I . Let \mathfrak{A}_q be the G_{I_q} -normal closure of $\{c\}$, that is, $\text{icl}(I_q \cup G_{I_q}(\mathbf{c}))$. We apply Lemma 4.1.8 with $J = I_q$.

We now consider two cases. First, assume that d is not in $\bigcup_{q=1}^n \mathfrak{A}_q$. Let $d \in A_{f,j}^{m+1}$, where $A_{f,j}^{m+1}$ is a petal in the G_I -decomposition of \mathfrak{A} . Applying moves $_{m+1}$ to \mathfrak{A} we obtain that there exists $A_{f,j'}^{m+1}$ with $j \neq j'$. By strata decomposition there exists $\tau \in \text{aut}(M/\mathfrak{A}^m)$ such that $\tau(A_{f,j}^{m+1}) = A_{f,j'}^{m+1}$.

Since q is arbitrary, Lemma 4.1.9 implies $\sum_{q=1}^n r(A_{f,j}^{m+1}, \mathfrak{A}_q - \mathfrak{A}^m) = 0$. But we need more, namely, $r(A_{f,j}^{m+1}, \bigcup_{q=1}^n \mathfrak{A}_q - \mathfrak{A}^m) = 0$. Assume the contrary, that there exist $a \in A_{f,j}^{m+1}$, $b \in \mathfrak{A}_q - \mathfrak{A}^m$, and $c \in \mathfrak{A}_p - (\mathfrak{A}^m \cup \mathfrak{A}_q)$ with $R(a, b, c)$. Not both b and c can be in \mathfrak{A}^{m+1} , because they are in different petals but petals on the same strata are fully independent. Then at least one of them, say b , is not in \mathfrak{A}^{m+1} . Let $b \in A_{u,v}^{m+k+1}$, then $a \in B_u^{m+k+1}$ and $B_u^{m+k+1} \not\subseteq \mathfrak{A}_q$, for a contradiction.

Now we finish the proof of the first case. Let $\rho = \tau \upharpoonright (\mathfrak{A}^m \cup A_{f,j}^{m+1}) \cup \text{id}_{\bigcup_{q=1}^n \mathfrak{A}_q - \mathfrak{A}^m}$. Recall, that $\tau \upharpoonright \mathfrak{A}^m$ is the identity, so ρ moves only $A_{f,j}^{m+1}$ to $A_{f,j'}^{m+1}$ and fixes $\mathfrak{A}^m \cup \bigcup_{q=1}^n \mathfrak{A}_q$. By our last remark, $r(A_{f,j'}^{m+1}, \mathfrak{A}^m \cup \bigcup_{q=1}^n \mathfrak{A}_q) = r(A_{f,j}^{m+1}, B_f^{m+1})$. Taking into account $A_{f,j}^{m+1} \cup B_f^{m+1} \cong_{B_f^{m+1}} A_{f,j'}^{m+1} \cup B_f^{m+1}$, we obtain ρ is a partial isomorphism and can be extended to an isomorphism ρ' of M , as its domain is closed in M . So, ρ' fixes \mathbf{a} and \mathbf{c} , but, by choice of case, moves d , for a contradiction.

Theorem 4.1.10 by doing the second case: $d \in \bigcup_{q=1}^n \mathfrak{A}_q$, say, $d \in \mathfrak{A}_1$. Let a'_n be independent over \mathbf{ac} . Let $\rho \in \text{aut}(M/\mathbf{c})$ fix a_q for $q < n$ and move a_n to a'_n . Then $\rho \in G_{I_1}$, so $\rho(d) = d$ and $d = g(a_1, \dots, a_{n-1}, a'_n)$, that contradicts f is truly n -ary. ■

Remark 4.1.11. Note that depending on whether \hat{T}_μ triples, there may or may not be a truly n -ary function. [EF12, Theorem 3.1] show that for any \hat{T}_μ the geometry (i.e. of the countable saturated model M), with any finite set X with $|\text{acl}_M(X)|$ infinite named, is isomorphic to that of the ω -stable version of the construction²⁰. Since $\text{acl}(\emptyset)$ is easily made infinite e.g. [BP20, Lemma 5.26], it is easy to construct examples with the same geometry.

We show now that for our general situation $\text{acl}_M(X)$ is infinite for any finite X For this, it is sufficient to show that for any finite set B there are infinitely many pairwise non-isomorphic good pairs A/B . Let $B = \{b_1, \dots, b_k\}$. Let $n \geq \max\{3, k\}$ and $A_n = \{a_1, \dots, a_n\}$. Let $\langle c_1, \dots, c_n \rangle$ be a sequence over B , that contains each element of B . We put $R(a_i, a_{i+1}, c_i)$ for each $i \in \{1, \dots, n-1\}$ and $R(a_n, a_1, c_n)$.

²⁰Mermelstein (personal communication) has shown this result extends to Steiner systems.

In contrast when $B = \emptyset$ the Hrushovski adequacy condition can be satisfied when $\mu(A/\emptyset) = 0$ for any A primitive over \emptyset . And it is not hard to show that amalgamation still holds [Bal21a]. But, in such a case the geometries are not elementarily equivalent as the formula $(\forall x)D_1(x)$ holds in the pregeometry with $\text{acl}(\emptyset) = \emptyset$ where D_n is a predicate that holds of n independent elements.

Thus the varied behavior of our examples show the coarseness of classifying only by geometry.

4.2. Counterexample

Let M be any model of \hat{T}_μ with $\mu(\alpha) = 2$. The following example satisfies $\text{dcl}^*I \neq \emptyset$ for $I = \{a_1, a_2\}$. This shows the assumption that $\mu(A/B) \geq 3$ for any good pair (A/B) , where $\delta(B) = 2$, is essential to show $\text{dcl}^*I = \emptyset$ (Theorem 4.1.2).

We sketch the motivation for the example. Recall, that in the decomposition of \mathfrak{A} into strata we have the 0-strata \mathfrak{A}^0 , that is obtained as $\mathfrak{A}^0 = \bigcup_{i=1}^2 (\mathfrak{A} \cap \text{acl}(a_i))$. We have used the hypothesis, $\mu(A/B) \geq 3$ for any good pair (A/B) with $\delta(B) = 2$, twice in the proof of Lemma 3.23. Recall Example 3.13, which shows that it is possible that \mathfrak{A} is $G_{\{I\}}$ -invariant and $A_{1,1}^2$ is $G_{\{I\}}$ -invariant. However, in that example the elements c_1 and c_2 are indiscernible over $\{a_1, a_2\}$. This happens because the elements a_1 and a_2 are indiscernible over $\{b_1, b_2\}$ and $\{a_1, a_2\} \cup \{b_1, b_2\} \cong_{\{b_1, b_2\}} \{c_1, c_2\} \cup \{b_1, b_2\}$. Below we modify $A_{1,1}^1$ so that a_1 and a_2 are no longer indiscernible over $A_{1,1}^1 \cup A_{1,2}^1$.

Let $tp(a_1/A_{1,1}^1) \neq tp(a_2/A_{1,1}^1)$: for instance the number of relations of a_1 with $A_{1,1}^1$ and the number of relations of a_2 with $A_{1,1}^1$ are different. Then $A_{1,1}^2$ contains a copy of a_1 , or a copy of a_2 , or both. But if a_1 and a_2 are distinguishable over B_1^1 inside \mathfrak{A}^1 (realize different types over B_1^1), then their copies α_1, α_2 in $A_{1,1}^2$ are distinguishable, too. But then the α_i can belong to $\text{dcl}^*(I)$.

The following construction describes an accessible case of the general strategy described in Remark 4.2.3.

Example 4.2.1. We consider the following example with universe $\{a_1, a_2\} \cup \{c_1, c_2, c_3\}$ and then 9 more points with d, δ, γ replacing c . We define the following relations:

- 0) $\mathfrak{A}^0 = I = \{a_1, a_2\}$.
- 1, 1) $A_{1,1}^1 = \{c_1, c_2, c_3\}$ with $R(a_1, c_1, c_3)$, $R(a_2, c_1, c_2)$, and $R(a_2, c_2, c_3)$.
- 1, 2) $A_{1,2}^1 = \{d_1, d_2, d_3\}$ with $R(a_1, d_1, d_3)$, $R(a_2, d_1, d_2)$, and $R(a_2, d_2, d_3)$.
- 2, 1) $A_{1,1}^2 = \{\alpha_1, \alpha_2, \gamma_1, \delta_1, \gamma_3, \delta_3\}$ with the following relations: $R(\alpha_1, \gamma_1, \gamma_3)$, $R(\alpha_2, \gamma_1, c_2)$, $R(\alpha_2, c_2, \gamma_3)$, and $R(\alpha_1, \delta_1, \delta_3)$, $R(\alpha_2, \delta_1, d_2)$, $R(\alpha_2, d_2, \delta_3)$.

Set $\mu(A_1^1) = \mu(A_1^2) = 2$. In the diagrams, we represent a triple satisfying R by a triangle.

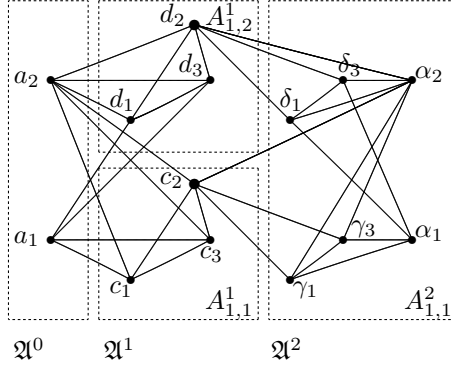
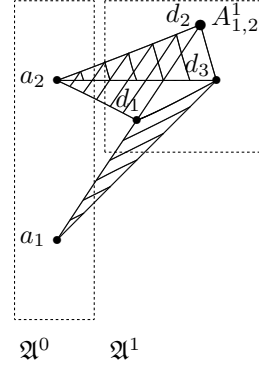
FIGURE 5. $\text{dcl}^*(I) \neq \emptyset$ FIGURE 6. \mathfrak{A}^1

Figure 6 shows by shaded triangles the R -triples in $I \cup \{d_1, d_2, d_3\}$. The petals $A_{1,1}^1$ and $A_{1,2}^1$ are isomorphic over I .

Clearly, $G_I(c_2) = \{c_2, d_2\}$, because there is no relation of either of these elements with a_1 and there are two relations of each one with a_2 . By mapping the point with the Greek label to the corresponding Roman one, we show that $A_{1,1}^2 = \{\alpha_1, \alpha_2, \gamma_1, \gamma_3, \delta_1, \delta_3\}$ is isomorphic to $\mathfrak{A}^1 - B_1^2 = \{a_1, a_2, c_1, c_3, d_1, d_3\}$ over $B_1^2 = \{c_2, d_2\}$.

It is routine to check that $\mathfrak{A}^1 - B_1^2$ is 0-primitive over B_1^2 . Obviously, $A_{1,1}^2$ is G_I -invariant. The element α_2 is a unique element in $A_{1,1}^2$ which is in 4 relations in \mathfrak{A}^2 , so $\alpha_2 \in \text{dcl}^*(I)$.

Remark 4.2.2. (1) In Example 4.2.1 moves₁ and dim₁ one hold; but $\mu_{1,1}^2 = 2$ so we cannot apply Lemma 3.23.A to conclude moves₂. In fact, $A_{1,1}^2$ is G_I -invariant.

(2) Note that this example is not a linear space (Section 5); if it satisfied the linear space axiom each of $A_{1,1}^1$ and $A_{1,2}^1$ would be a clique.

(3) α_2 is in $\text{dcl}^*(I)$ but not in $\text{sdcl}^*(I)$, because an automorphism which swaps a_1 and a_2 cannot preserve $A_{1,1}^1 \cup A_{1,2}^1$, since in \mathfrak{A}^1 , a_1 is in two relations and a_2 is in four relations. Thus this structure is G_I -invariant but not $G_{\{I\}}$ -invariant. In order to build $\mathfrak{A}^{G_{\{I\}}}$ we add new copies of $A_{1,1}^1$ and $A_{1,2}^1$:

2, 1) $A_{2,1}^1 = \{c'_1, c'_2, c'_3\}$ with $R(a_2, c'_1, c'_3)$, $R(a_1, c'_1, c'_2)$, and $R(a_1, c'_2, c'_3)$.

2, 2) $A_{2,2}^1 = \{d'_1, d'_2, d'_3\}$ with $R(a_2, d'_1, d'_3)$, $R(a_1, d'_1, d'_2)$, and $R(a_1, d'_2, d'_3)$.

Now there is an $f \in G_{\{I\}}$ with $f(a_0) = a_1$, $f(a_1) = a_0$ that maps $A_{1,i}^1$ to $A_{2,i}^1$. One can construct an $A_{2,1}^2$ containing α'_2 that is the image of α_2 under f .

(4) Note that $\{A_{1,1}^1, A_{1,2}^1\}$ is called a flower of $A_{1,1}^1$ over its base B_1^1 in Definition 4.3.1. Also note that $\{A_{2,1}^1, A_{2,2}^1\}$ is another flower of $A_{1,1}^1$ over its base B_1^1 . The difference is that if we arrange B_1^1 for the first considered flower as $\langle a_1, a_2 \rangle$, then for the second flower the arrangement of B_1^1 must be $\langle a_2, a_1 \rangle$.

(5) In Definition 4.3.3, we call the collection $\{\{A_{1,1}^1, A_{1,2}^1\}, \{A_{2,1}^1, A_{2,2}^1\}\}$ of all of these flowers a bouquet.

We now explain here the methodology and motivation for constructing a set with non-empty $\text{dcl}^*(I)$. It may be useful for further examples.

Remark 4.2.3. Let E_i be a subset of $\text{acl}(a_i)$, for $i \in \{1, 2\}$ such that $\delta(E_1) = \delta(E_2) = 1$ (that is, $E_i \leq M$). The most simple case is $E_i = \{a_i\}$. Let $B_1^1 = E_1 \cup E_2$. Then $\delta(B_1^1) = 2$. Let $A_{1,1}^1$ be any set that is good over B_1^1 . We put $\mu(A_{1,1}^1/B_1^1) = 2$. So, $\mathfrak{A}^1 = \text{icl}(I \cup E_1 \cup E_2) \cup A_{1,1}^1 \cup A_{1,2}^1$.

We choose one element b_i from $A_{1,i}^1$ for $i = 1, 2$. Let $B_1^2 = \{b_1, b_2\}$. Then $\delta(B_1^2) = \delta(\mathfrak{A}^1) = 2$ and there is a chain $B_1^2 = X_0 \leq X_1 \leq \dots \leq X_r = \mathfrak{A}^1$ such that X_{i+1} is a 0-primitive extension of X_i . So, X_1 is a 0-primitive extension of B_1^2 and is a subset of \mathfrak{A}^1 . We must choose $A_{1,1}^1$ and B_1^2 so that X_1 is good over B_1^2 . This is not true in general²¹, but Example 4.2.1 shows it can be done.

We are going to find $A_{1,2}^2$ inside \mathfrak{A}^1 in order to make $A_{1,1}^2$ G_I -invariant. Suppose there are m copies of X^1 over B_1^2 that are inside \mathfrak{A}^1 ; put $\mu(X_1/B_1^2) = m + 1$. Let $A_{1,1}^2$ be the $(m+1)$ -th copy of X_1 over B_1^2 . Obviously, $A_{1,1}^2$ is not in \mathfrak{A}^1 . We put $\mathfrak{A}^2 = \mathfrak{A}^1 \cup A_{1,1}^2$. If B_1^2 is fixed pointwise by G_I (that is, b_1 is definable in $A_{1,1}^1$ over I and b_2 its copy in $A_{1,2}^1$), then $\ell_1^2 = 1$ and $|J_1^2| = 1$.

The following is not essential to achieving Remark 4.2.3 but is mandated by the construction.

Claim 4.2.4. *The intersection $X_1 \cap B_1^1$ is not empty.*

Proof. This follows from the fact that $A_{1,1}^1$ and $A_{1,2}^1$ are free over \mathfrak{A}^0 . Indeed, assume that $X_1 \cap B_1^1 = \emptyset$. Then

$$0 = \delta(X_1/B_1^2) = \delta(X_1 \cap A_{1,1}^1/B_1^2) + \delta(X_1 \cap A_{1,2}^1/B_1^2) + \delta(X_1 - (A_{1,1}^1 \cup A_{1,2}^1)/B_1^2)$$

because there are no relations between $A_{1,1}^1$, $A_{1,2}^1$ and $\mathfrak{A}^0 - B_1^1$. Then each of these predimensions is equal to 0; that contradicts the definition of a good pair. ■

4.3. Bouquets and Flowers

In Remark 4.2.2.3, we noted that to make a G_I -normal structure $G_{\{I\}}$ -normal we required not only an image of a 0-primitive A with base B but an image $\pi(A)$ for a π in $G_{\{I\}}$ that fixes B setwise but not pointwise. The analysis of the case where there are good pairs A/B with $\delta(B) = 2$ and $\mu(A/B) = 2$ requires a much finer analysis of the second realization of A . We introduce here some further notation to describe the situation and illustrate them in Example 4.3.4.

²¹Here is a counterexample. To begin with we find a 0-primitive extension of a one element set, $\{b\}$. We consider A' as four points c_1, c_2, d_1, d_2 satisfying $R(c_1, c_2, b)$, $R(d_1, d_2, b)$, $R(c_1, d_1, b)$, $R(c_2, d_2, b)$. There are 5 points 4 edges and any subset has larger δ .

We would like to make this structure 0-primitive over a_1, a_2 . It needs one more trick. Replace A' by A by adding a point c_3 to A' and replacing the edge $R(c_1, c_2, b)$ by two edges $R(c_1, c_2, c_3), R(c_2, c_3, b)$. Then, A is 0-primitive over $\{b\}$. Now consider two new elements $A_0 = \{a_1, a_2\}$; we want A 0-primitive over A_0 . For this let the new relations be $R(c_1, c_2, a_1), R(d_1, d_2, a_2)$.

