Iterated elementary embeddings and the model theory of infinitary logic

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Abstract

We use iterations of elementary embeddings derived from countably complete ideals on ω_1 to provide a uniform proof of some classical results connecting the number of models of cardinality $\aleph_1$ in various infinitary logics to the number of syntactic types over the empty set. We introduce the notion of an analytically presented abstract elementary class (AEC), which allows the formulation and proof of generalizations of these results to refer to Galois types rather than syntactic types. We prove (Theorem 0.4) the equivalence of $\aleph_0$-presented classes and analytically presented classes and, using this, generalize (Theorem 0.5) Keisler's theorem on few models in $\aleph_1$ to bound the number of Galois types rather than the number of syntactic types. Theorem 0.6 gives a new proof (cf. [5]) for analytically presented AEC's of the absoluteness of $\aleph_1$ categoricity from amalgamation in $\aleph_0$ and almost Galois ω-stability.

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This paper combines methods of axiomatic and descriptive set theory to study problems in model theory. In particular, we use iterated generic elementary embeddings to analyze the number of models in $\aleph_1$ in various infinitary logics and for Abstract Elementary Classes (AEC). The technique here provides a uniform method for approaching and extending theorems that Keisler et al. proved in the 1970's relating the existence of uncountable models realizing many types to the existence of many models in $\aleph_1$ (Theorem 0.3). To formalize this uniformity we introduce the notion of an analytically presented AEC and show that it is a further but more useful variant for the well-known notion of PC_δ over $L_{\omega_1,\omega}$. (Theorem 3.3). This allows us to extend the Keisler-style results relating the number of types in $\aleph_0$ to the number of models in $\aleph_1$ from syntactic types to Galois types (Theorem 5.6). Finally, we prove some absoluteness results leading to the conclusion that categoricity in $\aleph_1$ is absolute for analytically

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presented AEC that satisfy the amalgamation property in $\aleph_0$ and are almost Galois-ω-stable (Theorem 0.6).

The arguments presented here are related to arguments by Farah, Larson, et al in [9, 10], in which iterated elementary embeddings were used to prove forcing-absoluteness results. Those papers focused on the large cardinal context. Here we work primarily in ZFC.

The approach of this paper is the following proof scheme which can be abbreviated as *consistency yields provability*: Prove that a model theoretic property Φ holds in a model N of ZFC$^\circ$. Of course if Φ is absolute between N and V this is an ancient technique to attain provability. But, here we extend the model N by ultralimits (one or many) models N^* satisfying Φ and such that Φ is absolute between N and V.

We refer the reader to [35, 1] for model-theoretic definitions such as amalgamation and Shelah's notion of *Abstract Elementary Class*, and for background on the notions used here. We use [1] as single reference with a uniform notation containing many results of Keisler, Shelah and others. Similarly, Gao's text [13] is used for descriptive set theory. For example, Theorem 0.2 is stated for atomic models of first order theories. The equivalence between the atomic model context and models of a complete sentence in $L_{\omega_1,\omega}$ is explained in Chapter 6 of [1]. Abstract Elementary Classes form a general context unifying many of the properties of such infinitary logics as $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$.

A fundamental result in the study of $\aleph_1$-categoricity for Abstract Elementary Classes is the following theorem of Shelah (see [1], Theorem 17.11).

0.1 Fact (Shelah). Suppose that $\mathbf{K}$ is an Abstract Elementary Class such that

- The Lówenheim-Skolem number, LS($\mathbf{K}$), is $\aleph_0$;
- $\mathbf{K}$ is $\aleph_0$-categorical;
- amalgamation fails[1] for countable models in $\mathbf{K}$.

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$. Then there are $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$ in $\mathbf{K}$.

Theorem 0.1 is one of the two fundamental tools to develop the stability theory of $L_{\omega_1,\omega}$. The second is the following theorem of Keisler (see [21, 1], Theorem 5.2.5).

0.2 Fact (Keisler). If a PC_δ over $L_{\omega_1,\omega}$ class $\mathbf{K}$ has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then for any countable fragment $L_{\mathcal{A}}$, then *every* member of $\mathbf{K}$ realizes only countably many $L_{\mathcal{A}}$-types over $\emptyset$.

The notion of ω-stability for sentences in $L_{\omega_1,\omega}$ is a bit subtle and is more easily formulated for the associated class $\mathbf{K}$ of atomic models of a first theory with first order elementary embedding as $\prec_{\mathbf{K}}$. For *countable* $A \subseteq M \in \mathbf{K}$,

[1] Unlike first order logic, this is a strictly stronger statement than 'amalgamation fails over subsets of models of $\mathbf{K}$.'

$S_{at}(A)$ denotes the set of first order types over A realized in atomic models[2]. $\mathbf{K}$ is ω-stable[3] if for each countable $M \in \mathbf{K}$, $|S_{at}(M)| = \aleph_0$.

Combining these two theorems, Shelah showed (under the assumption $2^{\aleph_0} < 2^{\aleph_1}$) that a complete sentence of $L_{\omega_1,\omega}$ which has less than $2^{\aleph_1}$ models in $\aleph_1$ has the amalgamation property in $\aleph_0$ and is ω-stable. Crucially, Shelah's argument relies on the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in two ways. It first uses a variation of the Devlin-Shelah weak diamond principle [7] for Theorem 0.1. Then using amalgamation, extending Keisler's theorem from types over the empty set to types over a countable model is a straightforward counting argument (using $2^{\aleph_0} < 2^{\aleph_1}$), as it is in this paper. In Section 3 we work on analogues of this analysis for AEC's in which the class of countable models is analytic.

Using the iterated ultrapower approach we prove in Section 2 a strengthening of Theorem 0.3 below, which is in turn an extension of Theorem 0.2 to the logic $L_{\omega_1,\omega}(\mathrm{aa})$. The result is suggested as an exercise (page 142) in [23]. That makes it clear that it was realized by the Keisler group in at least the late 70's that this theorem could be proved for a number of logics not covered by Theorem 0.2. Keisler in the last sections of [21] developed the notion of Skolem ultrapowers to study models of set theory and two cardinal theorems in $L_{\omega_1,\omega}$. Keisler's proof of Theorem 0.2, does not involve the ultrapower construction, but expands the vocabulary of the given logic $L_{\mathcal{A}}$ by adding an ordering and then doing an inductive construction involving the omitting types theorem. While we do provide new a proof of this result, the important innovation in our approach is to apply the *iteration* of ultrapowers of models of set theory to construct a tree of models of set theory that automatically make the $L_{\mathcal{A}}$-models defined in them non-isomorphic.

The version we prove (Theorem 2.3) gives a somewhat stronger distinction; the $2^{\aleph_1}$ many models we produce pairwise realize just countably many F-types in common. As uncountability can be expressed in stationary logic, Theorem 0.3 subsumes the corresponding version for $L(Q)$.

Theorem 0.3. *Suppose that* $\mathbf{K}$ *is the class of models of some fixed sentence of* $L_{\omega_1,\omega}(\mathrm{aa})$*, and that, for some countable set* F *of* $L_{\omega_1,\omega}(\mathrm{aa})$*-formulas, uncountably many* F*-types are realized in some model in* $\mathbf{K}$*. Then there are* $2^{\aleph_1}$ *many non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$*.*

In Section 3 we introduce the notion of an analytically presented AEC, given by the natural descriptive set theoretic definition on the countable models and a similar requirement that the elementary submodel relation on countable models be analytic, and prove the following (Theorem 3.3).

Theorem 0.4. *If* $\mathbf{K}$ *is an AEC in a countable language with countable Löwenheim-Skolem number, then* $\mathbf{K}$ *can be analytically presented if and only if its restriction to* $\aleph_0$ *is the restriction to* $\aleph_0$ *of a* $PC\Gamma(\aleph_0,\aleph_0)$*-AEC.*

[2] This definition does not extend to uncountable A, see page 138 of [1]

[3] This requirement that M is a model is essential; Example 3.17 of [1], covers of the multiplicative group of $\mathbb{C}$, is ω-stable but there are countable atomic A with $|S_{at}(A)| = 2^{\aleph_0}$

There are several important features of this theorem. On the one hand it allows one to determine the class is $PC\Gamma(\aleph_0, \aleph_0)$-presented by looking only at the countable models. On the other it is more than just another presentation. Keisler's [21] treatment of the next theorem is entirely in terms of syntactic types, while Galois types are the understood notion of type for analytically presented AEC.

We prove the following partial extension of Keisler's Theorem for analytically presented Abstract Elementary Classes. Hypothesis (3) below corresponds to one of the cases given by Burgess's theorem for analytic equivalence relations (see [13], Theorem 9.1.5, and Theorem 4.3 below).[4] Theorem 0.5 follows from Theorem 5.6 below.