So while the discussion here is fine for motivating the example it doesn't suffice to show that \mathfrak{A}^1 with pair $A_{1,1}^1, A_{1,1}^1$ must contain an $A_{1,1}^2$ good over one point from each.

Definition 4.3.1 (flower). *Let A/B be a good pair. A flower \mathcal{F} of A/B in a set \mathfrak{D} is the set of images of isomorphisms of A over B into a \mathfrak{D} which fix B pointwise. The elements of the flower are called petals²².*

Suppose $\mathfrak{D} \cap A \subseteq \mathfrak{A}$ and \mathfrak{A} is G -normal. A certificate \mathcal{C} of A/B (witnessing $\chi_{\mathfrak{A}}(A/B) = \mu(A/B)$) is a maximal disjoint set of images of isomorphisms of A over B into \mathfrak{A} which fix B pointwise.

When A/B is well-placed, a flower \mathcal{F} contains at least one certificate \mathcal{C} for $\chi_M(A/B) = \mu(A/B)$ and, since each intersection decreases δ , $|\mathcal{F}| \leq \mu(A/B) + \delta(B)$. Moreover any pair of petals from distinct certificates (or flowers) that intersect are in $\text{icl}(B) \subseteq \mathfrak{A}^m$, for the least m such that $B \subseteq \mathfrak{A}^m$.

Of course each petal $C \in \mathcal{F}$ is isomorphic to A over B . \mathfrak{D} will usually be fixed in context as either the generic M or a G -decomposable \mathfrak{A} (e.g. an \mathfrak{A}^m). Note that there only finitely many certificates of A/B in M ; a very crude upper bound is $\mu(A/B) \times |A|!$.

Note that in the description of the class \mathbf{L}_μ one put the upper bound on the cardinality of a certificate of a good pair A/B —it does not exceed $\mu(A/B)$.

When we write two structures C and D are equal we mean they have both the same domain and each symbol in the vocabulary has the same interpretation in each. For a substructure X of M , $\text{diag}_X(\mathbf{x})$ denotes the diagram of X , with respect to a fixed enumeration \mathbf{x} of the domain of X .

Notation 4.3.2. *Let $\mathbf{b} = \langle b_1, \dots, b_n \rangle$ enumerate B and $\gamma \in \text{aut}(B)$; write \mathbf{b}^γ for $\langle \gamma(b_1), \dots, \gamma(b_n) \rangle$.*

Any sequence \mathbf{c} that satisfies $\text{diag}_{A \cup B}(\mathbf{x}, \mathbf{b}^\gamma)$ determines an enumeration of a petal of the flower of A/B . The set enumerated by this sequence is a petal F_i^γ . Each F_i^γ may have multiple enumerations that satisfy the fixed diagram. A flower \mathcal{F}^γ of A/B is a maximal set $\{F_i^\gamma : i < r^\gamma\}$ of such petals.

Note that for fixed γ there may be different certificates. Any two such certificates must have at least one pair of intersecting petals (by maximality). But distinct flowers A/B and A'/B cannot have a common petal. If f and g map A and A' while fixing B pointwise, then $g^{-1} \circ f$ is an isomorphism from A to A' fixing B .

However, a different problem appears when we allow automorphisms that fix the base setwise but not pointwise. We must do this when considering $G_{\{I\}}$ since I itself can be the base.

For simplicity of reading we denote $G_{\{I\}}$ by G^* and $G_{\{B\}}^*$ (G_B^*) denotes the elements of G^* that fix B setwise (pointwise).

We now have a subclass of the 0-primitive extensions A where A/B is well-placed by \mathfrak{A}^m : the orbit of the flower of A/B under $G_{\{B\}}^*$. We call the orbit a bouquet.

Definition 4.3.3 (bouquet). *Let A/B be a good pair. The bouquet \mathcal{B} of A/B is the collection of all images $\{\pi(F_i) : F_i \in \mathcal{F}\}$ of each flower \mathcal{F} as π ranges through elements of $G_{\{B\}}^*$.*

Can two flowers in a bouquet contain a common petal? When does a bouquet contain more than one flower? Lemma 4.3.4 and 4.3.5 answer these questions.

²²In Construction 3.8 there were $\mu(A/B)$ petals, the $A_{i,f}^m$ and $C_{i,k}^m$. We no longer assume that A/B is well-placed and we allow the petals to intersect so we have less control over the number of petals; in particular it will vary with \mathfrak{D} .

Lemma 4.3.4. *Let A/B be a good pair and $\{\{F_i^\gamma : i < r_\gamma\} : \gamma \in G_{\{B\}}^*\}$ list its bouquet \mathcal{B} . If $i \neq j$ then $F_i^\gamma \neq F_j^\delta$ for each $\gamma, \delta \in G_{\{B\}}^*$ unless $\mathcal{F}_\gamma = \{F_t^\gamma : t < r_\gamma\}$ and $\mathcal{F}_\delta = \{F_t^\delta : t < r_\delta\}$ are the same flower.*

Proof. Assume that $F_i^\gamma = F_j^\delta$ for some $\gamma, \delta \in \text{aut}(G_{\{B\}}^*)$. We will show $\mathcal{F}_\delta = \mathcal{F}_\gamma$.

Let $\langle f_1, \dots, f_k \rangle$ be an enumeration of F_i^γ , such that

$$M \models \text{diag}_{A \cup B}(f_1, \dots, f_k, \gamma(b_1), \dots, \gamma(b_n))$$

Since $|F_i^\gamma| = |F_j^\delta|$ there is $\varepsilon \in S_k$ such that $\langle f_1, \dots, f_k \rangle = \langle f_{\varepsilon(1)}, \dots, f_{\varepsilon(k)} \rangle$ and

$$M \models \text{diag}_{A \cup B}(f_{\varepsilon(1)}, \dots, f_{\varepsilon(k)}, \delta(b_1), \dots, \delta(b_n))$$

Let $s < r_\gamma$ and let $\langle d_1, \dots, d_k \rangle$ enumerate the petal C_s^γ of a flower \mathcal{F}_γ of A over B . That is,

$$M \models \text{diag}_{A \cup B}(d_1, \dots, d_k, \gamma(b_1), \dots, \gamma(b_n)).$$

By the property of ε noted above and the definition of diagram, we have

$$M \models \text{diag}_{A \cup B}(d_{\varepsilon(1)}, \dots, d_{\varepsilon(k)}, \delta(b_1), \dots, \delta(b_n))$$

but $D = \{d_{\varepsilon(1)}, \dots, d_{\varepsilon(k)}\}$ is a petal of the flower \mathcal{F}_δ . Obviously, $F_s^\gamma = D$. So, each petal of the flower $\{F_t^\gamma : t < r_\gamma\}$ is also a petal of $\{F_t^\delta : t < r_\delta\}$.

The inverse inclusion is similar. Hence, the flowers are equal. \blacksquare

We can now conclude:

Lemma 4.3.5. *Assume that A/B is well-placed by some X and A is $G_{\{I\}}$ -invariant. Then*

- (1) *the bouquet of A/B consists of a single flower;*
- (2) *the bouquet of A/B is $G_{\{I\}}$ -invariant.*

Proof. 1) Assume to the contrary that the bouquet of $A_{j,1}^{m+1}$ over B_j^{m+1} consists of at least two flowers. Let $\pi \in G_{\{I\}}$ be an automorphism which moves one flower of the bouquet of $A_{j,1}^{m+1}$ over B_j^{m+1} to another one. Since $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant, $\pi(A_{j,1}^{m+1}) = A_{j,1}^{m+1}$; so, these two flowers have a common petal. By Lemma 4.3.4 these flowers are equal, for a contradiction.

2) By Lemma 3.19.1, B is $G_{\{I\}}$ -invariant, so each $g \in G_{\{I\}}$ fixes B setwise. But then the C/B -bouquet is just the $G_{\{I\}}$ -orbit of the unique flower \mathcal{F} of A/B , namely \mathcal{F} . \blacksquare

We now give several examples to clarify the relationship among these concepts.

Example 4.3.6. 1) Two certificates in the same flower: Let A/B be 0-primitive, C_j^i for $i < 3, j < 2$ isomorphic with A over B . For each i , $|C_0^i \cap C_1^i| = 1$; these are the only intersections. Let $\mathfrak{D} = B \cup \bigcup_{i < 3, j < 2} C_j^i$. A/B is well-placed by D . $\{A\} \cup \{C_j^i : i < 3, j < 2\}$ is the flower of A/B . But each of $\{A\} \cup \{C_j^i : i < 3\}$ for $j = 0$ and $j = 1$ is a certificate (Actually, there are 8 certificates.)

2) Two flowers in the same bouquet: Let $B = \{b_1, b_2\}$ and $C_i = \{c_1^i, c_2^i, c_3^i\}$ with $R(b_1, c_1^i, c_2^i)$, $R(b_2, c_2^i, c_3^i)$, $R(b_2, c_3^i, c_1^i)$, for $i = 1, 2$, and let $\mu(C_i/B) = 2$.

Let $D_i = \{d_1^i, d_2^i, d_3^i\}$ with $R(b_2, d_1^i, d_2^i)$, $R(b_1, d_2^i, d_3^i)$, $R(b_1, d_3^i, d_1^i)$, for $i = 1, 2$. There is a $\pi \in G_{\{I\}}^*$ that swaps b_1 and b_2 and takes $\{C_1, C_2\}$ to $\{D_1, D_2\}$. Recall that flowers are given by maps that fix B pointwise. Note C_1 and D_1 are in the same orbit under $G_{\{B\}}^*$ but not G_B^* .

There are two flowers: $\{C_1, C_2\}$ over $\langle b_1, b_2 \rangle$ and $\{D_1, D_2\}$ over $\langle b_2, b_1 \rangle$. (They are distinct because b_1 occurs in two relations in the D_i and one in the C_i .)

4.4. $G_{\{I\}}$: elimination of imaginaries fails

Context We showed that $\text{dcl}^*(I) = \emptyset$ and so $\text{sdcl}^*(I) = \emptyset$, provided that μ triples: $\mu(B, C) \geq 3$ for $\delta(B) = 2$. So \hat{T}_μ does not admit elimination of imaginaries. Now we are going to show that the symmetric $\text{sdcl}^*(I)$ is empty for any μ satisfying Hrushovski's original conditions and so elimination of imaginaries fails. That is, we now omit the adequacy hypothesis that governed Section 4.1. There may now be definable truly binary functions but elimination of imaginaries still fails. The innovation is to consider the action of $G_{\{I\}}$ rather than G_I , sdcl rather than dcl .

Recall that in Example 4.2.1 $d(G_I(a_1)) = 1$, since $G_I(a_1) = \{a_1\}$. The situation differs when we consider $G_{\{I\}}$. In this case, working in a $G_{\{I\}}$ -normal set, $G_{\{I\}}(a_1) = \{a_1, a_2\}$, so, $d(G_{\{I\}}(a_1)) = 2$. Similarly, while a_2 is in $\text{dcl}(I)$, $\text{sdcl}(I) = \emptyset$. In general, the G_I -invariant set generated by a set U is contained in the $G_{\{I\}}$ -invariant set U generates.

While in the proof of Theorem 4.1.2 we showed dim_m for $m \geq 1$, here we shall prove sdim_m for $m \geq 0$. Allowing $m = 0$ has a crucial role for application of Claim 4.4.13 in the proof of Theorem 4.4.1, showing any $G_{\{I\}}$ -invariant subset of \mathfrak{A}^2 is safe. In Example 4.2.1 one can see the difference between a flower and a bouquet and how the notion of bouquet works for the proof of Theorem 4.4.1 (Remark 4.2.2(3)–(5)).

Theorem 4.4.1. *If \hat{T}_μ is as in Definition 0.1, then there is no symmetric \emptyset -definable truly n -ary function for $v \geq 2$, i.e., $\text{sdcl}^*(I) = \emptyset$ for any v -element independent set I . That is, there is no \emptyset -definable truly n -ary function whose value does not depend on the order of the arguments. Thus, \hat{T}_μ does not admit elimination of imaginaries. (See Theorem 2.13.)*

As a corollary, we obtain that $\text{sdcl}(J) = \bigcup_{a \in J} \text{sdcl}(a)$ for any independent set J .

In contrast to Section 4.1, we work now with a *global induction* on the height m_0 of $G_{\{I\}}$ -decompositions of finite, $G_{\{I\}}$ -invariant subsets \mathfrak{A} of $\text{acl}(I)$ with $I \subseteq \mathfrak{A} \leq M$. We show for each m_0 , for all such decompositions of height m_0 , for all $m \leq m_0$, sdim_m holds. While we analyze a specific $G_{\{I\}}$ -normal \mathfrak{A} containing I and a $G_{\{I\}}$ -decomposition of \mathfrak{A} into strata \mathfrak{A}^n as in Section 3, the contradiction will result in most involved case a second normal subset of \mathcal{M} . The analysis takes into account that the resulting \mathfrak{A}^n are now $G_{\{I\}}$ -invariant. For this we need to introduce the induction hypothesis in Lemma 4.4.3 on the dimension of $G_{\{I\}}$ -invariant sets.

Definition 4.4.2 (Safe). *Let $X \subseteq \mathfrak{A}$, a finite $G_{\{I\}}$ -invariant set. We say X is safe if $d(E) \geq 2$ for any $G_{\{I\}}$ -invariant set $E \subseteq X$ that is not a subset of $\text{acl}(\emptyset)$.*

The $G_{\{I\}}$ -decomposition \mathfrak{A}^m of \mathfrak{A} satisfies sdim_m if every $G_{\{I\}}$ -invariant subset of \mathfrak{A}^m is safe.

In addition to changing the group, the requirement $E \not\subseteq \mathfrak{A}^0$ has been replaced by $E \not\subseteq \text{acl}(\emptyset)$. So, the main differences between Theorem 4.1.2 and Theorem 4.4.1 are the following:

- There may be cases where $A_{f,k}^{m+1}$ is $G_{\{I\}}$ -invariant, because there is no longer the restriction that $\mu(C/B) \geq 3$ (In Example 3.13, Figure 4, $A_{1,1}^2$ is $G_{\{I\}}$ -invariant);

- Different $A_{j,i}^{m+1}$ may be shown safe for different reasons. (Lemma 4.4.12)
- Although $d(G_I(e))$ may be 1, we show $d(G_{\{I\}}(e)) \geq 2$ for any $e \in \mathfrak{A} - \text{acl}(\emptyset)$.

Note that X is $G_{\{I\}}$ -invariant implies X is G_I -invariant. Analogously to Notation 3.16 we write $s\ell_j^{m+1}$ for the number of images under $G_{\{I\}}$ of A_j^{m+1} that do not intersect \mathfrak{A}^m . Since $G_{\{I\}} \supseteq G_I$, $s\ell_j^{m+1} \geq \ell_j^{m+1}$. Results from Section 4.1 for G_I decompositions do not automatically extend. We will now prove sdim_{m_0} holds not by a dual induction but by distinct arguments depending on whether move_m holds at a given stage, which requires an even more global induction on all G -normal decompositions rather than the length of a fixed decomposition. The main lemma becomes:

Lemma 4.4.3. *Let \hat{T}_μ be as in Theorem 4.4.1. Then for every finite G -normal $\mathfrak{A} \subseteq \text{acl}(I)$ and every $G_{\{I\}}$ -decomposition $\langle \mathfrak{A}^i : i < m_0^{\mathfrak{A}} \rangle$ of \mathfrak{A} :*

for every $m \leq m_0$ sdim_m holds of \mathfrak{A} .

Theorem 4.4.1 follows from Lemma 4.4.3, because if there were a $u \in \text{sdcl}(I)$, then $G_{\{I\}}(u) = \{u\}$ and so $d(G_{\{I\}}(u)) \leq \delta(G_{\{I\}}(u)) = 1$.

We cannot prove that every $A_{j,k}^{m+1}$ is moved by $G_{\{I\}}$ (Remark 4.2.2). Rather, we show that if $A_{j,k}^{m+1}$ is $G_{\{I\}}$ invariant then each $s \in A_{j,k}^{m+1}$ satisfies $\dim(G_{\{I\}}(s)) \geq 2$. This turns out to be short (Lemma 4.4.8) if $\mu_j^{m+1} = \mu(A_{j,k}^{m+1}/B_j^{m+1}) = 2$. So we assume below that $\mu_j^{m+1} \geq 3$. The global induction for Theorem 4.4.1 turns a failure of sdim_{m+1} into another $G_{\{I\}}$ -invariant set $\tilde{\mathfrak{A}}^* = \mathfrak{A}^{m-1} \cup \tilde{A}_{1,j}^m$. The height of $\tilde{\mathfrak{A}}^*$ is m , but $\tilde{A}_{1,j}^m$ contains an element e' such that $d(G_{\{I\}}(e')) \leq 1$, which violates the inductive hypothesis, sdim_m , for $\tilde{\mathfrak{A}}^*$. However, $\tilde{\mathfrak{A}}^*$ need not be contained in \mathfrak{A} .

The proof is a lengthy induction. We start with the following claim which is blatantly false for G_I .

Claim 4.4.4. *The statement sdim_0 holds: every $G_{\{I\}}$ -invariant subset of \mathfrak{A}_0 has dimension at least 2.*

Proof. Without loss, since $e \in \text{acl}(\emptyset)$ implies $d(G_{\{I\}}(e)) = 0$, let $e \in \mathfrak{A}^0 \setminus \text{acl}(\emptyset)$. Then $e \in \text{acl}(a_1) \cup \text{acl}(a_2) \cup \dots \cup \text{acl}(a_v)$, say, $e \in \text{acl}(a_1)$. Since $e \in \text{acl}(a_1) \setminus \text{acl}(\emptyset)$, we obtain $a_1 \in \text{acl}(e)$.