Theorem 0.5. *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

1. *the set of reals coding countable structures in* $\mathbf{K}$ *and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$ *are both analytic (we say analytically presented);*
2. $\mathbf{K}$ *satisfies amalgamation for countable models;*
3. *there is a countable model in* $\mathbf{K}$ *over which there is a perfect set of reals coding inequivalent Galois types.*

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$. *Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$.

Finally, we turn our attention to absoluteness and prove some results leading to the following theorem, which is a slight improvement of Theorem 4.1 of [5].

Theorem 0.6. *If* $\mathbf{K}$ *is an analytically presented almost Galois* ω*-stable AEC satisfying amalgamation in* $\aleph_0$*, then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is equivalent to a* $\utilde{\Pi}^1_2$*-sentence, and therefore absolute.*

Sections 1, 2 and 4 consist primarily of background material; the main results are presented in Sections 3, 5 and 6. In Section 1 we lay out the method of iterated ultrapowers of models of set theory; Section 2 applies this method to classes defined syntactically in various infinitary logics. Section 3 analyzes 'analytically presented' AEC. Section 4 discusses the descriptive set theory of analytic functions and equivalence relations. Section 5 extends Keisler's theorem to relate the number of Galois types to the number of models. In Section 6 we address absoluteness issues, leading up to the issue of absoluteness of $\aleph_1$-categoricity for almost Galois ω-stable analytically presented AEC satisfying amalgamation.

Most of the results in this paper were proved in 2011. The paper [5] answers some questions that were left open after our first round of results were completed. After [5] was completed we found new proofs of variations of absoluteness results which were moved to [5] from the first version of this paper.

[4] Burgess's theorem is also used by Shelah in Chapter I of [35], on a different equivalence relation.

These new proofs use the techniques of that paper, and appear here. Thus Theorem 6.9 is a weak version of part (2) Theorem 3.18 of [5] and Theorem 6.10 is a slight refinement of Theorem 4.2 of [5].

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1 Iterations

The main technical tool in this paper is the iterated generic elementary embedding induced by countably complete ideals on ω_1. We will concentrate on two such ideals, the nonstationary ideal (NS_{ω_1}) and the ideal of countable sets (`Ctble`). Though this will not be relevant here, we note that iterated embeddings via NS_{ω_1} play a fundamental role in Woodin's $\mathbb{P}_{\max}$ forcing [37] (much of this section is a condensed version of Section 1 of [27]). Iterated elementary embeddings were also used by Keisler in Chapter 33 of [21], in a manner similar to Fact 1.10 below.

The iterations constructed here could be developed using the construction of carefully specified extensions of models of set theory. See [22, 16, 8] for background on these methods. We illustrate this technique in [5].

Given an ideal I on ω_1, I^+ denotes $\mathcal{P}(\omega_1) \setminus I$. An ideal I on ω_1 is *normal* if for any $A \in I^+$, if $f\colon A \to \omega_1$ is regressive (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A$), then f is constant on a set in I^+. Fodor's Lemma (see, for instance, [18]) says that NS_{ω_1} is normal; `Ctble` is not.

Suppose that M is a model of a sufficient fragment of ZFC (such as the fragment ZFC$^\circ$below), and that, in M, I is a countably complete ideal on ω_1. Forcing over M with the Boolean algebra $\mathcal{P}(\omega_1)/I$ gives rise to an M-ultrafilter[5] $U \subseteq \mathcal{P}(\omega_1)^M \setminus I$ with the property that whenever $\{A_i : i \in \omega\}$ is a collection of sets in M with union ω_1^M, some A_i is in U. The ultrafilter U induces the equivalence relation of mod-U equivalence on functions in M with domain ω_1^M. The generic ultrapower $\mathrm{Ult}(M, U)$ consists of the corresponding equivalence classes, where for any two such functions f and g, $[f]_U \in g_U$ in $\mathrm{Ult}(M, U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha) \in g(\alpha)\} \in U$. By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M, U)$ with its Mostowski collapse. By the countable completeness of I in M, the corresponding elementary embedding $j\colon M \to \mathrm{Ult}(M, U)$ (where each element of M is mapped to the equivalence class of its corresponding constant function on ω_1^M) has critical point (i.e., first ordinal moved) ω_1^M (see Fact 1.6 and the discussion before). We say that such an embedding is *derived by forcing with $\mathcal{P}(\omega_1)/I$ over M.*

1.1 Remark. If I is normal in M, the identity function represents the ordinal ω_1^M in the ultrapower. It follows then by the definition of $\mathrm{Ult}(M, U)$ that for each $A \in \mathcal{P}(\omega_1)^M$, $A \in U$ if and only if $\omega_1^M \in j(A)$. In fact, each ordinal $\gamma \in \omega_2^M$ is represented in $\mathrm{Ult}(M, U)$ by a function of the form $f(\alpha) = o.t.(g[\alpha])$,

[5] An M-ultrafilter on ω_1 is a maximal proper filter contained in $\mathcal{P}(\omega_1)^M$; in the cases we are interested in, the filter is not an element of M.

where $g\colon \omega_1 \to \gamma$ is a surjection (and *o.t.* stands for "order type"), so (in the case where I is normal) the ordinals of $\mathrm{Ult}(M, U)$ contain an isomorphic copy of ω_2^M (which is less than or equal to $j(\omega_1^M)$, since each such f has range contained in ω_1^M) as an initial segment. We call such a function f a *canonical function* for γ. While it is possible to have well-founded ultrapowers of the form $\mathrm{Ult}(M, U)$ using NS_{ω_1} (at least assuming the existence of large cardinals), this does not always happen.

When I is `Ctble`, no function in M can represent ω_1^M in $\mathrm{Ult}(M, U)$, so the longest wellfounded initial segment of the ordinals of $\mathrm{Ult}(M, U)$ is isomorphic to ω_1^M. Generic ultrapowers with NS_{ω_1} and `Ctble` then give the two types of extensions produced by Hutchinson in [16].

Since we want to deal with structures whose existence can be proved in ZFC, we define a useful fragment of ZFC.

1.2 Definition. The fragment ZFC$^\circ$ is the theory ZFC $-$ Powerset $-$ Replacement + "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1).

1.3 Remark. In [21], Keisler uses Zermelo set theory plus a certain form of ω_1-Choice where we use ZFC$^\circ$. As far as this paper is concerned, there is no significant difference between the two theories.

The theory ZFC$^\circ$ holds in every structure of the form $H(\kappa)$ or V_κ, where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$ (recall that $H(\kappa)$ is the collection of sets whose transitive closures have cardinality less than κ).

1.4 Remark. While one can prove stronger preservation results for ZFC$^\circ$, we note the following, which suffices for the applications in this paper. Suppose that θ is a regular cardinal and P is a partial order in $H(\theta)$ such that the following hold in any forcing extension by P :

- θ is a regular cardinal greater than $2^{2^{\aleph_1}}$;
- every element of the $H(\theta)$ of the forcing extension is the realization of a P-name in $H(\theta)$ of the ground model.

Then any forcing extension of $H(\theta)$ (of V) by P is a model of ZFC$^\circ$. Therefore, if X is a countable elementary submodel of $H(\theta)$ with P as a member, then any forcing extension of the transitive collapse of X satisfies ZFC$^\circ$. The conditions above on P and θ are satisfied if $2^{2^{|P|}} < \theta$. If P is c.c.c. then $2^{(|P|^{\aleph_1})} < \theta$ suffices.

For us, the importance of ZFC$^\circ$ is that it proves Fact 1.5 below, which implies that M is elementarily embedded in $\mathrm{Ult}(M, U)$ whenever M is a model of ZFC$^\circ$ and U is an M-ultrafilter on ω_1^M. The proof of the fact is a direct application of the ω_1-Replacement-like scheme in ZFC$^\circ$.

1.5 Fact (ZFC°). Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \Rightarrow \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

We let $j[x]$ denote $\{j(y) \mid y \in x\}$. One direction of Fact 1.6 below follows from the fact that every partition in M of ω_1^M into ω many pieces must have one piece in the ultrafilter U, so, if x is countable then every function from ω_1 to x in M (i.e., every representative of a member of $j(x)$) must be constant on a set in U and so must represent a member of $j[x]$). For the other direction, note that if x is uncountable then any injection from ω_1 to x represents an element of $j(x) \setminus j[x]$ in the ultrapower $\mathrm{Ult}(V, U)$.