Let $g_i \in G_{\{I\}}$ be such that $g_i(a_1) = a_i$. Such a g_i exists because the a_i are independent and strong minimality implies there is a unique non-algebraic type over the empty set. Then $a_i \in \text{acl}(g_i(e))$. Thus $\{a_1, a_2, \dots, a_v\} \subseteq \text{acl}(\{g_i(e) : i = 1, \dots, v\}) \subseteq \text{acl}(G_{\{I\}}(e))$. So, $d(G_{\{I\}}(e)) = v \geq 2$. ■

By Lemma 4.1.4 (the inductive step, ‘ moves_m implies dim_m ’, from Section 4.1) applied to a $G_{\{I\}}$ -decomposition, when $A_{j,i}^{m+1}$ moves, we have:

Claim 4.4.5. *If $m \geq 1$ and $A_{j,i}^{m+1}$ is not $G_{\{I\}}$ -invariant, sdim_m implies $A_{j,i}^{m+1}$ is safe.*

This is the first divide; it tells us two things. 1) We can prove sdim_{m+1} by showing individual $G_{\{I\}}$ -invariant petals are safe (Definition 4.4.2) and 2) sdim_1 is true as in Corollary 4.1.5.

For the remainder of Section 4.4 we assume sdim_m holds for each $G_{\{I\}}$ -normal \mathfrak{A} . We show that for any such \mathfrak{A} , any $G_{\{I\}}$ -invariant $A_{j,i}^{m+1}$ is safe.

We now establish some tools used below as well as show in Claim 4.4.8 that for each $m \geq 1$ and for $G_{\{I\}}$ -invariant $A_{j,1}^{m+1}$, sdim_m and $\mu_j^{m+1} = 2$ imply $A_{j,1}^{m+1}$ is safe. In order to explain the main idea of the rest of the proof we review Example 3.13, Figure 4, where $A_{1,1}^2$ is $G_{\{I\}}$ -invariant. Clearly, there is an isomorphism ρ_0 of $A_{1,1}^2$ to $C_{1,1}^2 = \{a_1, a_2\} = I$ over $B_1^2 = \{b_1, b_2\}$. Since we put $I \leq M$, so is B_1^2 , then ρ_0 can be extended to an automorphism ρ of M . Thus, we have found an automorphism which takes the $G_{\{I\}}$ -invariant petal $A_{1,1}^2$ into \mathfrak{A}^1 . Moreover, $\rho(A_{1,1}^2) = C_{1,1}^2 = I$ is obviously $G_{\{I\}}$ -invariant. Thus, $A_{1,1}^2$ has a $G_{\{I\}}$ -invariant copy inside \mathfrak{A}^1 and by the inductive hypothesis, sdim_1 , this copy is safe. Now the key points are Observation 4.4.7 and Lemma 4.4.8, which allow the transfer of safeness of $\rho(A_{1,1}^2)$ to $A_{1,1}^2$. In general, given a $G_{\{I\}}$ -invariant petal $A_{j,1}^{m+1}$ we find an automorphism ρ and prove that ρ takes $A_{j,1}^{m+1}$ into \mathfrak{A}^m or possibly into another G -normal set $\tilde{\mathfrak{A}}^m$ of height m . We show that $\rho(A_{j,1}^{m+1})$ is $G_{\{I\}}$ -invariant and then by the induction hypothesis is safe. Finally, we apply Lemma 4.4.8 to show that $A_{j,1}^{m+1}$ is safe. Finding ρ is easy, showing that $\rho(A_{j,1}^{m+1})$ is in a G -normal set of height m is quite simple in Lemmas 4.4.11 and 4.4.19, but is more difficult in Lemma 4.4.23. Much of the argument, including Subsection 4.3, is aimed at proving that $\rho(A_{j,1}^{m+1})$ is $G_{\{I\}}$ -invariant.

Notation 4.4.6. *Extending Notation 3.20 we write A and B for $A_{j,i}^{m+1}, B_j^{m+1}$. C represents a $C_{j,q}^{m+1}$ for arbitrary q , where $C_{j,q}^{m+1}$ for $q = 1, \dots, \nu = \mu_j^{m+1} - 1$ list the isomorphic over B copies of $A_{j,1}^{m+1}$ in \mathfrak{A} that are subsets of \mathfrak{A}^m . We may write C^1, \dots, C^ν , when the stratum m and j are fixed. Recall that μ_j^x is short for $\mu(A_{j,1}^x/B_j^x)$.*

We state next observation for G as G_I or $G_{\{I\}}$ to emphasize it holds for either group. Our application will be to $G_{\{I\}}$.

Observation 4.4.7. *Consider the action of G on M . Suppose A and C are G -invariant subsets of $\mathfrak{A} \leq M$ with $A \cap \mathfrak{A}^m = \emptyset$ and $C \subseteq \mathfrak{A}^m$, and ρ is an automorphism of M that takes A onto C . If for an arbitrary $\alpha \in G$,*

$$\hat{\alpha}(\mathfrak{A}^m \cup A) = \alpha \mathfrak{A}^m \cup (\rho^{-1} \upharpoonright C \circ \alpha \upharpoonright C \circ \rho \upharpoonright A).$$

extends to an element (also called $\hat{\alpha}$) of G then ρ^{-1} injects each orbit of G on C into an orbit of G on A as follows. For any $e \in A$ and α ; if $\alpha(\rho(e)) = e' \in C$ then $\hat{\alpha}(e) = \rho^{-1}(\hat{\alpha}(\rho(e))) = \rho^{-1}(e')$ defines an injection from the G -orbit of $\rho(e)$ to that of e . Consequently, the image $\rho(X)$ of any G -invariant subset X of A is a union of G -orbits and hence G -invariant.

Part 1) of Lemma 4.4.8 tells us that the special case where $\delta(B) = 2$ we can extend an isomorphism ρ from A to C to an automorphism of M and then deduce the safety of A from the safety of C . Part 2) asserts that the deduction of safety is fine provided ρ extends to an automorphism. A major task in this section will be establishing that ρ satisfy this condition.

Lemma 4.4.8. *Let $A = A_{j,1}^{m+1}$ be $G_{\{I\}}$ -invariant and*

- (1) ρ an isomorphism fixing $B = B_j^{m+1}$ pointwise and taking $A_{j,1}^{m+1}$ to $C \subseteq \mathfrak{A}^m$.
 - (a) If C is $G_{\{I\}}$ -invariant and there is an automorphism $\alpha' \in G_{\{I\}}$ moving $\rho(e)$ to e' for some $e \in A_{j,1}^{m+1}$, $e' \in C$, then there exists $\hat{\alpha} \in G_{\{I\}}$ moving e to $\rho^{-1}(e')$. So, if C is safe so is $A_{j,1}^{m+1}$.

- (b) In particular, if $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant, sdim_m holds, and $\mu_j^{m+1} = 2$, then $d(G_{\{I\}}(s)) = 2$ for each $s \in A_{j,1}^{m+1}$.
- (2) More generally, let $A = A_{j,1}^{m+1}$ be $G_{\{I\}}$ -invariant and ρ is an automorphism of M moving $B = B_j^{m+1}$ inside \mathfrak{A}^m and taking $A_{j,1}^{m+1}$ to a $G_{\{I\}}$ -invariant $D = \rho(A_{j,1}^{m+1})$ with $D \subseteq \mathfrak{A}^m$. Then if D is safe so is $A_{j,1}^{m+1}$.

Proof. 1a) By Observation 4.4.7, we must show that the $\hat{\alpha}$ on $A^m \cup A$ defined in the Observation extends to an element of $G_{\{I\}}$. Note

$$\hat{\alpha} \upharpoonright A^m \cup A = \alpha \upharpoonright A^m \cup (\rho^{-1} \upharpoonright C \circ \alpha \upharpoonright C \circ \rho \upharpoonright A).$$

is well-defined on $A^m \cup A$, since ρ fixes B pointwise and C is fixed setwise by α . Since $R(A, \mathfrak{A}^m) = R(A, B)$ and ρ is a B -isomorphism from A to C , $\hat{\alpha} \upharpoonright \mathfrak{A}^m \cup A$ is an automorphism of $\mathfrak{A}^m \cup A$. And, since $\mathfrak{A}^m \cup A \leq M$, $\hat{\alpha}$ extends to the required map in $G_{\{I\}}$.

1b) Since $B \subseteq \text{acl}(I)$ but $B \subsetneq \text{acl}(\emptyset)$, sdim_m , and the conditions on μ (in Definition 1.1.7) imply $2 \leq d(B) \leq \delta(B) \leq 2$. Thus, ρ extends to an automorphism of M . Note that all petals over B are disjoint, because $B \leq M$, so the total number of petals that are isomorphic to A over B is equal to $\mu_j^{m+1} = 2$, namely, they are A and C . Since $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant, B_j^{m+1} is $G_{\{I\}}$ -invariant. Now we prove that C is $G_{\{I\}}$ -invariant. By Lemma 4.3.5 the bouquet on $A_{j,1}^{m+1}/B_j^{m+1}$ is equal to the flower on $A_{j,1}^{m+1}/B_j^{m+1}$ and has only two elements. The global $G_{\{I\}}$ -isomorphism ρ guarantees the same holds for C . Consequently, $G_{\{I\}}$ fixes each of A, B, C setwise. By 1a) $\rho^{-1} \upharpoonright C$ induces a $G_{\{I\}}$ -isomorphism from $G_{\{I\}}(e)$ into $G_{\{I\}}(\rho(e))$. The induction hypothesis gives $d(G_{\{I\}}(c)) = 2$ for any $c \in C$, e.g., $\rho(e)$; so $d(G_{\{I\}}(e)) = 2$.

2) Let $\alpha \in G_I$ fix D setwise. Now consider

$$\hat{\alpha} \upharpoonright (\mathfrak{A}^m \cup A) = \alpha \upharpoonright \mathfrak{A}^m \cup (\rho^{-1} \upharpoonright D \circ \alpha \upharpoonright D \circ \rho \upharpoonright A).$$

$\hat{\alpha}$ is well-defined and fixes I as in case 1a). Since $R(A, \mathfrak{A}^m) = R(A, B)$ and ρ is an isomorphism of BA to $\rho(B)D$, so $\rho(A)$ is good over $\rho(B)$. But since $\mathfrak{A}^m \leq M$, this implies $R(\rho(A), \mathfrak{A}^m) = R(\rho(A), \rho(B))$. So, $\hat{\alpha} \upharpoonright \mathfrak{A}^m \cup A$ is an automorphism of $\mathfrak{A}^m \cup A$. And, since $\mathfrak{A}^m \cup A \leq M$, $\hat{\alpha}$ extends to the required map in $G_{\{I\}}$. By Observation 4.4.7 and since ρ is an automorphism, if D is safe, so is A . ■

The following notation will be used to study the relationship between a $G_{\{I\}}$ -invariant set and the set it determines (Definition 3.18). If $\mu(A/B) \geq 3$ we will have the following situation.

Notation 4.4.9. We extend Notation 4.4.6 to consider two levels. We will let \mathfrak{D} range over subsets of the G -decomposable \mathfrak{A} ; in applications they will usually be initial segments of the decomposition. Let A, B denote a good pair well-placed by $\mathfrak{D} \leq M$ such that A is $G_{\{I\}}$ -invariant. C denotes an arbitrary petal of the flower of A/B . We write $\hat{A}, \hat{B}, \hat{\mathfrak{D}}$ for a similar triple determined (Definition 3.18) by the first.

Here is the way in which this situation arises. Suppose a $G_{\{I\}}$ -invariant $A_{j,1}^{m+1}$ with base B_j^{m+1} determines $A_{i,1}^m$. Then each of the $C_{j,k}^{m+1}$ intersects $A_{i,1}^m$. When $A_{i,1}^m$ is also G -invariant, then we get a new iteration. To avoid writing/reading subscripts, we call the first level A, B, C and the second $\hat{A}, \hat{B}, \hat{C}$. Similarly \mathfrak{D} and

$\hat{\mathfrak{D}}$ refer to (are instantiated as) \mathfrak{A}^m and \mathfrak{A}^{m-1} . We introduce this notation to avoid the distraction of the multiple super/sub scripts and focus on certain relationships which will appear several times in the sequel. In the crucial case where $\mu(\hat{A}/\hat{B}) = 2$, we will be able to extend the partial isomorphism ρ over \hat{B} taking \hat{A} to its unique copy $\rho(\hat{A}) = \hat{C} \subseteq \hat{\mathfrak{D}}$ to an automorphism of M also called ρ .

Lemma 4.4.10. *In the situation of Notation 4.4.9, suppose $\mu(\hat{A}/\hat{B}) = 2$. Let ρ be a partial isomorphism over \hat{B} from \hat{A} to its unique copy $\rho(\hat{A}) = \hat{C} \subseteq \hat{\mathfrak{D}}$. Then $\rho(B)$ is $G_{\{I\}}$ -invariant.*

Proof. By Lemma 4.4.8.1b, ρ extends to an automorphism of M . Suppose $\pi \in G_{\{I\}}$ fixes \hat{B} setwise. By Lemma 3.23.A, $B \subseteq \hat{A} \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0)$ and so

$$\rho(B) \subseteq \rho(A \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0)) = \hat{C} \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0).$$

Obviously, $\rho(B \cap \text{acl}(\emptyset)) = B \cap \text{acl}(\emptyset)$ is $G_{\{I\}}$ -invariant, because B is $G_{\{I\}}$ -invariant. By Lemma 3.23, $B - \text{acl}(\emptyset) = B \cap A_{i,1}^m$ is $G_{\{I\}}$ -invariant. We know that both \hat{A} and \hat{C} are $G_{\{I\}}$ -invariant, and that $\rho(B \cap \hat{A}) \subseteq \hat{C}$. Assume for contradiction that $\rho(B \cap A_{i,1}^m)$ is not $G_{\{I\}}$ -invariant, witnessed by $\pi \in G_{\{I\}}$ such that

$$\pi(\rho(B \cap \hat{A})) \neq \rho(B \cap \hat{A}).$$

Then we put

$$\tau = \pi \upharpoonright \mathfrak{A}^{m-1} \cup (\rho^{-1} \circ \pi \circ \rho) \upharpoonright \hat{A}$$

Obviously, τ can be extended to an automorphism of M and $\tau(B \cap \hat{A}) \neq B \cap \hat{A}$, contradicting $G_{\{I\}}$ -invariance of B . \blacksquare

Relying heavily on our induction hypothesis, sdim_m , we show a $G_{\{I\}}$ -invariant $A_{j,1}^{m+1}$ with only one element determines an $A_{i,1}^m$ with at least two elements and $\mu_i^m \geq 3$. While part 1) of Lemma 4.4.11 is essential in this paper, parts 2) and 3) foreshadow the main argument below.

Claim 4.4.11. *If $|A_{j,1}^{m+1}| = 1$ and is $G_{\{I\}}$ -invariant then*

- (1) $B = B_j^{m+1} \leq M$, and each $C^k = C^{m+1,k}$ is contained in $A_{i,1}^m \cup B_i^m$.
- (2) Moreover, $\mu_j^{m+1} \geq 3$;
- (3) and $\mu_i^m \geq 3$.

Proof. 1) Lemma 3.19.1 asserts $A_{j,1}^{m+1} = \{e\}$ determines some $A_{i,1}^m$ and by sdim_m , Lemma 3.19.2 yields $B \subseteq A_{i,1}^m$. By sdim_m again, $d(B) = \delta(B) = 2$; so $B \leq M$. Since $B \subseteq A_{i,1}^m$ has relations in \mathfrak{A}^m only with elements of $A_{i,1}^m$ and its base B_i^m and each C^k is a singleton, each $C^k \subseteq A_{i,1}^m \cup B_i^m$.

2) Assume to the contrary that $\mu_j^{m+1} = 2$. By Lemma 4.3.5.1, $A_{j,1}^{m+1}$, B_j^{m+1} , and $C_{j,1}^{m+1}$ are $G_{\{I\}}$ -invariant. Since $C_{j,1}^{m+1} \subseteq \mathfrak{A}^m$ and is $G_{\{I\}}$ -invariant, $C_{j,1}^{m+1}$ is safe by induction. But $|C_{j,1}^{m+1}| = 1$, so $C_{j,1}^{m+1} = \{c\}$. Then $d(G_{\{I\}}(c)) = d(\{c\}) \leq \delta(\{c\}) = 1$, for a contradiction.

3) Assume to the contrary that $\mu_i^m = 2$. Using the notation and result of Lemma 4.4.10, we are given a partial isomorphism ρ taking $A_{i,1}^m$ to $C_{i,1}^m \subseteq \mathfrak{A}^{m-1}$. Moreover, $A_{i,1}^m, B_i^m, C_{i,1}^m, \rho(A_{j,1}^{m+1})$ are all $G_{\{I\}}$ -invariant. By sdim_m , $d(B_i^m) \geq 2$ and so $2 \leq \delta(B_i^m) \leq \mu_i^m = 2$ and $A_{i,1}^m$ is a 0-primitive extension of B_i^m ; thus, $B_i^m \cup A_{i,1}^m \leq M$. So ρ can be extended to an automorphism $\hat{\rho}$ of M . The automorphism $\hat{\rho}$ is not in $G_{\{I\}}$ as it doesn't respect strata. Indeed, it may not fix \mathfrak{A} setwise.

Clearly, $\hat{\rho}(B), \rho(C^1), \dots, \rho(C^\nu) \subseteq \hat{\rho}(A_{i,1}^m \cup B_i^m) = C_{i,1}^m \cup B_i^m \subseteq \mathfrak{A}^{m-1}$. Since $B \leq M$, $\hat{\rho}(B) \leq M$ so by Lemma 3.7 $\mu(\rho(B), \rho(C^1)) = \mu(B, A_{j,1}^{m+1}) = \nu + 1$, so $\hat{\rho}(A_{j,1}^{m+1}) = \{e'\}$ is a $(\nu + 1)$ th copy of $\hat{\rho}(C^1)$ over $\hat{\rho}(B)$. Note that $\hat{\rho}(A_{j,1}^{m+1})$ is the unique such copy which is not in $C_{i,1}^m$.