1.6 Fact. Suppose that M is a model of ZFC°, and that $j \colon M \to \mathrm{Ult}(M, U)$ is an elementary embedding derived from forcing over M with $\mathcal{P}(\omega_1)/I$ for some countably complete ideal in M. Then for all $x \in M$, $j(x) = j[x]$ if and only if x is countable in M.

If M is a countable model of ZFC° then there exist M-generic filters for each partial order in M. Furthermore, if $j \colon M \to N$ is an ultrapower embedding dervied by forcing with $\mathcal{P}(\omega_1)/I$ for some ideal in M (where M and N may be ill-founded), then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable (recall that the ultrapower uses only functions from M), and there exist N-generic filters for all partial orders in N. We can then continue choosing generic filters in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition; which allows a choice of ideal at each stage α.

1.7 Definition. Let M be a model of ZFC° and let γ be an ordinal less than or equal to ω_1. An *iteration* of M of length γ consists of models M_α ($\alpha \leq \gamma$), sets U_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} \colon M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

- $M_0 = M$;
- each U_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/I_\alpha)^{M_\alpha}$, where I_α is, in M_α, a countably complete ideal on ω_1;
- each $j_{\alpha\alpha}$ is the identity mapping;
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by U_α;
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system

  $$\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\},$$

  and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

The models M_α in Definition 1.7 are called *iterates* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j\colon M \to M^*$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, U_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \gamma, \beta < \gamma\rangle$$

of M, and that M^* is the final model of this iteration.

1.8 Remark. We emphasize that for any countable model M of ZFC° and any ideal I in M on ω_1^M there are $2^{\aleph_0}$ many M-generic ultrafilters for $(\mathcal{P}(\omega_1)/I)^M$. It follows that there are $2^{\aleph_1}$ many iterations of M of length ω_1.

1.9 Remark. As noted above, the ordinals of $\mathrm{Ult}(M, U)$ always contain an isomorphic copy of ω_2^M as an initial segment, whenever M is a countable (well-founded or illfounded) model of ZFC° and U is an M-normal ultrafilter. It follows from this that whenever

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \omega_1, \beta < \omega_1\rangle$$

is an iteration of M which uses normal ideals at each stage, $\omega_1^{M_{\omega_1}}$ contains a closed copy of ω_1 corresponding to the members of the set $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$. This set is called the *critical sequence* of the iteration.

Fact 1.10 below says that the final model of an iteration of length ω_1 is correct about uncountability. It is an immediate consequence of Fact 1.6 and the definition of iterations. These ultrapower arguments provide alternative methods for various results Keisler obtains by omitting types. This approach gives a proof of Corollary B on page 138 of [21]. Corollary A on page 137 can also be proved by considering ideals on other cardinals. The last sentence of Fact 1.10 follows from Remark 1.1. The (well-known) absoluteness of the existence of a model in $\aleph_1$ of an arbitrary sentence is $L_{\omega_1,\omega}$ (i.e., Theorem 2.1) follows easily from Fact 1.10.

1.10 Fact. Suppose that M is a model of ZFC°, and that M_{ω_1} is the final model of an iteration of M of length ω_1. Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable. Furthermore, ω_1^M is a proper initial segment of $\omega_1^{M_{\omega_1}}$.

Fact 1.11 records the fact that one can easily make M_{ω_1} correct about stationarity for subsets of its ω_1 if one uses NS_{ω_1} (again, this is due to Woodin [37]). Note that the notion of stationarity makes sense for any uncountable set (so in particular, for $\omega_1^{M_{\omega_1}}$ as below, even if it is ill-founded) : $Y \subseteq [X]^{\aleph_0}$ is stationary if and only if every for every function $F\colon X^{<\omega} \to X$ there is a nonempty element of Y closed under F.

1.11 Fact. Suppose that M is a model of ZFC°, $\{B_\xi : \xi < \omega_1\}$ is a partition of ω_1 into stationary sets and

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle \tag{1}$$

is an iteration of M of length ω_1 using NS_{ω_1} at each stage. Suppose that for every $\alpha < \omega_1$ and every $A \in (\mathcal{P}(\omega_1) \setminus NS_{\omega_1})^{M_\alpha}$ there is a $\xi < \omega_1$ such that, for all $\beta \in \omega_1 \setminus \alpha$,

$$\beta \in B_\xi \Rightarrow j_{\alpha,\beta}(A) \in G_\beta.$$

Then for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary.

The rest of this section concerns the subsets of ω appearing in iterates of models of ZFC°. Suppose that M is countable ω-model of ZFC°, and that $f\colon \omega_1^M \to \mathcal{P}(\omega)$ is a function in M. If U is an M-ultrafilter, then $[f]_U$ is a subset of ω in $\mathrm{Ult}(M,U)$, where for each $i \in \omega$, $i \in [f]_U$ if and only if $\{\alpha < \omega_1^M \mid i \in f(\alpha)\} \in U$. Suppose now that f is injective, and that I is a countably complete ideal on ω_1^M in M. Then no condition in $(\mathcal{P}(\omega_1)/I)^M$ can decide all of $[f]_U$, so U can be chosen so that $[f]_U$ is not equal to any element of $\mathcal{P}(\omega)^M$. By dovetailing countably many such diagonalizations at a time, one can apply this idea in a straightforward way to produce ω_1 many iterations of M of length ω_1 whose finals models (call them M_α $(\alpha < \omega_1)$) have the property that for all $\alpha < \alpha' < \omega_1$, $\mathcal{P}(\omega)^{M_\alpha} \cap \mathcal{P}(\omega)^{M_{\alpha'}} = \mathcal{P}(\omega)^M$.

The next two Theorems provide two refinements of the construction in the previous paragraph which provides the 'many' models. The first (Theorem 1.12) replaces Keisler's omitting types proof of Theorem 0.2 by one in the language of iterated generic elementary embeddings. The proof is a straightforward application of the methods of the previous paragraph. The main point is that when building $U_{\sigma,0}$ and $U_{\sigma,1}$, one must ensure that no subset of ω in $M_{\sigma^\frown\langle i\rangle}$ (for each $i \in \{0,1\}$) is equal to $j_{\langle\rangle,\tau^\frown\langle 1-i\rangle}(f)(\omega_1^{M_\tau})$ for any initial segment of τ of σ. Doing this requires diagonalizing against countably many subsets of ω (in previously constructed models) and functions in M_σ representing subsets of ω, so is easily achieved. If M_σ $(\sigma \in 2^{\omega_1})$ are as in Theorem 1.12, and σ and σ' are distinct elements of 2^{ω_1}, then the range of $j_{\langle\rangle,\sigma}(f)$ contains a subset of ω which is not in $M_{\sigma'}$.

Theorem 1.12. *Suppose that M is a countable ω-model of* ZFC°*, and that $f\colon \omega_1^M \to \mathcal{P}(\omega)$ is an injective function in M. Then there exist*

- *ω-models M_σ of* ZFC°*, for $\sigma \in 2^{\leq\omega_1}$;*
- *M_σ-ultrafilters $U_{\sigma,i}$ for $\sigma \in 2^{<\omega_1}$ and $i \in \{0,1\}$;*
- *elementary embeddings $j_{\sigma,\tau}\colon M_\sigma \to M_\tau$, for each pair σ, τ from $2^{\leq\omega_1}$ such that σ is an initial segment of τ;*

such that

- $M_{\langle\rangle} = M$;
- *each* $U_{\sigma,i}$ *is derived from forcing over* M_σ *with* $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^{M_\sigma}$*, and* $j_{\sigma,\sigma^\frown\langle i\rangle}$ *is the corresponding embedding;*
- *for each* σ, $\langle (M_{\sigma\restriction\alpha}, U_{\sigma\restriction\alpha}) : \alpha \leq |\sigma|\rangle$ *forms an iteration;*
- *whenever* σ *is a proper initial segment of* τ*, and* $i = \tau(|\sigma|)$*,*
$$j_{\langle\rangle,\sigma^\frown\langle 1-i\rangle}(f)(\omega_1^{M_\sigma}) \notin M_\tau.$$

The second refinement strengthens the result to give $2^{\aleph_1}$ many uncountable iterates with pairwise just countably many subsets of ω in common. This gives a stronger conclusion than Theorem 1.12, albeit with a proof invoking more sophisticated set theoretic notions. Although Theorem 1.13 uses models of $\mathrm{MA}_{\aleph_1}$ (the restriction of Martin's Axiom which asserts the existence of a filter meeting any $\aleph_1$ many maximal antichains from a c.c.c. partial order), our applications of it will be ZFC theorems, as such models can be proved to exist in ZFC by forcing over countable models of ZFC°.

Theorem 1.13 (Larson [26])**.** *If* M *is a countable model of* ZFC° $+\ \mathrm{MA}_{\aleph_1}$ *and*
$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle$$
and
$$\langle M'_\alpha, G'_\beta, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle$$
are two distinct iterations of M *using* NS_{ω_1}*, then*
$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} = \mathcal{P}(\omega)^{M_\beta},$$
where β *is least such that* $G_\beta \neq G'_\beta$*.*

For the reader's convenience, we sketch the proof of the version of Theorem 1.13 for iterations of length 1 (which appears in [11]). Suppose that M is a countable model of ZFC° $+\ \mathrm{MA}_{\aleph_1}$ and let G and G' be two distinct M-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then there exist disjoint sets A, A' in $(\mathcal{P}(\omega_1 \setminus NS_{\omega_1})^M$ such that $A \in G$ and $A' \in G'$. Let $N = \mathrm{Ult}(M, G)$ and $N' = \mathrm{Ult}(M, G')$, and fix $x \in \mathcal{P}(\omega)^N \setminus M$ and $x' \in \mathcal{P}(\omega)^{N'} \setminus M$. Then there exist functions $f\colon A \to \mathcal{P}(\omega)^M$ and $f'\colon A' \to \mathcal{P}(\omega)^M$ representing x in N and x' in N' respectively. Applying Fodor's Lemma we see that, since x and x' are not in M, there exist $B \subseteq A$ and $B' \subseteq A'$ in G and G' respectively on which f and f' (respectively) are injective. Applying Fodor's Lemma again we can thin B and B' to sets C and C' on which the ranges of f and f' are disjoint and contain only infinite, co-infinite sets, by subtracting nonstationary sets. Finally, it is a consequence of $\mathrm{MA}_{\aleph_1}$ (see [19], for instance) that for any two disjoint sets of infinite, co-infinite subsets of ω, there is a subset of ω which intersects each member of the first set infinitely, and no member of the second set infinitely. Thus if M satisfies $\mathrm{MA}_{\aleph_1}$ there is such a $z \subseteq \omega$ in M with respect to the ranges of $f\restriction C$ and $f\restriction C'$, which means that $x \cap z$ is infinite and $x' \cap z$ is not.

2 $L_{\omega_1,\omega}$(aa)

In this section we indicate how to use the method here to give uniform proofs a number of variants of Theorem 0.2. Note that L(aa) does not satisfy the Löwenheim-Skolem theorem down to $\aleph_0$ so it is not PC_δ over $L_{\omega_1,\omega}$. In fact, L(aa) is not an AEC. Since it has Löwenheim-Skolem number $\aleph_1$, if it were an AEC it would have Hanf number at most $\beth_{(2^{\omega_1})^+}$. But it has the same Hanf number as 2nd order logic [24]. Thus, Theorem 2.3 is not a consequence of Theorem 0.2. Here we just notice an application of our methods to give a uniform proof of known results. We significantly strengthen Theorem 0.2 with Theorem 5.6. The arguments here are sensitive to the absoluteness of the syntax of the logic so we begin with a careful description of that syntax.

Briefly, the logic $L_{\omega_1,\omega}$ is the extension of first order logic where one allows conjunctions and disjunctions of countable sets of formulas so that only finitely many free variables appear in the union of the set of formulas. Each formula in $L_{\omega_1,\omega}$ has a *rank*, the number (less than ω_1) of steps it takes to construct the formula from atomic formulas (see the appendix to [2]). More explicitly, we may think of sentences of $L_{\omega_1,\omega}$ as well-founded trees, with an associated notion of rank. An ill-founded model of ZFC$^\circ$ can contain objects which it thinks are sentences of $L_{\omega_1,\omega}$ which are really not, i.e., if the rank of the sentence as computed in the model is an ill-founded ordinal of the model. On the other hand, if a (real) sentence ϕ of $L_{\omega_1,\omega}$ exists in an ω-model M of ZFC$^\circ$, then M computes the rank correctly, and is therefore well-founded at least up the rank of ϕ. Furthermore, M correctly verifies whether the models that it sees satisfy ϕ. In both cases, the computation of the rank and the verification of the truth value, M runs exactly the same process that is carried out in V.

Shelah's logic $L_{\omega_1,\omega}$(aa) [33, 6] extends $L_{\omega_1,\omega}$ by adding the quantifier aa, where (aa $x \in [X]^{\aleph_0}$) ϕ means that for stationarily many countable $x \subseteq X$, ϕ holds, i.e., for any function $f\colon X^{<\omega} \to X$, there is a countable $x \subseteq X$ closed under f such that x satisfies ϕ. Note that "there exist uncountably many $x \in X$ such that ϕ holds" (i.e., the quantifier Q) can be expressed using aa. If M is a model of ZFC$^\circ$ as in conclusion of Fact 1.11, i.e., such that for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary, then if X is a set in M of cardinality $\aleph_1$ (in M) and Y is a subset of $[X]^{\aleph_0}$ in M, then $M_{\omega_1} \models$ "Y is stationary" if and only if Y is stationary

The second parts of the equivalences in the following theorems are $\underset{\sim}{\Sigma}^1_1$, and therefore absolute. The forward directions simply involve taking the transitive collapse of a countable elementary submodel of suitable initial segment of the universe. The reverse directions involve building iterations as in the previous section (using Fact 1.11 for correctness about stationarity). Since the final models of these iterations are well-founded up to at least the ω_2 of the corresponding original models, they verify correctly truth for ϕ and for members of the set F for the models that they see.

The existence of a model of ϕ of size $\aleph_1$ in the following theorem is also easy given the completeness theorem for L(aa) [6].

Theorem 2.1. *Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of ϕ of size $\aleph_1$ is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $\{\phi, \omega\}$ which thinks there is a model of ϕ of size $\aleph_1$.*

Theorem 2.2. *Given a countable fragment F of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of size $\aleph_1$ realizing $\aleph_1$ many distinct F-types is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $F \cup \{F, \omega\}$ which thinks there is a model of size $\aleph_1$ distinct realizing $\aleph_1$ F-types.*

Theorem 2.3 shows that each of the equivalent statements in the previous theorem implies that there are $2^{\aleph_1}$ models of size $\aleph_1$, such that each pair jointly realize only countably many F-types. Theorem 0.3 can easily be proved directly from the construction in Theorem 1.12, and is an immediate corollary to Theorem 2.3. As discussed before Theorem 1.13, the stronger conclusion of Theorem 2.3 (as opposed to Theorem 0.3) comes from the application of Theorem 1.13.

Theorem 2.3. *Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.*

Proof. Let ϕ be a sentence of $L_{\omega_1,\omega}(\mathrm{aa})$, let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$ and let N be a model of cardinality $\aleph_1$ realizing uncountably many F-types. Let X be a countable elementary submodel of $H((2^{(2^{\aleph_1})^+})^+)$ containing $\{\phi, N\}$ and the transitive closure of $\{F\}$. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse. Let M' be a forcing extension of M satisfying Martin's Axiom via a c.c.c. partial order of cardinality $(2^{\aleph_1})^+$. Then, like M, M' is a wellfounded model of ZFC$^\circ$ (see Remark 1.4). By choosing a pair of distinct generic ultrafilters (using NS_{ω_1}) for each iterate (and bookkeeping as in Fact 1.11) one can produce a $2^{<\omega_1}$-tree of iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M' of length ω_1 (as in Remark 1.8) such that the final model of each of these iterations (call them M_f $(f \in \mathcal{P}(\omega_1))$) is correct about stationarity for subsets of its version of ω_1. Each M_f has a corresponding image of N_0, N_f, and M_f is correct about which F-types are realized in N_f. By Theorem 1.13, the models M_f pairwise have just countably many subsets of ω in common. Since F-types can be associated to subsets of ω using an enumeration of F in M, the models N_f will pairwise realize just countably many F-types in common. □

If one assumes in addition that $2^{\aleph_0} < 2^{\aleph_1}$, then, as in Theorem 18.16 of [1], one gets that if there exists a model of cardinality $\aleph_1$ realizing uncountably many types over some countable subset of $L_{\omega_1,\omega}(\mathrm{aa})$, then there exists a $2^{\aleph_1}$-sized family of nonisomorphic models. That is, if there is an uncountable model N with a countable subset A over which uncountably many types are realized, then there are models N_f $(f \in \mathcal{P}(\omega_1))$ all containing the same countable set A and all realizing different sets of types over A, so that any isomorphisms of any two N_{f_1} and N_{f_2} into a third N_{f_3} must map A pointwise to different sets (which is impossible if $2^{\aleph_1} > 2^{\aleph_0}$).