As C_1^m is $G_{\{I\}}$ -invariant, by the ‘consequently’ of Observation 4.4.7, $\rho(B)$ is a $G_{\{I\}}$ -invariant set and so is $\{\hat{\rho}(C^1), \dots, \hat{\rho}(C^\nu)\}$. By Lemma 4.3.5 the bouquet of $A_{j,1}^{m+1}/B_j^{m+1}$ consists of one flower. Again by Observation 4.4.7, the bouquet of $\hat{\rho}(A_{j,1}^{m+1})/\hat{\rho}(B_j^{m+1})$ consists of one flower $\{\hat{\rho}(C^1), \dots, \hat{\rho}(C^\nu), \hat{\rho}(A_{j,1}^{m+1})\}$. Since $\hat{\rho}(C^1), \dots, \hat{\rho}(C^\nu)$ are in the $G_{\{I\}}$ -invariant set $C_{i,1}^m$, $\hat{\rho}(A_{j,1}^{m+1})$ is a $G_{\{I\}}$ -invariant set, because it is disjoint from $C_{i,1}^m$. Thus, e' as a unique element of $\hat{\rho}(A_{j,1}^{m+1})$, is fixed by $G_{\{I\}}$.

If $\hat{\rho}(A_{j,1}^{m+1}) \subseteq \mathfrak{A}$, by the definition of decomposition it is contained in $\mathfrak{A}^m \leq M$. But this means $e' \in \text{sdcl}^*(I)$ contradicting the induction assumption that \mathfrak{A} satisfies sdim_m , since $G_{\{I\}}(e') = \{e'\}$, implies that

$$d(G_{\{I\}}(e')) = d(\{e'\}) \leq \delta(\{e'\}) = 1.$$

The final possibility is that $\hat{\rho}(A_{j,1}^{m+1}) \subseteq M - \mathfrak{A}$. Now we use our ‘global induction’. Let $\tilde{\mathfrak{A}} = \mathfrak{A}^{m-1} \cup \{e'\}$. Then $\tilde{\mathfrak{A}} \leq M$ (since $\delta(\{e'\}/\rho(B)) = 0$) and $\tilde{\mathfrak{A}}$ is $G_{\{I\}}$ -invariant. Thus $\tilde{\mathfrak{A}}$ admits a decomposition of height m and $\{e'\} = \tilde{A}_{1,1}^m$. But $G_{\{I\}}(e') = \{e'\}$ contradicting the inductive hypothesis that sdim_m hold for all decompositions. \blacksquare

The argument for Lemma 4.4.11.3 shows the main idea of the proof of (Lemmas 4.4.19, 4.4.23) for $m \geq 2$. We cut an intermediate strata out, preserving the top, in order to obtain a counterexample with smaller height. There are three possibilities for $A_{j,i}^{m+1}$: item 1) of Lemma 4.4.12 details that we have finished the proof for two of them and item 2) specifies the conditions for further analysis.

Lemma 4.4.12. *Suppose \mathfrak{A} satisfies sdim_m .*

- (1) $A_{j,1}^{m+1}$ is safe if either
 - (a) $A_{j,1}^{m+1}$ is not $G_{\{I\}}$ -invariant or
 - (b) $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant and $\mu_j^{m+1} = 2$
- (2) If $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant and $\mu_j^{m+1} \geq 3$ then $A_{j,1}^{m+1}$ determines $A_{i,1}^m$ for some i . Moreover, $B_j^{m+1} \cap A_{i,1}^m \neq \emptyset$ and $B_+ = B_j^{m+1} - A_{i,1}^m \subseteq \text{acl}(\emptyset)$.

Proof. Case 1) follows from Lemmas 4.4.5 and 4.4.8.1a. Claim 4.4.11.1 shows a strong form of case 2) ($B_+ = \emptyset$) when $|A_{j,1}^{m+1}| = 1$. If $|A_{j,1}^{m+1}| > 1$ the hypotheses of Lemma 3.22 hold, so we may apply Lemma 3.21 and then Lemma 3.23. Thus, $\delta(B_+) \leq 1$. Since \mathfrak{A}^m witnesses \mathfrak{A} satisfies sdim_m , and $B_+ - \text{acl}(\emptyset)$ is $G_{\{I\}}$ -invariant, if $B_+ - \text{acl}(\emptyset)$ were nonempty it would have dimension 2. Thus, $B_+ \subseteq \text{acl}(\emptyset)$. \blacksquare

So we need only study case 2 of Lemma 4.4.12. We describe the case $m + 1 = 2$ to illuminate a major issue in the remainder of the proof.

Claim 4.4.13. *Assume that $A_{j,1}^2$ is $G_{\{I\}}$ -invariant, then $\mu_j^2 = 2$ and $A_{j,1}^2$ is safe.*

Proof. If some $A_{j,1}^2$ is $G_{\{I\}}$ -invariant and $\mu_j^2 \geq 3$, Claim 4.4.12.2 gives that $A_{i,1}^1$ is $G_{\{I\}}$ -invariant for some i ; this contradicts Lemma 3.11 (moves₁). So, $\mu_j^2 \leq 2$.

Since $\delta(B) \geq d(B) \geq 2$ by sdim_1 , we obtain that $\mu_j^2 = 2$ and sdim_2 follows from Lemma 4.4.8. \blacksquare

The difficulty is that this argument depended on every $A_{j,i}^1$ being moved; not merely being safe. In order to deal with this, we introduce a new system of indexing which is expounded more fully in Notation 4.4.20. Note that a $G_{\{I\}}$ -invariant \overleftarrow{A}^0 engenders by Lemma 3.23 a decreasing sequence of $G_{\{I\}}$ -invariant petals \overleftarrow{A}^i such that \overleftarrow{A}^i determines \overleftarrow{A}^{i+1} which continues as long as $\overleftarrow{\mu}^i = \mu(\overleftarrow{A}^i/\overleftarrow{N}^i) \geq 3$. However, we know that no petal $A_{u,v}^1$ is $G_{\{I\}}$ -invariant. So this sequence must terminate with an $s \leq m-1$ such that $\overleftarrow{\mu}^s = 2$. We begin the study of such sequences with the case $\overleftarrow{\mu}^1 = 2$, where the chain has only two levels: *The next five Lemmas, 4.4.14 through 4.4.19, complete the proof when $\mu_i^m = 2$.* We apply the technical Lemma 4.4.14 in the proof of Lemma 4.4.15.

Lemma 4.4.14. *Let $D, E \subseteq_\omega M$ satisfy $\delta(D) = 0$; then $\delta(D/E) \leq 0$. Thus, if $E \leq M$ then $\delta(D/E) = 0$.*

Proof. Monotonicity of δ implies the first inequality

$$\delta(D/E) \leq \delta(D/E \cap D) = \delta(D) - \delta(E \cap D) = -\delta(E \cap D) \leq 0$$

and the second equality holds since any subset of M has non-negative dimension. \blacksquare

Lemma 4.4.15. *Suppose $\mu_j^{m+1} \geq 3$, $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant and determines $A_{i,1}^m$. Then $C^{m+1,q} \subseteq A_{i,1}^m \cup \text{icl}(B_i^m)$ for each q with $1 \leq q \leq \nu^{m+1}$, i.e. $C_+^{m+1,q} \subseteq \text{icl}(B_i^m)$.*

Proof. Let B denote B_j^{m+1} , \hat{B} denote B_i^m , \hat{A} denote $A_{i,1}^m$. We write C , for a fixed but arbitrary q , $C = C^{m+1,q}$, and $C_- = C_-^{m+1,q} = (C^{m+1,q} - \mathfrak{A}^{m-1})$. First we show $C_- = C^{m+1,q} \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ is contained in \hat{A} .

Assume that C_- intersects some other petal A' on strata m . By monotonicity and since C is 0-primitive over B , taking into account $B \cup C \subseteq \mathfrak{A}^m$,

$$(*) \quad \delta((C \cap A')/\mathfrak{A}^m - A') \leq \delta((C \cap A')/(B \cup (C - A'))) < 0.$$

But, also

$$(**) \quad \delta((C \cap A')/\mathfrak{A}^m - A') = \delta((C \cap A')/\mathfrak{A}^{m-1}) \geq 0.$$

(**) holds because $R(A', \mathfrak{A}^m - A') = R(A', \mathfrak{A}^{m-1})$ as all petals in $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ are fully independent over \mathfrak{A}^{m-1} and because $\mathfrak{A}^{m-1} \leq M$. But (*) and (**) are contradictory, so $C_- \subseteq A_{i,1}^m = \hat{A}$.

Our goal is to show $C_+^{m+1,q} \subseteq \text{icl}(\hat{B})$, where $\hat{B} = B_i^m$. So, suppose for contradiction that for some q with $1 \leq q \leq \nu^{m+1}$ where $\nu^{m+1} = \mu^{m+1} - 1$, and with $C_+ = C_+^{m+1,q} = C^{m+1,q} \cap \mathfrak{A}^{m-1}$, we have $C_+ - \text{icl}(\hat{B}) \neq \emptyset$. Since C/B is a good pair and $C = (C_+ - \text{icl}(\hat{B})) \cup (C_+ \cap \text{icl}(\hat{B})) \cup C_-$:

$$(9) \quad 0 > \delta((C_+ - \text{icl}(\hat{B}))/B \cup C_- \cup (C_+ \cap \text{icl}(\hat{B}))).$$

Observe the easy claim:

Claim 4.4.16. *Inequality 9 simplifies to:*

$$(10) \quad \delta((C_+ - \text{icl}(\hat{B}))/B \cup C_- \cup (C_+ \cap \text{icl}(\hat{B}))) = \delta(C_+ - \text{icl}(\hat{B})/B_+ \cup (C_+ \cap \text{icl}(\hat{B}))).$$

Proof. Since $B_- \cup C_- \subseteq A_{i,1}^m$, each relation between $B_- \cup C_-$ and \mathfrak{A}^{m-1} is a relation on $B_- \cup C_-$ as a subset of \hat{A} and the base, \hat{B} , of \hat{A} . So we can delete $B_- \cup C_-$ from the base of Equation 9. \blacksquare

In the special case that $B_+ = \emptyset$, the Lemma 4.4.15 is easy. By monotonicity of δ and by $\text{icl}(\hat{B}) \leq M$ we obtain

$$\delta(C_+ - \text{icl}(\hat{B})/C_+ \cap \text{icl}(\hat{B})) \geq \delta(C_+ - \text{icl}(\hat{B})/\text{icl}(\hat{B})) \geq 0$$

The last contradicts (9).

More generally, we only know $\delta(B_+) = 0$ so we must consider more carefully the connections of C_+ and B_+ . Note $0 < \delta(C_+/B) \leq \delta(C_+/B_+)$ by the definition of good and monotonicity.

We apply the identity $(\delta(X/Y \cup Z) = \delta(X \cup Y/Z) - \delta(Y/Z))$ by putting $X = C_+ - \text{icl}(\hat{B})$, $Y = (B_+ - \text{icl}(\hat{B}))$, and $Z = (B_+ \cap \text{icl}(\hat{B})) \cup (C_+ \cap \text{icl}(\hat{B})) = (B_+ \cup C_+) \cap \text{icl}(\hat{B})$. Thinking of B_+ as $(B_+ - \text{icl}(\hat{B})) \cup (B_+ \cap \text{icl}(\hat{B}))$, we rewrite the right hand side of (10) as follows:

$$\begin{aligned} (11) \quad & \delta(C_+ - \text{icl}(\hat{B})/B_+ \cup (C_+ \cap \text{icl}(\hat{B}))) = \\ (12) \quad & = \delta((C_+ - \text{icl}(\hat{B})) \cup (B_+ - \text{icl}(\hat{B}))/((C_+ \cup B_+) \cap \text{icl}(\hat{B})) - \\ (13) \quad & - \delta(B_+ - \text{icl}(\hat{B})/(C_+ \cup B_+) \cap \text{icl}(\hat{B})). \end{aligned}$$

Now we show the subtracted term, $\delta(Y/Z)$ (Line (13)) is 0. We apply monotonicity of δ in Line (14) and Lemma 4.4.14 in Line (15).

$$\begin{aligned} (14) \quad & \delta(B_+ - \text{icl}(\hat{B})/(C_+ \cup B_+) \cap \text{icl}(\hat{B})) \leq \delta(B_+ - \text{icl}(\hat{B})/B_+ \cap \text{icl}(\hat{B})) = \\ (15) \quad & = \delta(B_+/B_+ \cap \text{icl}(\hat{B})) \leq 0 \end{aligned}$$

On the other hand, applying monotonicity of δ and $\text{icl}(\hat{B}) \leq M$ we obtain

$$\delta(B_+ - \text{icl}(\hat{B})/(C_+ \cup B_+) \cap \text{icl}(\hat{B})) \geq \delta(B_+ - \text{icl}(\hat{B})/\text{icl}(\hat{B})) \geq 0$$

So, Line 13 is 0. By Line 10 and Claim 4.4.16, Line 11 is negative, so Line (12) is negative, too. Below we sequentially apply the distributive law, monotonicity of δ and $\text{icl}(\hat{B}) \leq M$ to Line (12).

$$\begin{aligned} & \delta((C_+ - \text{icl}(\hat{B})) \cup (B_+ - \text{icl}(\hat{B}))/((C_+ \cup B_+) \cap \text{icl}(\hat{B}))) = \\ & = \delta((C_+ \cup B_+) - \text{icl}(\hat{B})/((C_+ \cup B_+) \cap \text{icl}(\hat{B}))) \geq \\ & \geq \delta((C_+ \cup B_+) - \text{icl}(\hat{B})/\text{icl}(\hat{B})) \geq 0 \end{aligned}$$

Obtained contradiction completes the proof of Lemma 4.4.15. \blacksquare

Lemma 4.4.19, Claim 4.4.16, Lemma 4.4.22 and Lemma 4.4.23 rely indirectly on the following ostensibly technical claim about the location of A , with A in Claim 4.4.17 getting different interpretations. It is the crucial point that allows us to anchor (Definition 4.4.20.4) our inductive analysis in $\text{acl}(\emptyset) \cup B$ where B is the base of good pair rather than the \mathfrak{A}_r , where the sequence in Definition 4.4.20 stops.

Claim 4.4.17. *Let A and B be disjoint finite subsets of M , with A good over B and $\delta(B) = d(B) \geq 1$. Then $A \cap \text{acl}(\emptyset) = \emptyset$.*

Proof. Let $d \in A \cap \text{acl}(\emptyset)$ and $D = \text{icl}(d)$. Then $\delta(D) = 0$. Suppose for contradiction that $D \subseteq A$. Since $B \leq M$, Lemma 4.4.14 implies $0 \leq \delta(D/B) \leq \delta(D/\emptyset) = 0$; this contradicts the definition of a good pair, as D must equal A and then, since $\delta(A) - \delta(D) = 0$, A is 0-primitive over $\emptyset \subsetneq B$.

Let $D_0 = D \cap B$ and $D_1 = D \cap (B \cup A)$. Since, D , B and $B \cup A$ are all strong in M , so are D_0 and D_1 . So, $\delta(D) = 0$ implies $\delta(D_0) = \delta(D_1) = 0$. By Lemma 4.4.14 $0 = \delta(D_1/B)$. Repeating the reasoning of the first paragraph with D_1 playing the role of D , we obtain the same contradiction; so, A is 0-primitive over \emptyset . ■

We avoid the subscripts and isolate in Lemma 4.4.18 the connections imposed by determination, which drive the proof, and to emphasize that these results do not require any inductive hypotheses.

In combination with Lemma 4.4.17 (which is used in Lemma 4.4.19), Claim 4.4.16, Lemma 4.4.22 and Lemma 4.4.23, we now extend Lemma 4.4.10 from petals to flowers. While this larger set being G -invariant is *a priori* weaker, we recover the result for petals by a substantial induction.

Lemma 4.4.18. *Let A be well-placed over \mathfrak{D} by B and \hat{A} be well-placed over $\hat{\mathfrak{D}} \leq \mathfrak{D}$ by \hat{B} . Further suppose that A is $G_{\{I\}}$ -invariant, A determines \hat{A} , and $\mu(\hat{A}/\hat{B}) = 2$. Further, let A, C^1, \dots, C^ν list the (by Lemma 4.3.5) flower associated with A/B . Let ρ be a partial isomorphism from \hat{A} over \hat{B} to its unique copy $\hat{C} \subseteq \hat{\mathfrak{D}}$. Then*

- (1) ρ extends to an automorphism of M .
- (2) there is a unique $G_{\{I\}}$ -invariant flower over $\rho(C)/\rho(B)$ and $\rho(A)$ is $G_{\{I\}}$ -invariant.