3 Analytically Presented Classes

In this section, we single out a class of AEC's that can be treated by the methods of descriptive set theory. We work with an abstract elementary class $\mathbf{K}$ in a countable vocabulary τ with Löwenheim number $\aleph_0$. As in [13] we code τ-structures on ω by functions $f\colon \omega \to 2$, where f is the characteristic function of the (suitably coded by pairing functions) of the relational predicates and (graphs of) function symbols of τ. In this way the set of codes for τ-structures is a closed subset of 2^ω. For any given $L_{\omega_1,\omega}(\tau)$-sentence ϕ, the set of codes for models of ϕ is Borel, and, conversely, any set of countable τ-structures (invariant under isomorphism) for which the corresponding set of codes is Borel is the class of models of some $L_{\omega_1,\omega}(\tau)$-sentence. (These facts are Lemma 11.3.3 and Theorem 11.3.6 of [13].)

Definition 3.1. *We say that an abstract elementary class* $\mathbf{K}$ *is* analytically presented *if its vocabulary and Löwenheim-Skolem number are countable, and if the sets of codes for countable models in* $\mathbf{K}$*, and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$*, are both analytic.*

This requirement is not as *ad hoc* as it might seem. Shelah's presentation theorem (Theorem 4.15 of [1]) asserts that any AEC of τ-structures with countable Löwenheim-Skolem number can be presented as the reducts to τ of models of a first order theory in a countable language τ' which omit a family of at most $2^{\aleph_0}$-types, and the class of pairs of elementary submodels has a definition of the same form. In [1] these are called $PC\Gamma(\aleph_0, \aleph_0)$ classes when the collection of omitted types is countable.[6] Keisler writes PC_δ *over* $L_{\omega_1,\omega}$ for this notion to emphasize that it can also be described as the class of τ-structures satisfying reducts to τ of a countable conjunction (thus a single sentence) of $L_{\omega_1,\omega}(\tau')$-sentences. (Note that for Keisler's class we have to omit only countably many types by Chang's trick as in Theorem 6.1.8 of [1].)

Example 3.2. Sentences ϕ in $L_{\omega_1,\omega}$ define $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC with $\prec_{\mathbf{K}}$ taken as elementary substructure in the smallest fragment containing ϕ. Sentences ϕ in $L_{\omega_1,\omega}(Q)$ are more problematic, as being an elementary submodel in the smallest fragment containing ϕ is not preserved under unions (a union of countable sets may become uncountable). And of course many $L_{\omega_1,\omega}(Q)$ sentences have Löwenheim-Skolem number $\aleph_1$. But if we restrict to $\mathbf{K}$ the class of models of a sentence ϕ where the Q-quantifier is only used negatively and we use $\leq^*$ (i.e., small sets can't grow; see Notation 6.4.6 of [1]) then $(\mathbf{K}, \leq^*)$ is $PC\Gamma(\aleph_0, \aleph_0)$. Examples of such classes include Zilber's pseudoexponentiation, Shelah's counterexample to absoluteness of $\aleph_1$-categoricity in $L(Q)$ (Theorem 17.7 of [1]), and Example 5.3 below.

We now show that 'analytically presented' is in one sense another *nom de plume* for $PC\Gamma(\aleph_0, \aleph_0)$. But Theorem 3.3 actually yields much more. Given a class presented as a $PC\Gamma(\aleph_0, \aleph_0)$ we apply Theorem 3.3 to get an analytic

[6] Shelah writes $PC_{\aleph_0}$ or $PC(\aleph_0, \aleph_0)$, suppressing the type omission.

presentation. Then we use Theorem 5.6 on the analytic presentation of the class and similarly to Theorem 0.2 bound the number of Galois types. But it is a significantly stronger conclusion. Since the submodel relation remains the same, $L_{\mathcal{A}}$-elementary submodel, there are at least as many Galois types $L_{\mathcal{A}}$-types so the bound on the number of types is tighter.

Theorem 3.3. *An abstract elementary class* $\mathbf{K}$ *is analytically presented if and only if its restriction to countable models is the restriction to countable models of a* $PC\Gamma(\aleph_0, \aleph_0)$ *class.*

The proof of Theorem 3.3 starts here and ends with the proof of Lemma 3.6. A straightforward induction (Lemma 11.3.3 of [13]) shows that any $L_{\omega_1,\omega}$-definable set of countable models is invariant Borel (a Borel class whose membership is preserved by any permutation of the universe). Any $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC is analytically presented, as omission of a countable family of types in τ' is Borel, and taking the reduct to τ makes the class of countable models analytic. (*Mutatis mutandis* we show the analogous result for pairs of countable models (M, N) with $M \prec_{\mathbf{K}} N$.)

The converse is more complicated and we proceed by two lemmas. We first show that if an AEC $\mathbf{K}$ of τ-structures is analytically presented then the countable models of $\mathbf{K}$ are the countable models of a $PC\Gamma(\aleph_0, \aleph_0)$ class. Lemma 3.4 is the restriction of Theorem 3.3 for countable models. This result is reported as folklore[7] in Theorem 1.3.1(a) of [36]).) We haven't found a published proof; we give a detailed argument since the details are necessary for the proof for uncountable models in Lemma 3.6.

For notational simplicity in this proof, we assume τ *contains a single binary relation* R. As in [13], membership in a class of τ-structures X that is analytically definable can be coded as: there is a tree T_X (contained in $2^{<\omega} \times \omega^{<\omega}$) such that $M = (\omega, R) \in X$ if and only if for some $f \in \omega^\omega$, $(g_R, f) \in [T_X]$ is a path through T_X, where $g_R \in 2^\omega$ codes the characteristic function of R.

Lemma 3.4. *The countable* τ*-models of an analytically presented class can be represented as reducts to* τ *of a sentence in* $L_{\omega_1,\omega}(\tau')$ *for appropriate* $\tau' \supseteq \tau$. *(i.e. as noted above, the countable models of a* $PC\Gamma(\aleph_0, \aleph_0)$*.)*

Moreover the class of countable pairs (M, N) *such that* $M \prec_{\mathbf{K}} N$ *is also a* $PC\Gamma(\aleph_0, \aleph_0)$*-class.*

Proof. Extend τ to τ' by adding unary functions s, f, g, a constant symbol 0 and for each n, a $2n$-ary relation symbol S_n. Let θ_0 be an $L_{\omega_1,\omega}(\tau')$ sentence such that if M is a model of θ_0:

1. Every element of M is equal to a unique expression of the form $s^n(0)$, for some $n \in \omega$.

   Notation: For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}$ to denote the sequence $s^{\sigma(0)}(0) \ldots s^{\sigma(n-1)}(0)$ of elements of M. When convenient we will write $\mathbf{n}$ for $s^n(0)$.

[7] Lopez-Escobar [30] describes Scott's role in understanding the connection between invariant-Borel and $L_{\omega_1,\omega}$-definability but the analytic set version doesn't appear there.

2. g and f map M into M.

3. g is the characteristic function of R via a pairing function.

4. $S_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_X$.

5. For every n, $S_n(g_R \restriction n, f \restriction n)$.

Now, checking through the definitions one sees that (M, R) is in X if and only if (M, R) can be expanded to a model of θ_0. Namely, if $(\omega, R) \in X$, choose g_R as just before the statement of Lemma 3.4; interpret 0 as 0 and s as the successor function on M. Choose f with $(g_R, f) \in T_X$. Interpret S_n by condition 4. Conversely, suppose $(\omega, R, s, 0, g, f) \models \theta$. Identifying $s^n(0)$ with n, if g and f are maps from M to N, we can identify f and g with maps from ω to ω. Suppose under this indentification, $(g, f) \in [T_X]$. So g is a code for a relation R^* with $(\omega, R^*) \in X$. But (ω, R^*) is isomorphic to (ω, R) and X is invariant so $(\omega, R) \in X = p[T]$.

To complete the proof of Lemma 3.4 we need to show that

$$Y = \{(M, N, R) : M \prec_{\mathbf{K}} N, |M| = |N| = \aleph_0\}$$

is also defined as a $PC\Gamma(\aleph_0, \aleph_0)$-class. A pair of models (M, W, R) (W denotes the submodel, R is the relation) is coded by a characteristic function $p_{R,W} \in [2^{<\omega} \times 2^{<\omega}]$. In this case we begin with a vocabulary $\hat{\tau}$ obtained by adding a unary predicate W for the smaller model to τ. We expand $\hat{\tau}$ to $\hat{\tau}'$ as we expanded τ to τ'. Again, there is a tree T_Y (contained in $2^{<\omega} \times 2^{<\omega} \times \omega^{<\omega}$) such that $M = (\omega, W, R) \in Y$ if and only if for some $h \in \omega^\omega$, $(p_{R,W}, h) \in [T_Y]$ (is a path through T_Y). The argument is exactly as before in this larger vocabulary; add now a set of $2n$-ary predicates W_n to code the tree T_Y. Let θ_1 be the $\hat{\tau}'$ sentence expressing this. $\square_{3.4}$

3.5 Remark. We could just prove the second part of Lemma 3.4 by applying the first part to the class of models (M, W, R). But the argument given is needed to extend to the uncountable case.

Finally we show that there is a further vocabulary $\tilde{\tau}$ which contains uniformly definable analogues of the extra predicates in $\hat{\tau}$ and a $\tilde{\tau}$-sentence $\tilde{\theta}_0$ such that the τ-reducts of $\tilde{\theta}_0$ (in all cardinalities) are exactly the members of $\mathbf{K}$. Moreover if we add a unary predicate W to $\tilde{\tau}$ (as we extended $\hat{\tau}$ to $\hat{\tau}'$) to get $\tilde{\tau}'$, there is a sentence $\tilde{\theta}_1$ such that the $\tilde{\tau}'$-structure $(M, N, \ldots)$ satisfies $\tilde{\theta}_1$ if and only if its $\tau \cup \{W\}$-reduct satisfies $M \prec_{\mathbf{K}} N$. For this extension to uncountable models think of each model as a direct limit[8] of finitely generated (and hence countable) submodels and use the idea of the proof of Lemma 3.4 to verify that these finitely generated submodels reduct to members of $\mathbf{K}$ and that the submodel relation is $\prec_{\mathbf{K}}$. We need to rewrite and extend the argument rather

[8]Recall that although AEC's are defined in terms of unions of chains, any AEC is closed under $\prec_{\mathbf{K}}$-direct limits.

than merely quote Lemma 3.4, because we appeal to the analyticity on every (at least λ of them) finitely generated (hence countable) τ-substructure of a model M with cardinality λ. Thus we introduce parameterized versions of the functions in Lemma 3.4. This argument is inspired by the proof of Shelah's presentation theorem (Theorem 4.15 of [1]): we use the functions $t_{\boldsymbol{a}}$ to artificially create infinite finitely generated substructures. For example, the $p_{R,W}$ of Lemma 3.4 becomes the parameterized family of functions $p_{\mathbf{c},\mathbf{d}}$ to represent a model pair $(U_{\mathbf{c}}, U_{\mathbf{c},\mathbf{d}})$ consisting of the finitely generated substructures indexed by $\mathbf{c}$ and $\mathbf{cd}$ respectively.

Lemma 3.6. *All τ-models of an analytically presented AEC $\mathbf{K}$ can be represented as reducts to τ of a sentence $\tilde{\theta}_0$ in $L_{\omega_1,\omega}(\tilde{\tau})$ for appropriate $\tilde{\tau} \supseteq \tau$.*

Moreover, if $\tilde{M}$ is a $\tilde{\tau}$-substructure of $\tilde{N}$ and both $\tilde{M}$ and $\tilde{N}$ satisfy $\tilde{\theta}_0$ then $\tilde{M}\restriction\tau \prec_{\mathbf{K}} \tilde{N}\restriction\tau$.

Further, the class of pairs of τ-structures (M, N) such that $M \prec_{\mathbf{K}} N$ is the class of reducts to $\tau \cup \{W\}$ of models of $\tilde{\theta}_1$, where $\tilde{\theta}_1$ is $\tilde{\theta}_0 \wedge \tilde{\theta}_0\restriction W$.

Proof. The countable models of $\mathbf{K}$ are τ-structures coded by a tree T_X as in the paragraph before Lemma 3.4. Extend τ to $\tilde{\tau}$ by adding a unary predicate N, constant symbol 0 and unary function symbol s, for each m an $m+1$-ary relation symbol $U^m(x, \mathbf{y})$, $m+1$-ary function symbols $t^m(x, \mathbf{x}), f^m(x, \mathbf{x}), g^m(x, \mathbf{x})$, $1 + k + \ell$-ary functions $p^m(x, \mathbf{x}, \mathbf{y})$, $h^m(x, \mathbf{x}, \mathbf{y})$, $W(x, \mathbf{x}, \mathbf{y})$ and $m+1$-ary relations $U^m(x, \mathbf{x})$, $1 + k + \ell$-ary relations $W^{k,\ell}(x, \mathbf{x}, \mathbf{y})$ and for each n, $2n$-ary relation symbols S_n and W_n. (For ease of reading below, we often omit the superscripts on $U^m, f^m, g^m, t^m \ldots$; the reader should infer that the length of the parameter sequence determines the suppressed superscript.)

Let $\tilde{\theta}_0$ be an $L_{\omega_1,\omega}(\tau')$ sentence such that if M is a model of $\tilde{\theta}_0$:

1. Every element of $N(M)$ is equal to a unique expression of the form $s^n(0)$.

2. Every element of M is equal to an expression of the form $t^n(0, \boldsymbol{a})$, for some $n \in \omega$ and $\boldsymbol{a} \in M$ with length m. $U_{\boldsymbol{a}} = U(M, \boldsymbol{a}) = \{(t(\boldsymbol{a}))^i(0) : i < \omega\}$. The map $t_{\boldsymbol{a}} : \mathbf{n} \mapsto s^n_m(0, \boldsymbol{a})$ is a bijection.

3. Each $U_{\boldsymbol{a}}$ is the universe of τ-structure.

   Notation: For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}$ to denote the sequence $s^{\sigma(0)}(0) \ldots s^{\sigma(n-1)}(0)$ of elements of M. When convenient we will write $\mathbf{n}$ for $s^n(0)$.

   For a finite sequence σ (of length n) of natural numbers, we will write $\hat{\sigma}_{\boldsymbol{a}}$ to denote the sequence $(t(\boldsymbol{a}))^{\sigma(0)}(0) \ldots (t(\boldsymbol{a}))^{\sigma(n-1)}(0)$ of elements of M.

   For each of the parameterized functions we abbreviate, e.g. $\lambda x g(x, \boldsymbol{a})$ by $g_{\boldsymbol{a}} : U_{\boldsymbol{a}} \mapsto N(M)$.

4. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $W_{\mathbf{c},\mathbf{d}} = W^{k+\ell}(M, \mathbf{c}, \mathbf{d}) = U^k_{\mathbf{c}}$.

5. If $\boldsymbol{a}$ has length k, $g_{\boldsymbol{a}}$ and $f_{\boldsymbol{a}}$ map $U^k(M, \boldsymbol{a})$ into ω.

6. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $p_{\mathbf{c},\mathbf{d}}$ and $h_{\mathbf{c},\mathbf{d}}$ map $U^{k+\ell}(M, \mathbf{c}, \mathbf{d})$ into ω.
7. $g_{\boldsymbol{a}}$ is the characteristic function of $R \restriction U_{\boldsymbol{a}}$ via a pairing function.
8. For any disjoint sequences $\mathbf{c}, \mathbf{d}$ of length k and ℓ, $p_{\mathbf{c},\mathbf{d}}$ is the characteristic function of the model pair $(W^{k+\ell}_{\mathbf{c}}, U^{k+\ell}_{\mathbf{cd}})$ and the relation $R \restriction U^{k+\ell}_{\mathbf{cd}}$ via a pairing function.
9. We code $U_{\boldsymbol{a}} \in \mathbf{K}$:
   (a) $S_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_X$.
   (b) For every n, $S_n(g_{\boldsymbol{a}} \circ t_{\boldsymbol{a}} \restriction n, f_{\boldsymbol{a}} \circ t_{\boldsymbol{a}} \restriction n)$.
10. For $\boldsymbol{a} \subset \boldsymbol{a}'$ we code $U_{\boldsymbol{a}} \prec_{\mathbf{K}} U_{\boldsymbol{a}'}$:
    (a) $W_n(\hat{\sigma}, \hat{\sigma'})$ if and only if $(\sigma, \sigma') \in T_Y$.
    (b) If $\mathbf{c} \subset \mathbf{d}$, for every n, $W_n(p_{\mathbf{c},\mathbf{d}} \circ t_{\mathbf{cd}} \restriction n, h_{\mathbf{c},\mathbf{d}} \circ t_{\mathbf{cd}} \restriction n)$.