Proof. i) In fact, we will make the extension of ρ fix $\text{acl}(\emptyset) \cap \mathfrak{A}^0$. Let $\mathcal{W}_1 = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \hat{B} \cup \hat{A}$ and $\tilde{\mathcal{W}} = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \hat{B} \cup \hat{C} \subseteq \mathfrak{D}$. Then, by Claim 4.4.17, with $\hat{C} = \rho(\hat{A})$ playing the role of A , $\hat{C} \cap \text{acl}(\emptyset) = \emptyset$. Now, since $\mu(\hat{A}/\hat{B}) = 2$, \hat{A} and \hat{C} are isomorphic by ρ not only over \hat{B} but over $\mathcal{W}_2 = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \hat{B}$. (We know $R(\hat{A}, \mathfrak{D}) = R(\hat{A}, \hat{B})$). So if the isomorphism is not over $A^0 \cap \text{acl}(\emptyset) \cup \hat{B}$, there is a relation between \hat{C} and $(A^0 \cap \text{acl}(\emptyset)) - \hat{B}$. But then $\delta((\mathfrak{A}^0 \cap \text{acl}(\emptyset))/\hat{C} \cup \hat{B}) < 0$, contradicting $\hat{C} \cup \hat{B} \leq M$.) Note that $\mathcal{W}_2 \leq M$ because

$$\begin{aligned} 2 = d(\hat{B}) &\leq d(\hat{B} \cup (\mathfrak{A}^0 \cap \text{acl}(\emptyset))) \leq \delta(\hat{B} \cup (\mathfrak{A}^0 \cap \text{acl}(\emptyset))) \\ &\leq \delta(\hat{B} - \text{acl}(\emptyset)) + \delta(\mathfrak{A}^0 \cap \text{acl}(\emptyset)) = 2 + 0 = 2 \end{aligned}$$

As $\mathcal{W}_2 \leq M$, ρ extends to an automorphism of M fixing \mathcal{W}_2 , also denoted ρ .

ii) By Lemma 4.4.10, $\rho(B)$ is $G_{\{I\}}$ -invariant. Suppose $\pi \in G_{\{I\}}$, and so fixes B setwise. By Lemma 3.23.A, $B \subseteq \hat{A} \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0)$ and so

$$\rho(B) \subseteq \rho(\hat{A} \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0)) = \hat{C} \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0).$$

Put

$$\tilde{\pi} = \pi \upharpoonright \mathfrak{D} \cup ((\rho^{-1} \upharpoonright \hat{C}) \circ (\pi \upharpoonright \hat{C}) \circ (\rho \upharpoonright \hat{A}))$$

Since $\mathfrak{D} \cup \hat{A} \leq M$, we can extend $\tilde{\pi}$ to $\pi' \in G_{\{I\}}$. By Lemma 4.3.5, there is a unique flower \mathcal{F} over A/B . π' maps it to a flower $\rho(\mathcal{F})$ over $\rho(C)/\rho(B)$, which must also be unique. As, π'^{-1} of a second flower over $\rho(C)/\rho(B)$ would contradict the uniqueness of \mathcal{F} .

Since A, C^1, \dots, C^ν enumerate the $G_{\{I\}}$ -invariant flower \mathcal{F} , its $G_{\{I\}}$ -invariant-image (by Lemma 4.3.5.2) under ρ is $\{\rho(A), \rho(C^1), \dots, \rho(C^\nu)\}$. Since $B_i^m \leq M$,

Lemma 4.4.15 implies all the $C^q \subseteq \hat{C} \cup \hat{B}$. But then, since ρ fixes both $\hat{B} = B_i^m$ and \hat{C} (since $\mu(\hat{A}/\hat{B}) = 2$), each $\rho(C^q)$ is contained in $\hat{C} \cup \hat{B}$ which is $G_{\{I\}}$ -invariant and $\rho(A) \cap (\hat{C} \cup \hat{B}) = \emptyset$. So $\rho(A)$ is invariant as the complement within the invariant flower $\rho(\mathcal{F})$ of the set of the $\rho(C^q)$ that are contained in $\hat{C} \cup \hat{B}$. ■

We continue the inductive proof of sdim_m for all $m < m_0$. The idea is similar to Claim 4.1.6.2, where with $|A_{j,1}^{m+1}| = 1$ we have constructed $\tilde{\mathfrak{A}}$, a counterexample with smaller decomposition height, but the situation is more complicated. We have reduced to the case where $\mu_j^{m+1} \geq 3$ and some for $m' < m$, $\mu_i^{m'} = 2$. Building on Lemma 4.4.10, we first consider the special case when $\mu_i^m = 2$. Note that by Lemma 4.4.11.3 the hypothesis $|A_{j,1}^{m+1}| > 1$ is essential.

Claim 4.4.19. *Suppose $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant and in addition that $|A_{j,1}^{m+1}| > 1$ and $\mu_i^m = 2$. Then, $A_{j,1}^{m+1}$ is safe.*

Proof. Recall that we are doing a global induction to show \mathfrak{A} satisfies sdim . The next constructions will allow us to show $A_{j,1}^{m+1}$ is safe by finding an isomorphic copy of it with lower height.

Lemma 3.23 implies that since $A_{j,1}^{m+1}$ is not a singleton and is $G_{\{I\}}$ -invariant, $\mu_j^{m+1} \geq 3$ implies $B_j^{m+1} \subseteq A_{j,1}^m \cup (\text{acl}(\emptyset) \cap \mathfrak{A}^0)$. By Lemma 4.4.10 the bouquet of $A_{i,1}^m$ over B_i^m is $G_{\{I\}}$ -invariant. Since both $A_{j,1}^{m+1}$ and $A_{i,1}^m$ are $G_{\{I\}}$ -invariant, Lemma 4.3.5 implies the bouquets of $A_{j,1}^{m+1}$ over B_j^{m+1} and of $A_{i,1}^m$ over B_i^m are each $G_{\{I\}}$ -invariant. The second of these consists of two petals: $A_{i,1}^m$ and $C^{m,1}$. Since the bouquet and $A_{i,1}^m$ are $G_{\{I\}}$ -invariant so is $C^{m,1}$.

We now apply Lemma 4.4.18, taking A as $A_{j,1}^{m+1}$, \hat{A} as $A_{i,1}^m$, B as B_j^{m+1} , \hat{B} as B_i^m , C^i as $C_{j,i}^{m+1}$, \hat{C} as $C^{m,1}$, \mathfrak{D} as \mathfrak{A}^{m-1} , and ρ as constructed in Lemma 4.4.18 to conclude: $\rho(B_j^{m+1})$ is $G_{\{I\}}$ -invariant, the flower over $\rho(C)/\hat{B}$ is $G_{\{I\}}$ -invariant, and $\rho(A)$ is $G_{\{I\}}$ invariant.

Clearly $\chi_M(\rho(A_{j,1}^{m+1})/\rho(B_j^{m+1})) \leq \mu_j^{m+1} = \mu(A_{j,1}^{m+1}/B_j^{m+1})$. The $\rho(C^{m+1,q})$ gives us $\mu_j^{m+1} - 1$ witnesses. Since $\rho(B^{m+1,q}) \subseteq \mathfrak{A}^{m-1} \leq M$, $\rho(A_{j,1}^{m+1})$ cannot split over \mathfrak{A}^{m-1} (Definition 3.6). Similarly $\mathfrak{A} \leq M$ implies $\rho(A_{j,1}^{m+1})$ cannot split over \mathfrak{A}^{m-1} . We now have three cases depending on the exact location of $\rho(A_{j,1}^{m+1})$.

Case 1. $\rho(A_{j,1}^{m+1}) \subseteq \mathfrak{A}^{m-1}$. Immediately, the induction hypothesis sdim_m (in fact, sdim_{m-1}) implies $\rho(A_{j,1}^{m+1})$ is safe.

To complete the proof, we show an extension of Lemma 4.4.8.

Case 2: $\rho(A_{j,1}^{m+1}) \subset \mathfrak{A}$ and $\rho(A_{j,1}^{m+1}) \cap \mathfrak{A}^{m-1} = \emptyset$: Since $\mathfrak{A}^{m-1} \leq M$, we must have $\delta(\rho(A_{j,1}^{m+1})/\mathfrak{A}^{m-1}) = 0$. But $(\rho(A_{j,1}^{m+1})/\rho(B_j^{m+1}))$ is a good pair. We know $(B_j^{m+1})_- \subseteq A_{1,1}^m$ so $(\rho(B_j^{m+1}))_- \subseteq C^{m,1} \subseteq \mathfrak{A}^{m-1}$. And $(\rho(B_j^{m+1}))_+ \subseteq \mathfrak{A}^0$. So $\rho(B_j^{m+1}) \subseteq \mathfrak{A}^{m-1}$. Since \mathfrak{A} is $G_{\{I\}}$ -normal and $\rho(A_{j,1}^{m+1})$ is well-placed by $\rho(B)$ over \mathfrak{A}^{m-1} the construction places $\rho(A_{j,1}^{m+1})$ in \mathfrak{A}^m . So by sdim_m , $\rho(A_{j,1}^{m+1})$ is safe.

Case 3. $\rho(A_{j,1}^{m+1}) \subset M - \mathfrak{A}$: Then we put $\tilde{\mathfrak{A}}^m = \mathfrak{A}^{m-1} \cup \rho(A_{j,1}^{m+1})$. Note that $\tilde{\mathfrak{A}}^m$ is a $G_{\{I\}}$ -normal with height m . Applying the global induction hypothesis sdim_m to $\tilde{\mathfrak{A}}^m$, we see $\rho(A_{j,1}^{m+1})$ is safe.

Thus, in each case $\rho(A_{j,1}^{m+1})$ is safe. So, by Lemma 4.4.8.3, $A_{j,1}^{m+1}$ is safe. ■

Because ‘ $A_{j,1}^{m+1}$ determines $A_{i,1}^m$ ’ produces a decreasing chain of complicated sub/superscripts, we introduce a notation for a descending sequence, which is relative to given $G_{\{I\}}$ -invariant petal $A_{j,1}^{m+1}$, and describes the ‘root’ below $A_{j,1}^{m+1}$ that controls its intersection with $\text{sdcl}^*(I)$. Recall that capital Roman letters (A, B) denote petals, while script letters \mathfrak{A} denote initial segments of a tree-decomposition.

Definition 4.4.20. [Determined Sequences] We write $\Upsilon(A)$ for the petal determined (Definition 3.18) by A . Then $\Upsilon^k(A)$ denotes the k th iteration of this operation.

- (1) As usual, $\mathfrak{A}^m = \bigcup_{i \leq m} \mathfrak{A}^i$.
- (2) Fix $\overleftarrow{A}^0, \overleftarrow{A}^1$ such that $\overleftarrow{A}^0 = A_{j,1}^{m+1}$ determines $\Upsilon(\overleftarrow{A}^0) = \overleftarrow{A}^1 = A_{i,1}^m$ (Definition 3.18). For fixed q , $\overleftarrow{C}^{0,q} = C^{m+1,q}$, $\overleftarrow{B}^0 = B_j^{m+1}$.

The crucial inductive definition is

$$\overleftarrow{A}^{k+1} = \begin{cases} \Upsilon(\overleftarrow{A}^k), & \text{if } \mu(\overleftarrow{A}^k, \overleftarrow{B}^k) \geq 3 \\ \text{stop}, & \text{if } \mu(\overleftarrow{A}^k, \overleftarrow{B}^k) = 2 \end{cases}$$

So, $\overleftarrow{A}^k = A_{t_k,1}^{m+1-k}$ for some t_k for each $k \leq s$. Increment indices for B, C, μ in the same way. E.g. $\overleftarrow{C}^{k,q}$ is $C_{j,q}^{m+1-k}$ in the notation for decompositions.

- (3) The order of $\overleftarrow{A}^0 = A_{j,1}^{m+1}$ is the least index s such that $\mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$.
- (4) Suppose the order of \overleftarrow{A}^0 is s . We define the root that supports \overleftarrow{A}^0 from the bottom up.

$$\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{B}^s.$$

For $k \leq s$, we define \mathcal{W}_k by downward induction.

$$\mathcal{W}_k = \mathcal{W}_{s+1} \cup \overleftarrow{A}^s \cup \dots \cup \overleftarrow{A}^k = \mathcal{W}_{k+1} \cup \overleftarrow{A}^k \subseteq \overleftarrow{\mathfrak{A}}^k.$$

Recall that increasing the superscript of an $\overleftarrow{\mathfrak{A}}$ moves to lower strata. Since we are analyzing $A_{j,1}^{m+1}$, for any $k < m+1$, $\overleftarrow{\mathfrak{A}}^0 = \mathfrak{A}^{m+1}$, $\overleftarrow{\mathfrak{A}}^{k+1}$ is the initial segment preceding $\overleftarrow{\mathfrak{A}}^k$ in the original decomposition. In particular,

$$\overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m+1-(s+1)} = \mathfrak{A}^{m-s} = \bigcup_{k \leq m-s} \mathfrak{A}^k.$$

Also, the $\overleftarrow{C}^{k,q} = C_{t_k}^{m+1-k,q} \subseteq \overleftarrow{\mathfrak{A}}^{k+1}$ are isomorphic over \overleftarrow{B}^k copies of \overleftarrow{A}^k .

Since $\mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$ and \overleftarrow{B}^s is safe, $2 \leq d(\overleftarrow{B}^s) \leq \delta(\overleftarrow{B}^s) \leq \mu(\overleftarrow{A}^s, \overleftarrow{B}^s) = 2$. So the next lemma is easy.

Lemma 4.4.21. Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leq k \leq s \rangle$ stops with $\mu(\overleftarrow{A}^s / \overleftarrow{B}^s) = 2$, then $\delta(\mathcal{W}_k) = 2$ for each $k \leq s+1$.

Proof. Since $\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{B}^s \leq M$ and $\delta(\overleftarrow{B}^s) = 2$, $\delta(\mathcal{W}_{s+1}) = 2$. Then we can finish by induction, because at each step we consider a 0-primitive extension. \blacksquare

Lemma 4.4.22. Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leq k \leq s \rangle$ stops with $\mu(\overleftarrow{A}^s / \overleftarrow{B}^s) = 2$ then

- (1) each $\mathcal{W}_k \leq M$ and is G -invariant;

(2) $\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$ for every $q \in \{1, \dots, \mu(\overleftarrow{A}^0/\overleftarrow{B}^0) - 1\}$.

Proof. 1) To start the induction, note $\overleftarrow{B}^s \leq M$ since $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$. $G_{\{I\}}$ -invariance follows from the definition of determined, noting that \overleftarrow{A}^k is $G_{\{I\}}$ -invariant by Lemma 3.23 as $\mu(\overleftarrow{A}^{k-1}/\mu(\overleftarrow{B}^{k-1})) \geq 3$. But each $\mathcal{W}_{k+1} \leq \mathcal{W}_k$ since all have dimension 2.

2) By Lemma 4.4.15, for each $k \leq s$, $q \leq \mu(\overleftarrow{A}^k/\overleftarrow{B}^k)$, $\overleftarrow{C}^{k,q} \subseteq \overleftarrow{A}^{k+1} \cup \text{icl}(\overleftarrow{B}^{k+1})$. Since each $\overleftarrow{B}^{k+1} \subseteq \mathcal{W}_k \leq M$, this implies $\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$. ■

Now we consider the general case of Lemma 4.4.19. The key difficulty is that we cannot deduce $\rho(B)$ is $G_{\{I\}}$ -invariant in one step as in Lemma 4.4.10. We have a sequence that stops with an \overleftarrow{A}^s such that $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$ so that there is an automorphism ρ mapping \overleftarrow{A}^s into \overleftarrow{C}^s . With this ρ fixed we argue inductively that each \overleftarrow{A}^k for $s \geq k \geq 0$ is safe.

But, we must perform a dual induction with the proof that $\rho(A)$ is $G_{\{I\}}$ -invariant.

Lemma 4.4.23. *Suppose the sequence $\langle \overleftarrow{A}^k : 0 \leq k \leq s \rangle$ stops with $\mu(\overleftarrow{A}^s/\overleftarrow{B}^s) = 2$. Then \overleftarrow{A}^k is safe for each $k \leq s$. In particular, when $k = 0$, we see $A_{j,1}^{m+1}$ is safe.*

Proof. We use Definition 4.4.20 of \mathcal{W}_k . By Lemma 4.4.17, as used in Lemma 4.4.19, fix an automorphism ρ of M , that sends \overleftarrow{A}^s to its unique copy \overleftarrow{C}^s and which fixes $\mathcal{W}_{s+1} = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{B}^s$ pointwise. Recall $\mathcal{W}_s = \mathcal{W}_{s+1} \cup \overleftarrow{A}^s$ and that $\overleftarrow{C}^s \subseteq \overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m-s}$. Let $\widetilde{\mathcal{W}}_s = \mathcal{W}_s \cup \overleftarrow{C}^s = (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{B}^s \cup \overleftarrow{C}^s$. Then $\rho(\mathcal{W}_s) = \widetilde{\mathcal{W}}_s$. For $k \leq s+1$, we build on Definition 4.4.20.4 of \mathcal{W}_k . We define

$$\widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_s \cup \rho(\overleftarrow{A}^{s-1}) \cup \dots \cup \rho(\overleftarrow{A}^k).$$

Note that $\widetilde{\mathcal{W}}_k$ need not be contained in \mathfrak{A} . In particular, $\mathcal{W}_s \subseteq \overleftarrow{\mathfrak{A}}^s$ while $\widetilde{\mathcal{W}}_s - \overleftarrow{\mathfrak{A}}^{s+1} = \rho(\overleftarrow{A}^{s-1})$.

Finally, to obtain a $G_{\{I\}}$ -normal structure with a well-defined height, we define:

$$\widetilde{\mathcal{R}}_k = \overleftarrow{\mathfrak{A}}^{s+1} \cup \rho(\overleftarrow{A}^s) \cup \dots \cup \rho(\overleftarrow{A}^k).$$

Note that the height of $\widetilde{\mathcal{R}}_s = \overleftarrow{\mathfrak{A}}^{s+1} \cup \rho(\overleftarrow{A}^s)$ is $m-s$ because $\rho(\overleftarrow{A}^s) \subseteq \overleftarrow{\mathfrak{A}}^{s+1}$ and $\overleftarrow{\mathfrak{A}}^{s+1} = \mathfrak{A}^{m-s}$. Moving from $\widetilde{\mathcal{R}}_{k+1}$ to $\widetilde{\mathcal{R}}_k$ increases the height at most by 1; that is why the height of $\widetilde{\mathcal{R}}_0$ is at most m . Since \overleftarrow{A}^k determines \overleftarrow{A}^{k+1} , Lemma 4.4.12.2 implies that $\overleftarrow{B}^k \subseteq (\mathfrak{A}^0 \cap \text{acl}(\emptyset)) \cup \overleftarrow{A}^{k+1}$: obviously, then,

$$(*) \rho(\overleftarrow{B}^k) \subseteq \widetilde{\mathcal{W}}_{k+1} = \widetilde{\mathcal{W}}_s \cup \rho(\overleftarrow{A}^s) \cup \dots \cup \rho(\overleftarrow{A}^{k+1}) \subseteq \widetilde{\mathcal{R}}_k.$$

By Claim 4.4.22 $\overleftarrow{C}^{k,q} \subseteq \mathcal{W}_{k+1}$. Thus, $\rho(\overleftarrow{C}^{k,q}) \subseteq \widetilde{\mathcal{W}}_{k+1}$. Obviously, $\rho(\overleftarrow{A}^k) \cap \widetilde{\mathcal{W}}_{k+1} = \emptyset$ because $\overleftarrow{A}^k \cap \mathcal{W}_{k+1} = \emptyset$. By Lemma 4.3.5 the bouquet of $\rho(\overleftarrow{A}^k)$ over $\rho(\overleftarrow{B}^k)$ consists just of one flower.