Now (*): $M \models \tilde{\theta}_0$ if and only if M is a direct limit of finitely generated $\tilde{\tau}$-substructures, which are in $\mathbf{K}$ by clause 9. The direct limit is with respect to the subsequence ($\lhd$) ordering of the finite indexing sequences and $\boldsymbol{a} \lhd \boldsymbol{a}'$ implies $U_{\boldsymbol{a}} \prec_{\mathbf{K}} U_{\boldsymbol{a}'}$ by clause 10. To see (*) note: If M is a direct limit then M is in $\mathbf{K}$ since $\mathbf{K}$ is closed under direct limits. To write M as a direct limit that witnesses $\tilde{\theta}_0$, choose the $U_{\boldsymbol{a}}$ by induction on $|\boldsymbol{a}|$. Demand that each $U_{\boldsymbol{a}} \prec_{\mathbf{K}} M$ is enumerated by $t^n_{\boldsymbol{a}}(0)$, and contains the $U_{\mathbf{b}}$ for each $\mathbf{b} \lhd \boldsymbol{a}$ and $|\mathbf{b}| < |\boldsymbol{a}|$.

Now we consider the moreover clause.

First we have $M' \restriction \tau$ is a direct limit of finitely generated partial $\tilde{\tau}$-structures $U_{\boldsymbol{a}}$ and $N' \restriction \tilde{\tau}$ is a $\prec_{\mathbf{K}}$-direct limit of $U_{\boldsymbol{a}}$ where $U_{\boldsymbol{a}}$ in the sense of M' equals $U_{\boldsymbol{a}}$ in the sense of N' for $\boldsymbol{a} \in M$ because M' is a $\tilde{\tau}$-substructure of N'. Each $U_{\boldsymbol{a}} \restriction \tau \prec_{\mathbf{K}} N' \restriction \tau$ so, since AEC's are closed under direct limits, the direct limit $M' \restriction \tau$ is a strong submodel of $N' \restriction \tau$. For the 'further' clause, just be careful in carrying out the expansion of N to a $\tilde{\tau}$ structure in the previous paragraph, that if $\boldsymbol{a} \in M'$, $U_{\boldsymbol{a}} \subseteq M$. The claim about $\tilde{\theta}_1$ is now evident. $\Box_{3.6}$

This completes the proof of Theorem 3.3. We have the following corollary.

Corollary 3.7. *If the countable models of an AEC* $\mathbf{K}$ *with Löwenheim number* $\aleph_0$ *are the countable models of an* $PC\Gamma(\aleph_0, \aleph_0)$*-class and the pairs of strong submodels are an* $PC\Gamma(\aleph_0, \aleph_0)$*-class then the class has a* $PC\Gamma(\aleph_0, \aleph_0)$*-representation.*

Proof. By one direction of Theorem 3.3, the given representation of the countable models of $\mathbf{K}$ implies it is analytically presented. By the other direction, the entire class has a $PC\Gamma(\aleph_0, \aleph_0)$-representation. $\Box_{3.7}$

To see the effect of this corollary, suppose one has a sentence in $L_{\omega_1,\omega}(Q)$ which has countable models that form an AEC. The translations to order structures of Keisler (e.g. [21], Theorem 5.1.8 of [1]) give us the hypotheses of the Corollary. But the class defined is not closed under unions of uncountable chains. So this is not the proper axiomatization; the more complicated parameterization in Lemma 3.6 is needed.

4 Analytic equivalence relations

We show below for analytically presented AEC that although Galois-ω-stability is not absolute, almost Galois-ω-stability is. The argument requires some background on analytic sets, with a focus on equivalence relations, including extracting more information from Burgess's proof of his eponymous theorem. Much of this material will not be used until Section 6. We begin by recalling the following classical fact, which follows easily from Luzin's First Separation Theorem and appears as Theorem 2.12.4 in [20].

Theorem 4.1. *If $f\colon \omega^\omega \to \omega^\omega$ is partial function which is analytic as a subset of $\omega^\omega \times \omega^\omega$, then f extends to a Borel partial function $f'\colon \omega^\omega \to \omega^\omega$.*

Notation. Suppose that E is an equivalence relation on ω^ω, and that A is a set of E-inequivalent members of ω^ω. We let $f_{E,A}$ be the function from B to A such that $xEf(x)$ for all $x \in B$, where B be the set of $x \in \omega^\omega$ which are E-equivalent to some member of A.

Our use of Theorem 4.1 is Lemma 4.2, which follows from the fact that every Borel partial function $g\colon \omega^\omega \to \omega^\omega$ has a hereditarily countable code c such that for any ω-model M of ZFC$^\circ$ with $c \in M$, $g\restriction(M \cap \omega^\omega)$ is in M (for instance, one such code c is the collection of Borel codes (see page 19 of [13]) for the sets $g^{-1}[\{x \in \omega^\omega \mid x(n) = m\}]$, for $(n, m) \in \omega \times \omega$). Similarly, there are many ways to code an analytic set. We rely on the method in the paragraph after Theorem 4.3, in which an analytic set A (contained in ω^ω or 2^ω or some product of such spaces) is coded by a Borel function f (a hereditarily countable code for f as just discussed can be taken to be a code for A, allowing any ω-model of ZFC$^\circ$ which contains it to compute its own version of A). Yet another version of coding appears in Section 5, where we discuss coding hereditarily countable sets by elements of 2^ω.

Given an equivalence relation E, and *E-inequivalent* set is a set of elements of the domain of E which are E-inequivalent.

Lemma 4.2. *Suppose that E is an analytic equivalence relation on an analytic subset of ω^ω, and that $A \subseteq \omega^\omega$ is an analytic E-inequivalent set. Let f' be a Borel partial function on ω^ω extending $f_{E,A}$ and let M be an ω-model of* ZFC$^\circ$ *containing a hereditarily countable code for f'. Then for all $x \in \mathrm{dom}(f') \cap M$, $f'(x) \in M$.*

In Sections 5 and 6, we will consider Galois types in light of Burgess's theorem on analytic equivalence relations.

Theorem 4.3 (Burgess)**.** *Suppose that E is an analytic equivalence relation on an analytic set $A \subseteq \omega^\omega$. If E has more than $\aleph_1$ many equivalence classes, then there exists a perfect E-inequivalent set $P \subseteq A$.*

We will extract some information from the proof of Burgess's theorem given in [13]. Following [13], we let LO be the set of linear orders on ω, and let WO be the set of well-orders on ω. For each countable ordinal α, we let WO_α be the set of well-orders of ω whose order type is at most α; each set WO_α is Borel as a subset of $\mathcal{P}(\omega \times \omega)$. A subset D of ω^ω is coanalytic if and only if there is a Borel function $f\colon \omega^\omega \to \mathrm{LO}$ such that $D = f^{-1}[WO]$ (this is Exercise 1.7.1 of [13]). It follows that every analytic set has a code in the sense mentioned before Lemma 4.2.

Now suppose that E is an analytic equivalence relation on an analytic set $A \subseteq \omega^\omega$. Then there exists a Borel function $f\colon \omega^\omega \times \omega^\omega \to \mathrm{LO}$ such that $E = (\omega^\omega \times \omega^\omega) \setminus f^{-1}[\mathrm{WO}]$. For each countable ordinal α, let

$$E_\alpha = (\omega^\omega \times \omega^\omega) \setminus f^{-1}[\mathrm{WO}_\alpha].$$

Then each E_α is a Borel relation, and $E = \bigcap_{\alpha<\omega_1} E_\alpha$. The proof of Theorem 9.1.2 of [13] (due to Burgess) shows that for club many $\alpha < \omega_1$, E_α is an equivalence relation. Let $C \subseteq \omega_1$ be such a club. By Silver's theorem on coanalytic equivalence relations (Theorem 5.3.5 of [13]), if any of the equivalence relations E_α ($\alpha \in C$) has uncountably many equivalence classes (when restricted to A), then it has perfectly many, in which case E also has perfectly many.

This does not finish the proof of Theorem 4.3 (see Theorem 9.1.4 of [13] for the remainder) but it is enough information for our purposes. The following remarks record the facts we will need in Section 6.

4.4 Remark. Suppose that E is an analytic equivalence relation on a analytic set $A \subseteq \omega^\omega$, and let f be as in our discussion of the proof of Theorem 4.3. Let M be an ω-model of ZFC° containing codes for E, A and f. Let E^M, A^M and f^M denote the sets constructed in M using these codes. Then E^M and A^M are (possibly proper) subsets of E and A respectively, and f^M is $f{\restriction}((\omega^\omega \times \omega^\omega) \cap M)$. Let $\{E_\alpha : \alpha \in C\}$ be as above, but constructed in M (so, in M, C is a club subset of its ω_1, which may be illfounded). Suppose that x and y are elements of A^M. If xE^My then xEy. Now suppose that x and y are E^M-inequivalent. Then, in M, the order type of $f^M(x,y)$ is the minimal countable ordinal α such that x and y are E_α-inequivalent. If this order type is wellfounded (in V) then it is in the wellfounded part of M, and x and y are E-inequivalent (in V). Otherwise they are E-equivalent.