We conclude Lemma 4.4.23 from the following, which we show below for each $k \leq s$:

- (1) $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$ -invariant;
- (2) $\overleftarrow{A}^{s+1} \cup \widetilde{\mathcal{W}}_k$, $\widetilde{\mathcal{W}}_k$, and $\rho(\overleftarrow{A}^k)$ are $G_{\{I\}}$ -invariant.

We prove these two assertions by simultaneous induction on k . The induction is downward from s and the base step is the third paragraph of the proof of

Lemma 4.4.19. So, we assume that (1)–(2) hold for $k+1$ and show that they hold for k .

(1) Recall that by Lemma 3.23.1, $\overleftarrow{B}^k \subseteq \overleftarrow{A}^{k+1} \cup (\mathfrak{A}^0 \cap \text{acl}(\emptyset))$. So,

$$\rho(\overleftarrow{B}^k) \subseteq \rho(\overleftarrow{A}^{k+1} \cup (\mathfrak{A}^0 \cap \text{acl}(\emptyset))) = \rho(\overleftarrow{A}^{k+1}) \cup (\mathfrak{A}^0 \cap \text{acl}(\emptyset)).$$

We consider $\rho(\overleftarrow{B}^k) \cap (\mathfrak{A}^0 \cap \text{acl}(\emptyset))$ and $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})$ separately. Since ρ fixes $\mathfrak{A}^0 \cap \text{acl}(\emptyset)$ pointwise and \overleftarrow{B}^k is $G_{\{I\}}$ -invariant, $\rho(\overleftarrow{B}^k \cap \mathfrak{A}^0 \cap \text{acl}(\emptyset))$ is $G_{\{I\}}$ -invariant.

We show an arbitrary $\pi \in G_{\{I\}}$ fixes $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})$ setwise. By the induction hypothesis, $\rho(\overleftarrow{A}^{k+1})$ is $G_{\{I\}}$ -invariant, so $\pi(\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1})) \subseteq \rho(\overleftarrow{A}^{k+1})$. Now we put

$$\tau = (\pi \upharpoonright \overleftarrow{\mathfrak{A}}^{k+2}) \cup (\rho^{-1} \circ \pi \circ \rho) \upharpoonright \overleftarrow{A}^{k+1}.$$

Obviously, this isomorphism extends to an automorphism from $G_{\{I\}}$. Since \overleftarrow{B}^k is $G_{\{I\}}$ -invariant, $\tau(\overleftarrow{B}^k) = \rho^{-1} \circ \pi \circ \rho(\overleftarrow{B}^k) = \overleftarrow{B}^k$; so $\pi(\rho(\overleftarrow{B}^k)) = \rho \circ \pi(\overleftarrow{B}^k) = \rho(\overleftarrow{B}^k)$.

(2) By the induction hypotheses $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, $\widetilde{\mathcal{W}}_{k+1}$, and $\rho(\overleftarrow{A}^{k+1})$ are $G_{\{I\}}$ -invariant. Whence, by (1) $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$ -invariant. Since $\widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_{k+1} \cup \rho(\overleftarrow{A}^k)$, it is sufficient to prove that $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$ -invariant to deduce that $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_k$ and $\widetilde{\mathcal{W}}_k$ are $G_{\{I\}}$ -invariant. So, we consider $\rho(\overleftarrow{A}^k)$. In fact, we repeat some reasoning from Lemma 4.4.19. We put $\overleftarrow{\nu}^k = \mu(\overleftarrow{A}^k / \overleftarrow{B}^k) - 1$.

Case 2a) $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$: Since $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$ -invariant, so is

$$(**) \bigcup_{q=1}^{\overleftarrow{\nu}^k} \rho(\overleftarrow{C}^{k,q}) \cup \rho(\overleftarrow{A}^k)$$

because it is the flower of $\rho(\overleftarrow{A}^k)$ over $\rho_0(\overleftarrow{B}^k)$. Clearly, each $\overleftarrow{C}^{k,q}$ intersects \overleftarrow{A}^{k+1} ; otherwise, there must be $c_1, c_2 \in \overleftarrow{C}^{k,q} \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+2}$ and $b \in \overleftarrow{B}^k \cap \overleftarrow{A}^{k+1}$ with $R(b, c_1, c_2)$. The last implies that $|\overleftarrow{A}^{k+1}| = 1$, for a contradiction. Thus, each $\rho(\overleftarrow{C}^{k,q})$ intersects $\rho(\overleftarrow{A}^{k+1})$, which is $G_{\{I\}}$ -invariant by the induction hypothesis. Let $\tau \in G_{\{I\}}$ be arbitrary. Since $\rho(\overleftarrow{B}^k)$ is $G_{\{I\}}$ -invariant $\tau(\rho(\overleftarrow{B}^k)) = \rho(\overleftarrow{B}^k)$ and so

$$\tau(\rho(\overleftarrow{A}^k)) \in \{\rho(\overleftarrow{C}^{k,q}) : q = 1, \dots, \overleftarrow{\nu}^k\} \cup \{\rho(\overleftarrow{A}^k)\}.$$

By construction, $\overleftarrow{A}^k \cap \overleftarrow{A}^{k+1} = \emptyset$; so $\rho(\overleftarrow{A}^k) \cap \rho(\overleftarrow{A}^{k+1}) = \emptyset$ and $\tau(\rho(\overleftarrow{A}^k))$ does not intersect $\tau(\rho(\overleftarrow{A}^{k+1})) = \rho(\overleftarrow{A}^{k+1})$. But, we showed in the last paragraph $\rho(\overleftarrow{C}^{k,q}) \cap \rho(\overleftarrow{A}^{k+1}) \neq \emptyset$, so $\tau(\rho(\overleftarrow{A}^k))$ cannot be equal to any of the $\rho(\overleftarrow{C}^{k,q})$. Hence, using (**), $\tau(\rho(\overleftarrow{A}^k)) = \rho(\overleftarrow{A}^k)$ and $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$ -invariant.

Since $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, by the global induction hypothesis $\rho(\overleftarrow{A}^k)$ is safe.

Case 2b) $\rho(\overleftarrow{A}^k) \not\subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$: As $\rho(\overleftarrow{A}^k)$ is a 0-primitive extension of $\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$, $\rho(\overleftarrow{A}^k) \cap (\overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}) = \emptyset$. By (*), $\rho(\overleftarrow{B}^k) \subseteq \overleftarrow{\mathfrak{A}}^s \cup \widetilde{\mathcal{W}}_{k+1}$; moreover $\rho(\overleftarrow{B}^k) \cap \rho(\overleftarrow{A}^{k+1}) \neq \emptyset$.

Claim 4.4.24. *In case 2b, for each $k < s$, $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1} - \overleftarrow{\mathfrak{A}}^{k+2}$ or $\rho(\overleftarrow{A}^k) \cap \mathfrak{A} = \emptyset$. In either case $\overleftarrow{\mathfrak{A}}^{k+1} \cup \widetilde{\mathcal{W}}_{k+2}$ is $G_{\{I\}}$ -normal.*

Proof. We have $\rho(\overleftarrow{B}^k) \subseteq \rho(\overleftarrow{A}^{k+1}) = \overleftarrow{C}^{k+1} \subseteq \overleftarrow{\mathfrak{A}}^{k+2} \cup \widetilde{\mathcal{W}}_{k-1}$, $\rho(\overleftarrow{A}^k)$ is 0-primitive over $\rho(\overleftarrow{B}^k)$, and $\overleftarrow{\mathfrak{A}}^{k+1} \leq M$. Thus, $\rho(\overleftarrow{A}^k)$ is 0-primitive over $\overleftarrow{\mathfrak{A}}^{k+1}$ and based on

$\rho(\overleftarrow{B}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1}$. If $\rho(\overleftarrow{A}^k) \subseteq \mathfrak{A}$ by, construction, $\rho(\overleftarrow{A}^k) \subseteq \overleftarrow{\mathfrak{A}}^{k+1} - \overleftarrow{\mathfrak{A}}^{k+2}$. If not, since $\rho(\overleftarrow{A}^k)$ cannot split (Definition 3.6) over \mathfrak{A} , $\mathfrak{A} \cap \rho(\overleftarrow{A}^k) = \emptyset$ and so $\overleftarrow{\mathfrak{A}}^{k+1} \cup \{\widetilde{\mathcal{W}}_k\}$ is $G_{\{I\}}$ -normal. ■

Since \overleftarrow{v}^k copies of $\rho(\overleftarrow{A}^k)$ over its base are inside $\widetilde{\mathcal{W}}_{k+1}$, $\rho(\overleftarrow{A}^k)$ is $G_{\{I\}}$ -invariant. This completes the proof of Case 2b.

Since $\rho(A_{j,1}^{m+1}) = \rho(\overleftarrow{A}^0) \subseteq \widetilde{\mathcal{W}}_0 \subseteq \widetilde{\mathcal{R}}_0$ and the height of $\widetilde{\mathcal{R}}_0$ is at most m , by the global induction $\rho(A_{j,1}^{m+1})$ is safe; by Lemma 4.4.8.2, we conclude $A_{j,1}^{m+1}$ is safe. We finish Lemma 4.4.23. ■

This completes the proof of Lemma 4.4.3 showing sdim_m for $m \leq m_0$; thus we have the main conclusion, Theorem 4.4.1.

5. STEINER SYSTEMS

In this section we study the strongly minimal k -Steiner systems discovered in [BP20]. A k -Steiner system is a collections of points and lines so that two points determine a line and all lines have the same finite length k . A quasigroup (binary operation with unique solutions of $ax = b$ and $xa = b$) such that every 2-generated sub-quasigroup has k elements determines a k -Steiner system where the lines are the two generated subalgebras. Our interest in the existence of definable truly binary functions arose from the discovery that while a Steiner system with line length three admits a quasigroup operation definable in the vocabulary of the ternary collinearity predicate and Steiner systems with prime power length admit the imposition of quasigroups that preserve lines (e.g. [GW80]), it seemed very unlikely in the second case that those quasigroups were *definable from R* ([Bal21a]).

There are two examples of strongly minimal 3-Steiner systems in [BP20] and [Hru93, Section 5]. By explicitly adding multiplication to the vocabulary, [Bal21a] constructs strongly minimal quasigroups which determine k -Steiner systems for each prime power k . We show below that this separate operation is essential. The following problem/example inspired this research and is solved here.

Problem 5.1. We can impose a quasigroup structure on any 4-Steiner system. There are two obvious ways: one commutative, one not [Bal21a]. In fact, [GW75] a quasi-group can be imposed in any Steiner k -system when $k = p^n$ for a prime p . 1) Prove the operations of these quasi-groups are not R -definable in a strongly minimal 4-Steiner system (M, R) . 2) More generally, is there an \emptyset -definable truly binary function?

We now use \mathbf{K} rather than \mathbf{L} to emphasize the distinctions from Section 4. Having said that, $\mathbf{K}^* = \mathbf{L}^*$ and $\mathbf{K}_0^* = \mathbf{L}_0^*$ while $\mathbf{K}_0 \neq \mathbf{L}_0$. We work in a vocabulary τ with one ternary relation R , and assume always that R can hold only of three distinct elements and then in any order (a 3-hypergraph) and the basic definitions are in Section 1.2. In the language of $*$ -petals, μ *triples* if for every *non-linear* (Definition 5.6) $*$ -petal (C/B) with $\delta(B) = 2$ and $|C| > 1$, $\mu(C/B) \geq 3$.

Theorem 5.2. *Let $M \models T_\mu^S$ be a strongly minimal Steiner system described in Definition 0.1. Then*

- (1) *The naturally imposed quasigroups ([GW75]) on M are not \emptyset -definable in M .*

- (2) If $\mu(\alpha) \geq 2$ and μ triples, then there is no \emptyset -definable truly binary function in T_μ^S .
- (3) There is no symmetric \emptyset -definable truly v -ary function for $v \geq 2$, i.e., $\text{sdcl}^*(I) = \emptyset$ for any v -element independent set I .

After a short introduction establishing 1), we prove 2) and 3). The major obstacles to adapting the earlier proofs of these results in the Hrushovski case are a) the need to modify the notion of base (Lemma 5.4) and b) the analysis of distinct occurrences of R (e.g. Lemma 3.21).

The following example (Figure 7) shows that as in the Hrushovski case, we must pass to $G_{\{I\}}$ and strengthen the hypothesis to get $\text{sdcl}^*(I) = \emptyset$. Definable truly binary functions may appear when $\mu(A/B) = 2$ and $d(B) = 2$ is allowed. We put the following lines: $\{a_1, d_2, d_1\}$, $\{a_1, d_4, d_5\}$, $\{a_2, d_5, d_3, d_1\}$, and $\{d_2, d_3, d_4\}$. The elements c_i is the isomorphic copy of d_i over $\{a_1, a_2\}$, for each i . In order to construct $A_{1,1}^2$ we make α_i a copy of a_i , δ_i a copy of d_i and each γ_i a copy of c_i for each appropriate i , where the isomorphism under consideration is over $\{d_3, c_3\}$. Then $\alpha_1 \in \text{dcl}^*(a_1, a_2)$.

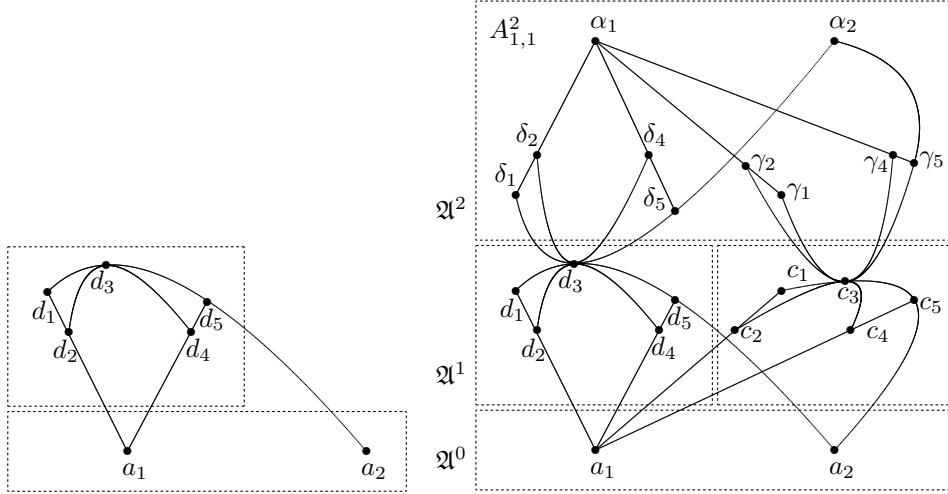


FIGURE 7. Example of $\text{dcl}^*(\{a_1, a_2\}) \neq \emptyset$

We defined linear spaces and the appropriate δ for studying them in Definition 1.2.2. In [BP20, Lemma 3.7] we showed that this δ is flat, submodular, and computes exactly on free products defined as in Definition 5.7. Thus, the notion of decomposition and the arguments for the basic properties of the standard Hrushovski construction in earlier sections go through below with minor changes. However, Lemma 5.11 shows some significant differences in the resulting decomposition. This finer analysis of the decomposition, which is the chief novelty of this section, powers the understanding of definable closure in these Steiner systems.

Recall from Conclusion 1.1.10: for each $3 \leq k < \omega$, there are continuum-many strongly minimal infinite linear spaces in the vocabulary τ that are Steiner k -systems. A crucial invariant for these systems is ‘line length’. The length of each line in a model of the Steiner system is $\mu(\alpha) + 2$ where α is from Notation 1.2.6.

However, there may be maximal cliques in a substructure A with smaller cardinality. We refer to such configurations as *partial lines*; a line of length $\mu(\alpha) + 2$ may be called *full* for emphasis. Following a convention established in [BP20], we think of two independent points as a *trivial (and therefore partial) line*.

The Hrushovski restraint in defining the μ -function: an integer $\mu(\beta) = \mu(A/B) \geq \delta(B)$ was relaxed in [BP20] to hold only when $|A - B| \geq 2$. To allow lines of length three, we required only $\mu(\beta) \geq 1$, if $\beta = \alpha$. Thus for the case when $\mu(\alpha) = 1$ we got a strongly minimal Steiner system with lines of length three. Obviously, there is a definable symmetric truly binary function H on pairs of distinct elements; $H(x, y)$ is the third point on the line determined by x and y and $H(x, x) = x$. So we restrict here to lines of length at least four.

Assumption 5.3. $\mu(\alpha) \geq 2$.

With longer line length k one can always introduce a k -ary *partial function* saying its value on $k - 1$ distinct elements is the remaining point on the line. But, now there is no clear way to give a uniform definition of a k -ary *function* on sequences with repetition. With the following variant on the results in Section 4, we show there is no such truly binary function in the vocabulary: $\{R\}$.

As noted in Remark 1.1.6 the original Hrushovski construction supports a happy coincidence. The minimal subset B of D (the base: Definition 1.2.5) such that A is 0-primitive over B is also the maximal subset such that every element of B is R related to some element of A . But for linear spaces, the two notions diverge. Allowing for and exploiting this difference is one of the two major changes from the proof for the Hrushovski construction in Section 4. We recast [BP20, Lemma 4.8] (Lemma 1.2.5) in case 2) of the next lemma.

Lemma 5.4. *Let $D \leq D \cup A \in \mathbf{K}_0$ be a 0-primitive extension with $D \cap A = \emptyset$. Then there are two cases:*

- (1) *If $A = \{a\}$ there is a unique line ℓ with $\ell \cap D \geq 2$. In that case, any $B \subseteq (\ell \cap D)$ with $|B| = 2$ yields a good pair (B, a) . Furthermore, $d \in D$ is in the relation R with the element a if and only if d is on ℓ .*
- (2) *If $|A| \geq 2$ then there is a unique maximal subset B of D with every point b in B incident with a line ℓ_b with $|\ell_b \cap A| \geq 2$ containing b .*

On the basis of Lemma 5.4 we add the new notion of *extended base*.

Definition 5.5. *Let A be a 0-primitive extension of D (in M), where we assume that $D \cap A = \emptyset$. If $A = \{a\}$, then the extended base for A is the maximal set $\tilde{B} = \ell \cap D$ where ℓ is the line through b_1, b_2 for any elements $b_1, b_2 \in D$ such that $R(b_1, b_2, a)$. Note that if $A = A_{j,i}^{m+1}$ and $D = \mathfrak{A}^m$ the extended base for A is $B_j^{m+1} = \{d \in \mathfrak{A}^m - \mathfrak{A}^{m-1} : R(b_1, b_2, d)\}$ for any $b_1, b_2 \in \mathfrak{A}^m$ with $R(b_1, b_2, a)$.*

If $A_{j,i}^{m+1} = \{a\}$ is 0-primitive over \mathfrak{A} with extended base $B = B_j^{m+1}$, any two element subset B_0 of B can act as a base. If $\ell_{m+j} < \mu(\alpha)$, the $C^{m+1,q}$ must be mapped into $B - B_0$.

Definition 5.6 will be clarified by Lemma 5.11 showing that all types of $*$ -petals have been described.

Definition 5.6. *Let $G \in \{G_I, G_{\{I\}}\}$, and let \mathfrak{A} be a G -normal set. Fix a decomposition of \mathfrak{A} into strata \mathfrak{A}^m constructed inductively as in Construction 3.8.*

- (1) We say $A = A_{j,1}^{m+1} = \{a\} \in \mathfrak{A} - \mathfrak{A}^m$ is an α -point if there exist $b_1, b_2 \in \mathfrak{A}^m$ with $R(b_1, b_2, a)$.
- (2) A set A is a linear cluster if $A = \{a \in \mathfrak{A}^{m+1} - \mathfrak{A}^m : R(b_1, b_2, a)\}$ for some $b_1, b_2 \in \mathfrak{A}^m$. We denote the linear cluster with extended base $\tilde{B} = B_f^m \subseteq \mathfrak{A}^m$ as $A_f^{m+1} = \bigcup A_{f,i}^{m+1}$ where the $A_{f,i}^{m+1}$ are the α -petals over \tilde{B} .
- (3) A $*$ -petal is either an $A_{f,i}^{m+1}$ with cardinality greater than 1 (called a non-linear petal) or a linear cluster.
- (4) We write Steiner-moves $_m$ if every non-linear petal $A_{f,k}^m$ is moved by some $g \in G_I$.
- (5) Recall that we say X is safe if $d(E) \geq 2$ for any $G_{\{I\}}$ -invariant set $E \subseteq X$ which is not a subset of $\text{acl}(\emptyset)$. The $G_{\{I\}}$ -decomposition \mathfrak{A}^m of \mathfrak{A} satisfies Steiner-sdim $_m$ if every $G_{\{I\}}$ -invariant subset of \mathfrak{A}^m is safe.

Now any $\gamma \in G$ that fixes A setwise fixes an extended base set-wise but it does not need to fix a base of an α -point even setwise.

Definition 5.7. [BP20, Lemma 3.14] Let $A \cap C = B$ with $A, B, C \in \mathbf{K}_0$. We define $D := A \oplus_B C$ as follows:

- (1) the domain of D is $A \cup C$;
- (2) a pair of points $a \in A - B$ and $b \in C - B$ are on a non-trivial line ℓ' in D if and only if there is line ℓ based in B such that $a \in \ell$ (in A) and $b \in \ell$ (in C). Thus $\ell' = \ell$ (in D).

- Lemma 5.8.**
- (1) If $D \supseteq A \cup B \cup C$ where A and C are 0-primitive over B , $B \leq D$, and there is a relation among elements $a_1 \in A - B$ and $a_2 \in C - B$ then both $|A - B|$ and $|C - B|$ are 1.
 - (2) Each \mathfrak{A}^m is partitioned into $*$ -petals and there is no non-trivial line (even through \mathfrak{A}^{m-1}) connecting distinct $*$ -petals. That is, the $*$ -petals are fully independently joined

Proof. 1) If $R(a_1, a_2, b)$ then $\delta(A/BC) < \delta(A/C)$, unless there is a line $\ell \subseteq D$ with $|\ell \cap B| \geq 2$ that contains both a_i . But each a_i is then the only element of a 0-primitive extension of B . 2) Thus the collection of $*$ -petals (i.e. non-linear petals and the linear clusters A_f^{m+1} of α -points) are fully freely joined as a partition of $\mathfrak{A}^{m+1} - \mathfrak{A}^m$. ■

Lemma 5.9. Fix a G -normal \mathfrak{A} and a decomposition of height at least 3, where $G \in \{G_I, G_{\{I\}}\}$. Every non-trivial partial line ℓ in \mathfrak{A} is either contained (except for at most one point) in a single petal of the topmost strata \mathfrak{A}^{m_0} or extends to a full line that intersects at most three strata.

Proof. Let m be least so that ℓ is based in \mathfrak{A}^m . If $|\ell| < \mu(\alpha) + 2$, adding a new point in \mathfrak{A}^{m+1} , that is related only to $\ell \cap \mathfrak{A}^m$ is a 0-primitive extension giving an α -petal $A_{f,i}^{m+1}$. By Corollary 3.7, $|\ell \cap \mathfrak{A}^{m+1}| = \mu(\alpha) + 2$. It is possible that one point of ℓ , but, by choice of m , not two, is in \mathfrak{A}^{m-1} . That is, it may be $|\ell \cap (\mathfrak{A}^{m+1} - \mathfrak{A}^{m-1})| = \mu(A_{j,1}^{m+1}/B_j^{m+1}) + 1$. This is the possibility that intersects three strata. If $m = m_0$, the line may remain partial but includes at most one point of \mathfrak{A}^{m-1} . ■

Definition 5.10. We say a petal $A_{j,1}^{m+1}$ Steiner-determines a $*$ -petal, if there is a non-linear petal $A_{i,f}^m$ or a linear cluster A_i^m which is the unique $*$ -petal based in

\mathfrak{A}^{m-1} that intersects $B_j^{m+1} - \mathfrak{A}^{m-1}$. (More precisely, $\langle A_{j,1}^{m+1}, B_f^m, \mathfrak{A}^m \rangle$ determines $\langle A_{i,f}^m, B_i^m, \mathfrak{A}^{m-1} \rangle$.)

Lemma 5.11. *Fix a decomposition of a G -normal set \mathfrak{A} , where $G \in \{G_I, G_{\{I\}}\}$. Suppose $A = \{a\}$ is an α -point of \mathfrak{A}^{m+1} based on $B = \{b_1, b_2\} \subseteq \mathfrak{A}^m$ and a subset of the linear cluster A_j^{m+1} . Let \check{B} be the extended base of a in \mathfrak{A}^m . Then,*

- (1) *If $m = 0$, $\{a\}$ is in a linear cluster A_j^1 with $|A_j^1| = \mu(\alpha) - |I|$. Since I is independent, this is possible only if $|I| = 2$.*
- (2) *Let $m > 0$. If a linear cluster satisfies $|A_j^{m+1}| = 1$ then $\check{B} - \mathfrak{A}^{m-1}$ is a subset of one $*$ -petal, say, $A_{f,i}^m$, which is not a linear cluster. So, A_j^{m+1} determines $A_{f,i}^m$ in this case.*
- (3) *Let $G = G_I$. Then Steiner-moves_m implies each α -point $\{a\}$ over \mathfrak{A}^m is moved by G_I .*
 - (a) *G_B acts as the symmetric group $S_{|A_j^m|}$ on a linear petal A_f^m based on $B = \{b_1, b_2\}$. Thus, G_I moves such α -points.*
 - (b) *A is a line based on $B \subsetneq A_{f,i}^m$ for some f, i . By Steiner-moves_m , $A_{f,i}^m$ is moved and a fortiori so is A .*
- (4) *Let A_j^{m+1} be a linear cluster which contains at least two elements (that is, at least two α -points) and which is G -invariant. If $d(\check{B}) \geq 2$ then $d(A_j^{m+1}) = 2$.*

Proof. 1) Suppose $m = 0$. We have $R(b_1, b_2, a)$; b_1, b_2 are algebraically independent; else $a \in \mathfrak{A}^0$. Moreover the definition of \mathfrak{A}^0 decrees $\neg R(b_1, b_2, b_3)$ for any 3 distinct $b_i \in \mathfrak{A}^0$. By Corollary 3.7, $\chi_M(\{a\}/\{b_1, b_2\}) = \mu(\alpha)$ yielding a linear cluster of cardinality $\mu(\alpha) - 2$.

2) By Lemma 3.7 the line ℓ passing through a, b_1, b_2 is full and is equal to $\check{B} \cup A_j^{m+1}$. Then $|\check{B}| = \mu(\alpha) + 2 - |A_j^{m+1}| \geq 3$, because $\mu(\alpha) \geq 2$ and $|A_j^{m+1}| = 1$. By Lemma 5.9 $|\check{B} \cap \mathfrak{A}^{m-1}| \leq 1$, so at least two elements of \check{B} are in $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. If these two elements belong to different $*$ -petals, then these $*$ -petals are not free over \mathfrak{A}^{m-1} , for a contradiction. (If there is a point on the line and in \mathfrak{A}^{m-1} or if there are three points in different petals, the petals are dependent over \mathfrak{A}^{m-1} .) Note that ℓ is not a linear cluster because $|\check{B} \cap \mathfrak{A}^{m-1}| \leq 1$, while a base for a linear cluster contains at least 2 elements.

3) Any α -point e is either on a linear petal with size ≥ 2 or $\text{icl}(G_I(e))$ intersects two distinct $*$ -petals:

3a) $|A_j^{m+1}| > 1$ and is a linear cluster: Then for $k \leq |A_j^{m+1}|$, all k -sequences from A_j^{m+1} realize the same quantifier-free type over \check{B} (and so over \mathfrak{A}^m since $R(A_j^{m+1}, \mathfrak{A}^m) = R(A_j^{m+1}, \check{B})$, so they are automorphic over \mathfrak{A}^m in \mathfrak{A} since $\mathfrak{A}^m A_j^{m+1} \leq \mathfrak{A}$).

3b) $|A_j^{m+1}| = 1$: Then $B_j^{m+1} \subseteq A_{f,i}^m$, which is a non-linear petal and so Steiner-moves_m implies $A = A_{j,1}^{m+1}$ is moved by G_I .

4) Since \check{B} is a partial line, $\delta(\check{B}) = 2$. So, $\check{B} \leq M$ because by the hypothesis $d(\check{B}) \geq 2$. Then $\text{icl}(A_j^{m+1}) \subseteq A_j^{m+1} \cup \check{B}$, because A_j^{m+1} is a 0-primitive extension of \check{B} and $\check{B} \leq M$. Since $|\text{icl}(A_j^{m+1})| \geq |A_j^{m+1}| \geq 2$ and $\text{icl}(A_j^{m+1})$ is contained in the line $A_j^{m+1} \cup \check{B}$, $d(A_j^{m+1}) = \delta(\text{icl}(A_j^{m+1})) = 2$. \blacksquare

Remark 5.12. Note that there are R -relations within a linear cluster; it lies on one line. And at least one linear cluster is G_I -invariant, the line through $I = \{a, b\}$;

others are easy to find. But Lemma 5.11 shows no α -point is in $\text{dcl}^*(I)$. There are partial lines of various lengths in the $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ that are not linear clusters. But each is within a single non-linear petal (Lemma 5.11). (This depends essentially on the decomposition of the ambient G -normal \mathfrak{A} ; every pair of points is contained in a nontrivial line in M , but perhaps not in \mathfrak{A} .)

Lemma 5.11.3a yields immediately the answer to the motivating Problem 5.1.1. Recall a quasigroup satisfies for all x and y , there exist unique l and r such that $lx = y$ and $xr = y$ (the multiplication table is a Latin square). [GW75] show that if Steiner system has line-length k , where k is a prime-power, then it is possible to impose a binary function $*$ on the universe such that:

(#) $a, b, a * b$ is on the line through a, b and $*$ is a quasigroup such that the restriction of $*$ to each line is generated by any two elements of the line.

However, this function cannot be definable (without parameters) in a strongly minimal structure (M, R) studied here. It suffices to find one line on which the function is not defined. This is straight forward since any finite configuration is strongly embedded in M . In detail,

Theorem 5.13. *No quasigroup $*$ restricted to each line and satisfying (#) is definable in a strongly minimal Steiner system from [BP20] with line length at least four.*

Proof. Take any independent pair $I = \{a_1, a_2\}$ contained in some \mathfrak{A}^m and suppose they generate the line $A = \{a_1, a_2, \dots, a_k\}$. Then $A - I \subseteq \mathfrak{A}^{m+1} - \mathfrak{A}^m$ is a linear cluster and by Lemma 5.11.3a, G_I induces the symmetric group on $A - I$.

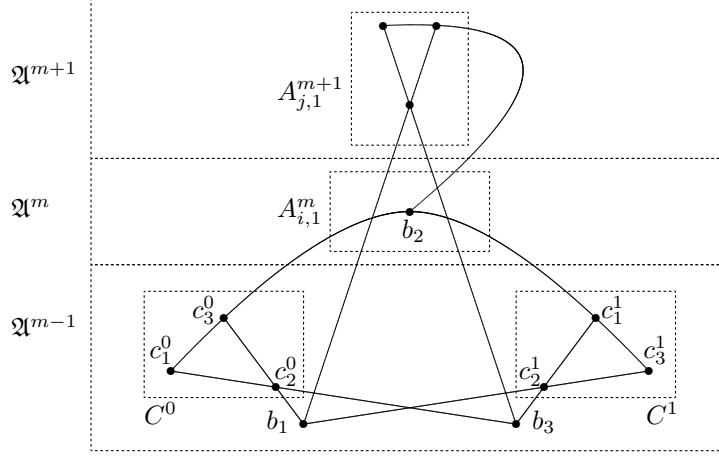
Suppose $a_1 * a_2 = a_i$ and $a_2 * a_1 = a_j$. Choose an element a_k of A distinct from a_i . There is a $g \in G_I$ with $g(a_i) = a_k$. The definition of a quasigroup is contradicted unless $a_i = a_j = a_k$; in that event replace a_k with an $a_{k'}$ distinct from all a 's previously considered; this is easy since $|A| \geq 4$. ■_{5.13}.

While this solution to Motivating Problem 5.1.1 invokes the decomposition, a more direct argument yields that result in [Bal21a]. However, here we have the much stronger consequence of the decomposition asked for in Problem 5.1.2, no truly n -ary function. For smoother reading, we mention results from Sections 3 and 4 that go through without any changes and pay attention to those results which requires some adaptations.

Lemmas 3.11 and 3.17 work for Steiner's systems. Lemma 5.11.(2) and (3).(b) yield a stronger version Lemma 3.19.(1): If $|A_{i,j}^{m+1}| = 1$ is G_I -invariant then $A_{i,k}^{m+1}$ determines a G_I -invariant non-linear petal $A_{f,i}^m$. Multiple realizations of α in $\mathfrak{A}^{m+1} - \mathfrak{A}^m$ represent distinct petals but only one $*$ -petal (linear cluster). We incorporate the role of Lemma 3.19.(2) in proving Lemma 3.22 into the proof of Lemma 5.14.

Comparing the argument for Lemma 3.19 with Figure 8 explains the main differences between Lemma 3.19 for Hrushovski's examples and Lemma 5.14 for Steiner systems. In Hrushovski's examples we obtain that b_2 is in two relations $R(b_2, c_1^0, c_3^0)$ and $R(b_2, c_1^1, c_3^1)$ with \mathfrak{A}^{m-1} , which contradicts $\mathfrak{A}^{m-1} \leq M$. But, in Steiner systems we have just one line ℓ , which contains points from different copies C^0 and C^1 of $A_{j,1}^{m+1}$.

Lemma 5.14. *Fix a decomposition of a G -normal set \mathfrak{A} , where $G \in \{G_I, G_{\{I\}}\}$. Suppose $B = B_j^{m+1}$ is the base of a non-linear petal $A_{j,1}^{m+1}$ which is G -invariant and $\ell_j^{m+1} + 1 < \mu_j^{m+1}$.*


 FIGURE 8. Example with one line and two C^d 's

- (1) Let $G = G_I$ and Steiner- \dim_m hold; or
- (2) let $G = G_{\{I\}}$ and Steiner-s \dim_m hold.

It is impossible that B has a non-empty intersection with a linear cluster A_f^m .

Proof. Suppose for contradiction that $(B \cap \ell) - \mathfrak{A}^{m-1} \neq \emptyset$, witnessed by b for some ℓ such that $A_f^m \cap \ell \neq \emptyset$ and $\ell \cap \mathfrak{A}^{m-1} = B_f^m = B'$, the extended base of b/\mathfrak{A}^{m-1} .

Step 1: We show B contains a single point b from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$. By Lemma 5.4.2, there exist $x_1, x_2 \in A_{j,1}^{m+1}$ with $R(x_1, x_2, b)$. Since $\ell_j^{m+1} + 1 < \mu_j^{m+1}$, there are (Figure 8) at least two disjoint embeddings C^0 and C^1 over B of $A_{j,i}^{m+1}$ into \mathfrak{A}^m ; the image C^i must contain copies c_1^i and c_2^i of x_1 and x_2 , which satisfy $R(c_1^i, c_2^i, b)$ for $i < 2$ and are disjoint from B . Without relying on the inductive hypotheses, the proof of Lemma 5.11.3.a shows that if the G -invariant B intersects a linear cluster A_f^m , $B \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ contains A_f^m . Since the $*$ -petals are freely joined, all the c_j^i are in \mathfrak{A}^{m-1} . So they must be in ℓ since any element in \mathfrak{A}^{m-1} related to b is in ℓ . And b is on a line with at least five elements²³ based in \mathfrak{A}^{m-1} .

In fact, $A_f^m = (\ell \cap \mathfrak{A}^m) - \mathfrak{A}^{m-1} \subseteq B$ must be a singleton. As, if b' is a second point in A_f^m , $(\{b, b'\}, c_1^i)$ realizes α with the base contained in B . But this is a contradiction, because $A_{j,1}^{m+1}$ is a non-linear petal based on B and C^i is isomorphic to $A_{j,1}^{m+1}$ over B .

Step 2: Having shown B contains a single point b from $\mathfrak{A}^m - \mathfrak{A}^{m-1}$ there are two cases. In the first case suppose this b and so its extended base B' are G -invariant. Since $\{b\}$ is G -invariant but not safe, this contradicts Steiner- \dim_m or Steiner-s \dim_m depending on $G = G_i$ or $G = G_{\{I\}}$.

We are left with the case that A_f^m is a singleton but not G -invariant, i.e. there exists $g \in G$ such that $g(A_f^m) \neq A_f^m$, but $|A_f^m| = 1$. Let $b = b_0, b_1, \dots, b_{k-1}$ enumerate the orbit of b under G . Then, for $u < k$ there is a $g_u \in G$ satisfying $R(g_u(x_1), g_u(x_2), g_u(b))$ and $g_u(x_1), g_u(x_2) \in A_{j,1}^{m+1}$, $g_u(b) \in B$ because both $A_{j,1}^{m+1}$ and B are G -invariant. Let $\langle C^i : i < \nu = \mu_j^{m+1} - 1 \rangle$ enumerate the copies of $A_{j,1}^{m+1}$

²³Note that this situation is impossible unless $\mu(\alpha) \geq 3$.

in \mathfrak{A}^m . Now, as in step 1, for each b_u, C^i , there are elements $d_1^{i,u}, d_2^{i,u} \in C^i \cap \mathfrak{A}^{m-1}$ satisfying $R(d_1^{i,u}, d_2^{i,u}, b_u)$. Again as in step 1, all the $C^i \subseteq \mathfrak{A}^{m-1}$ and for each u all the $d_w^{i,v}$ for $w < 2, i < \nu$ are on the same line. Now we consider the substructure $\bar{C} = B \cup \bigcup_{v < \nu} C^v$. If C^1 and C^2 are freely joined over B , $\delta(C^1 \cup C^2/B) = 0$. For each fixed b_u we have one new line ℓ_u with at least five points on it and the nullity of $\ell \cap (C^1 \cup C^2)$ is $4 - 2 = 2$. As no points are added this reduces $\delta(C^1 \cup C^2/B)$ by 1. (One line of length 3 in BC^2 in the computation of $\delta(C^1 C^2/B)$ has been replaced by two points added to ℓ_u .) Each additional C^i decrements another 1 so with respect to $b_u \in B \cap (\mathfrak{A}^m - \mathfrak{A}^{m-1})$ the line ℓ_u reduces $\delta(\bar{C}/B)$ by $(\nu - 1)$. But there are k such b_u and $\nu = \mu_{j,1}^{m+1} - 1$, so $\delta(\bar{C}/B) \leq k(1 - \nu) = k(1 - (\mu_{j,1}^{m+1} - 1)) = k(2 - \mu_{j,1}^{m+1})$. Hence,

$$\delta(\bar{C} \cup B) - \delta(B) = \delta(\bar{C}/B) \leq 2k - 2\mu_j^{m+1} \leq 2k - k\delta(B).$$

Consider the first and last terms and move $\delta(B)$ and $\delta(\bar{C} \cup B)$ to the opposite sides of the inequality; then divide by $k - 1$ to get

$$\delta(B) \leq \frac{2k - \delta(\bar{C} \cup B)}{k - 1} \leq \frac{2k - 2}{k - 1} = 2.$$

Recall, that B is safe, so $2 \leq d(B) \leq \delta(B) \leq \delta(\bar{C} \cup B)$. This justifies the second inequality. Thus, $d(B) = \delta(B) = 2$ and $B \leq M$ and all 0-primitive extensions of B must be independent; this contradicts the existence of the lines ℓ_u . ■

Lemma 3.21 1) and 3) concern only non-linear petals and so goes through without changes. However, a small new argument is needed for part 2).

Lemma 5.15. *Assume that $A_{j,1}^{m+1}$ is G -invariant, $|A_{j,1}^{m+1}| > 1$, and $|A_{i,f}^m| > 1$ for each i, f such that $A_{i,f}^m \cap B \neq \emptyset$. Then, for any d with $1 \leq d \leq \nu = \nu_j^{m+1}$:*

A For any i, f such that $A_{i,f}^m \cap B \neq \emptyset$, $C^d \cap A_{i,f}^m \neq \emptyset$, i.e., $C_-^d \neq \emptyset$.

B $\delta(B_-/B_+ \cup \bigcup_{1 \leq d \leq \nu} C_+^d) = \delta(B_-/B_+)$.

C If $C^d \cap \mathfrak{A}^{m-1} = \emptyset$, that is $C_+^d = \emptyset$, then there is a unique petal $A_{i,f}^m$ that contains both C^d and B_- . So, $A_{i,f}^m$ is G -invariant.

Proof. A) As in Lemma 3.21.1, for each f, i for each d , $A_{i,f}^m \cap B \neq \emptyset$ implies $C^d \cap A_{i,f}^m \neq \emptyset$. For B) note that if $\delta(B_-/B_+ \cup \bigcup_{1 \leq d \leq \nu} C_+^d) = \delta(B_-/B_+)$ fails it is because there is a line ℓ with $|\ell| \geq 3$ intersecting B_- and $B_+ \cup \bigcup_{1 \leq d \leq \nu} C_+^d$ with at most one point in B_+ . If $|\ell \cap B| = 2$, then each C^d is a linear petal. Since $A_{j,1}^{m+1} \cong_B C^d$, $A_{j,1}^{m+1}$ is also linear; contradiction. Then $|\ell \cap B| = |\ell \cap B_-| = 1$ and ℓ is based in \mathfrak{A}^{m-1} . Let $\{b\} = \ell \cap B_-$. Then $b \in A_{i,f}^m$ for some i and f , and $\{b\}$ is a linear petal over \mathfrak{A}^{m-1} , contradicting the hypothesis that $|A_{i,f}^m| > 1$ for each i, f such that $A_{i,f}^m \cap B \neq \emptyset$. C) follows as in Lemma 3.21.3. ■

From Lemma 5.14, we know that if B_j^{m+1} is the base of a G -invariant $A_{j,1}^{m+1}$, B_j^{m+1} intersects only non-linear petals. Lemma 3.23 relies on ‘ μ -triples’ but involves only calculations justified by the axiomatic properties of δ , so we can apply it here to conclude:

Corollary 5.16. *Fix a decomposition of a G -normal set \mathfrak{A} , where $G \in \{G_I, G_{\{I\}}\}$. Suppose a non-linear petal $A_{j,1}^{m+1}$ is G -invariant. Assume μ triples, (so $\mu_j^{m+1}(A/B) \geq 3$ when B is not a singleton).*

- (1) Let $G = G_I$ and Steiner-dim $_m$ hold; or
- (2) let $G = G_{\{I\}}$ and Steiner-sdim $_m$ hold.

Then, $A_{j,1}^{m+1}$ Steiner-determines a non-linear petal $A_{i,1}^m$.

We restate and prove Theorem 5.2 using essentially the same induction as in Section 4.1; the difference is that Lemma 5.11 makes the treatment of α -petals easier while we apply Corollary 5.16 for determinacy of non-linear petals.

Theorem 5.17 (no definable truly n -ary function). *Suppose T_μ^S is a Steiner-system as in Definition 0.1. Assume μ triples. Let I be a finite independent set that contains at least 2 elements. Fix a G -normal $\mathfrak{A} \leq M \models \hat{T}_\mu$ with height m_0 .*

Then for every $m \leq m_0$, $\mathfrak{A}^m \cap \text{dcl}^(I) = \emptyset$.*

Thus, $\text{dcl}^(I) \cap \mathfrak{A} = \emptyset$; so there is no n -ary \emptyset -definable function for $n \geq 2$ and \hat{T}_μ does not admit elimination of imaginaries.*

As a corollary, we obtain that $\text{dcl}(J) = \bigcup_{a \in J} \text{dcl}(a)$ for any independent set J .

As in Section 4.1, the decomposition for Theorem 5.17 is with respect to G_I .

Proof. We show Steiner-moves $_m$ and Steiner-dim $_m$ jointly imply Steiner-moves $_{m+1}$. Suppose for contradiction that $A_{j,1}^{m+1}$ is G_I -invariant. By Lemma 5.11.3, we can assume $A_{j,1}^{m+1}$ is non-linear. Then Corollary 5.16 implies that $A_{j,1}^{m+1}$ Steiner-determines a non-linear petal $A_{i,1}^m$; but this contradicts Steiner-moves $_m$.

Fix m with $1 \leq m \leq m_0$. Since Lemma 4.1.4 uses only the notions depending on abstract properties of the δ -function: Steiner-moves $_{m+1}$ and Steiner-dim $_m$ imply Steiner-dim $_{m+1}$. Thus by induction as in Section 4.1, we have Steiner-moves $_m$ for all $m \leq m_0$ and finish. \blacksquare

Before attacking the symmetric function case in general, we prove the Steiner version of Claim 4.4.11, describing the implications of the existence of a $G_{\{I\}}$ -invariant α -petal.

Lemma 5.18. *Let T_μ^S be a strongly minimal Steiner-system as described in Definition 0.1. Suppose sdim_m , $|A_{j,1}^{m+1}| = 1$ and $A_{j,1}^{m+1}$ is $G_{\{I\}}$ -invariant. Then*

- (1) $A_{j,1}^{m+1}$ determines some non-linear $A_{i,1}^m$; $B = B_j^{m+1} \leq M$, B and each $C^k = C^{m+1,k}$ is contained in $A_{i,1}^m$.
- (2) Moreover, $\mu_j^{m+1} \geq 3$;
- (3) and $\mu_i^m \geq 3$.

Proof. Lemma 5.11.2 shows $A_{j,1}^{m+1}$ determines some non-linear $A_{i,1}^m$. The analog of Lemma 4.4.11.(1), showing each $C^{m+1,q} \subseteq A_{i,1}^m \cup B_i^m$, has both a shorter proof and is stronger. Let $B = B_j^{m+1}$ be the extended base of $A_{j,1}^{m+1}$. By Lemma 5.9 $B \cap \mathfrak{A}^{m-1}$ contains at most one element, but since this set is $G_{\{I\}}$ -invariant and a one-element set cannot be safe, Steiner-sdim $_m$ implies that $B \cap \mathfrak{A}^{m-1} = \emptyset$. Also Steiner-sdim $_m$ implies $B \leq M$. So, each $C^{m+1,q} \subseteq B \subseteq A_{i,1}^m$. 2) and 3) now follow exactly by the argument in Claim 4.4.11. \blacksquare

As in Section 4.4, we now drop the μ -triples requirement and still show there is no symmetric definable function.

Theorem 5.19 (no definable symmetric function). *If T_μ^S is a Steiner-system as in Definition 0.1, then there is no symmetric \emptyset -definable v -ary function for $v \geq 2$, i.e., $\text{sdcl}^*(I) = \emptyset$ for any v -element independent set I .*

That is, there is no definable function of v variables whose value does not depend on the order of the arguments. Thus, \hat{T}_μ does not admit elimination of imaginaries. As a corollary, we obtain that $\text{sdcl}(J) = \bigcup_{a \in J} \text{sdcl}(a)$ for any independent set J .

Proof. We break the proof from Section 4.4 into several sections and indicate changes necessary for Steiner systems.

First, note Claim 4.4.4 obviously works for Steiner systems, i.e. Steiner-sdim_0 holds. The inductive proof of safety of $A_{j,1}^{m+1}$ from sdim_m , Claim 4.4.5, follows for non- $G_{\{I\}}$ -invariant non-linear petals (or α -petals) from δ -calculations as in Lemma 4.1.4.

We now fix on a $G_{\{I\}}$ -invariant $A_{j,1}^{m+1}$ that determines $G_{\{I\}}$ -invariant $A_{i,1}^m$. Our aim is to prove $A_{j,1}^{m+1}$ is safe. They respectively have μ_j^{m+1} and μ_i^m realizations in M . Results 4.4.7 through 4.4.10 establish the result when $\mu_i^m = 2$. This can only happen when $A_{j,1}^{m+1}$ is non-linear by Lemma 5.18, which is the analog of Lemma 4.4.11. These results are properties of automorphisms of finite structures and hold for the same reasons as in Subsection 4.4.

As in Lemma 4.4.12 we have reduced to the case that $\mu_j^{m+1} \geq 3$. But with Lemmas 5.16 and 5.18, while $A_{j,1}^{m+1}$ may be linear, every element of the sequence it determines is non-linear. Moreover, if $A_{j,1}^{m+1}$ is linear, $B_j^{m+1} \subseteq A_{i,1}^m$.

The analogs of Lemmas 4.4.14 through 4.4.19 complete the proof when $\mu_i^m = 2$. They go through in the Steiner case with little change. (Lemma 5.18 includes for Steiner systems the more difficult conclusion in Lemma 4.4.15.)

This leaves us with the analog of Lemmas 4.4.20 to 4.4.24, which formulate and carry out the complicated double induction. But again, one can check that the arguments go through with minor modifications. ■

6. FURTHER WORK

We worked throughout in this paper in a vocabulary with a single ternary relation symbol. We now explain a conjectured sufficient condition for the elimination of imaginaries in arbitrary finite and infinite vocabularies, using Hrushovski's δ and definition of \mathbf{K}_0 .

In [Ver02], the second author constructed a variant of Hrushovski's example with elimination of imaginaries. The idea is that for each $n \geq 3$ we add an n -ary relation R_n and put $\mu(\{a_1\}/\{a_2, \dots, a_n\}) = 1$, where the tuple (a_1, a_2, \dots, a_n) satisfies R_n . This gives us an $(n-1)$ -ary symmetric function. Thus we can construct *in an infinite vocabulary* a Hrushovski strongly minimal set which has elimination of imaginaries. The conjecture is that in some sense it is the only way to get a symmetric function in Hrushovski's examples. Recall that the constraint $\mu(A/B) \geq \delta(B)$ has a crucial role in proving the amalgamation property. However, as it was shown in [Ver06], for good pairs (A/B) satisfying $r(\{a\}, A \cup B, \{b\}) > 0$ for each $a \in A$ and $b \in B$ we may put $\mu(A/B)$ equal to any positive number while preserving the amalgamation property. A slight modification should construct definable truly n -ary functions. So, the exact formulation of the conjecture is the following:

Conjecture 6.1. *We take the class \mathbf{L}_0 to be all finite τ -structures that satisfy the hereditarily positive ϵ dimension defined in Axiom 1.1.1.2. Assume that there is a natural number N , such that $\mu(A/B) \geq \delta(B)$ for any good pair (A/B) with $\delta(B) \geq N$; then $\text{sdcl}^*(I) = \emptyset$ for any independent set I with $|I| \geq \max\{N, 5\}$.*

Thus, no Hrushovski construction in a finite relational vocabulary τ (that is, where \mathbf{K}_0 contains all finite τ -structures) has elimination of imaginaries.

We have provided several properties distinguishing among strongly minimal theories with flat geometries and provided some examples. Four directions of inquiry are 1) Can these proofs be uniformized by proving a theorem for all theories with strictly flat geometries rather than referring directly to the proof. [Mer20] makes a step in this direction by representing each strictly flat geometry by a Hrushovski construction. 2) Are there further useful distinctions among the theories of flat acl-geometries? 3) Are there further useful syntactic distinctions among the theories themselves? 4) Are there further applications in combinatorics using the methods developed here. Linear spaces and quasi-groups are only a glimpse at the structures that can be obtained when we remove the restriction that we are imposing the dimension function on *all* finite structures for a given vocabulary. Moreover as exemplified in [Bal21b], new phenomena are obtained by varying μ .

In a different direction, one can ask whether these methods might be useful higher in the complexity classification.

QUESTION 6.2. [Bal21a, Bal21b], constructs strongly minimal quasigroups using the graph of the quasigroup operation as in the study of model complete Steiner triple systems of Barbina and Casanovas [BC19]. As noted in Remark 5.27 of [BP20], their generic structure M differs radically from ours: $\text{acl}_M(X) = \text{dcl}_M(X)$.

Do the strongly minimal quasigroups in last paragraph satisfy elimination of imaginaries? Is it possible to develop a theory of q -block algebras for arbitrary prime powers similar to that for Steiner quasigroups in their paper? That is, to find a model completion for each of the various varieties of quasigroups discussed in [Bal21a]. Where do the resulting theories lie in the stability classification?

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