4.5 Remark. Let E, A, f, C and E_α ($\alpha \in C$) be as above. It is a standard fact (for which we do not know of a published proof) that if P is a perfect E-inequivalent subset of A, then P is E_α-inequivalent for some $\alpha \in C$. To see this, suppose instead that for each $\alpha \in C$ there exists a pair $x, y \in P$ such that $xE_\alpha y$. Let Y be a countable elementary submodel of $H((2^{\aleph_1})^+)$ containing E, A, f and P, and let M be the transitive collapse of Y. Let M' be a generic ultrapower of M via $\mathcal{P}(\omega_1)/\mathtt{Ctble}$ and let C' be the image of $C \cap (\omega_1 \cap M)$ in this ultrapower. Then $\omega_1^{M'}$ is illfounded, and there exist $x, y \in P \cap M'$ and an illfounded t in C' such that, in M', xE_ty. It follows that $f(x,y)$ is a linear order in M' which does not embed into t, and therefore does not embed into any wellfounded ordinal of M'. Therefore, $f(x,y)$ is illfounded, and (in V) xEy, giving a contradiction.

The proof of Burgess's theorem also gives the standard fact that for an analytic equivalence relation E on an analytic set A, the assertion that A contains a perfect E-inequivalent reals is Σ^1_2 in a code for E. We conclude this section by giving an alternate proof, using ultrapowers (there is an easier proof, in the case where A is Borel).

Theorem 4.6. *Let E be an analytic equivalence relation on an analytic set $A \subseteq \omega^\omega$, and let $f\colon (\omega^\omega \times \omega^\omega) \to \mathrm{LO}$ be a Borel function such that $E = f^{-1}[\mathrm{WO}]$. The following statements are equivalent.*

1. *There exists a perfect set of E-inequivalent elements of A.*
2. *There is a countable model M of ZFC° such that*
   - *ω_1^M is well-founded;*
   - *M contains codes for E, f and A;*
   - *in M, there exists a perfect set of E-inequivalent elements of A.*

Proof. Again, the forward direction just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix M, a tree $S \in M$ projecting to A and a tree $T \subseteq 2^{\omega\omega}$ in M without terminal nodes, such that the paths through T are E-inequivalent elements of A. Since ω_1^M is well-founded, M correctly witnesses the fact that no two paths through T give E-equivalent elements of A (i.e., M can assign a countable ordinal rank to the tree of attempts to find two paths through T giving E-equivalent members of A). Furthermore, every path through T in a $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$-ultrapower of M is a member of A, as witnessed in the ultrapower. The set of such paths is an uncountable analytic set, and therefore contains a perfect set. □

5 Galois types

In this section we review Shelah's notion of Galois[9] types and apply the descriptive set theoretic techniques presented above to study them in analytically presented AEC. We expound the use of Burgess's theorem to provide a trichotomy of stability classes for analytically presented AEC. In particular we explain the relevance of work by Hyttinen-Kesala and Kueker on the one hand and Baldwin-Larson-Shelah [5] on the other to describe the connections between 'almost Galois ω-stable' and 'Galois ω-stable'. Then we prove a partial generalization of Keisler's theorem that many types implies many models to analytically presented classes. The generalization does not include Theorem 2.3 because $L(\mathrm{aa})$-classes are not analytically presented.

Following page 68 of [1] or page 234 of [35] (vol. 1) we define a (class-sized) reflexive and symmetric relation $\sim_0$ on the set of triples of the form (M, a, N), where M and N are countable structures in $\mathbf{K}$ with $M \prec_{\mathbf{K}} N$, and $a \in N \setminus M$. We say that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$ if $M_0 = M_1$ and there exist

[9]Shelah now calls them 'orbital'; Grossberg coined the evocative 'Galois'.

a structure $N \in \mathbf{K}$ and strong embeddings[10] $f_0 \colon N_0 \to N$ and $f_1 \colon N_1 \to N$ such that $f_0 \restriction M_0 = f_1 \restriction M_1$ and $f_0(a_0) = f_1(a_1)$. We let $\sim$ be the transitive closure of $\sim_0$. The equivalence classes of $\sim$ are called *Galois types*; members of an equivalence class are said to be *Galois equivalent*. A Galois type is said to be *over* a countable $M \in \mathbf{K}$ if M is the first coordinate of each triple in the Galois type.

If an abstract elementary class $\mathbf{K}$ is given syntactically in a logic $\mathcal{L}$ and $\mathbf{K}$-embedding is given by a relation refining $\mathcal{L}$-elementary submodel, the Galois types over a countable M refine the syntactic types; in general there may be more Galois types than syntactic types (e.g. [3]).

In order to apply descriptive set theory to the study of Galois types in analytically presented AEC, we need to be pedantic about coding.

A set $R \subseteq \omega \times \omega$ *codes* a hereditarily countable set a if the structure (ω, R) is isomorphic to the structure $(\mathrm{tc}(\{a\}), \in)$, where $\mathrm{tc}(x)$ denotes the transitive closure of x. If $\mathrm{tc}(\{a\})$ is infinite, then a is coded by uncountably many $R \subseteq \omega\times\omega$ (however, each $R \subseteq \omega \times \omega$ codes at most one $a \in H(\aleph_1)$, as the rigidity of the set theoretic universe implies that $a = b$ whenever $(\mathrm{tc}(\{a\}), \in)$ and $\mathrm{tc}(\{b\}), \in)$ are isomorphic).

For notational convenience, we prefer codes in 2^ω as opposed to $\mathcal{P}(\omega \times \omega)$. Fixing a bijection $\pi \colon \omega \times \omega \to \omega$, we say then that $r \in 2^\omega$ *codes* $a \in H(\aleph_1)$ if the set $\{(n, m) \in \omega \times \omega \mid r(\pi(n, m)) = 1\}$ codes a in the above sense.

For an analytically presented AEC $\mathbf{K}$, the set B consisting of those $x \in 2^\omega$ coding triples of the form (M, a, N), for M, N countable structures in $\mathbf{K}$ with $M \prec_{\mathbf{K}} N$ and $a \in N \setminus M$, is an analytic set. We let E be the equivalence relation on B where xEy if and only if x and y code Galois equivalent triples. Then E is analytic. Given a countable $M \in \mathbf{K}$, we let E_M be the equivalence relation E restricted to the set B_M consisting of codes for triples whose first element is M. Then E_M is also analytic.

Each of the reals coding a triple in a Galois type can be said to be *representing* the Galois type. Given a real $x \in B$ and a $\mathbf{K}$-structure N^*, we say that N^* *realizes the Galois type represented by* x if, letting (M, a, N) be the triple coded by x, there is a strong embedding of N into N^* which fixes M pointwise. (we also say that N^* *realizes the Galois type over* M *represented by* x).

By Burgess's Trichotomy, for each countable $M \in \mathbf{K}$ there are either at most $\aleph_1$ many E_M-equivalence classes, or a perfect set of E_M-inequivalent reals[11]. For the syntactic types discussed in the Section 2 the intermediate possibility of $\aleph_1$-types without there being a perfect set of types is impossible, as for each countable fragment of ($L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$, $L_{\omega_1,\omega}(\mathrm{aa})$) the set of types is Borel (See 4.4.13 in [31].) Note that this intermediate possibility is obscured in the presence of the CH if Galois-stability is described in terms of the number of classes.

But even for analytically presented AEC all three parts of the trichotomy can

[10] In the context of an AEC $\mathbf{K}$, a *strong embedding* $f \colon N \to N'$ is an isomorphism between N and a $\mathbf{K}$-substructure of N'.

[11] This is basically folklore for $PC\Gamma(\aleph_0, \aleph_0)$ since as we noticed in Section 3, such classes are easily seen to be analytically presented so Burgess applies.

occur (see Example 5.3 below) and Keisler's Theorem 0.2 does not generalize in full. Following [35], we use the following definitions.

5.1 Definition. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be *Galois ω-stable* if for every countable $M \in \mathbf{K}$, E_M has countably many equivalence classes, and *almost Galois ω-stable* if for each countable $M \in \mathbf{K}$, E_M does not have a perfect set of equivalence classes.[12]

The analogue for Galois types of the first order theorem that ω-stability implies stability in all powers fails except under very restrictive conditions. Baldwin and Kolesnikov [3] exhibit a complete sentence (ϕ_3) that is ω-Galois stable but not Galois stable in $\aleph_1$. Recall than an AEC $(\mathbf{K}, \prec)$ has the *joint embedding property* (*JEP*) in a cardinal $\kappa \geq \mathrm{LS}(\mathbf{K})$ if any two models in $\mathbf{K}$ of cardinality κ have a $\prec$-extension (of cardinality κ) in common.

Example 5.2. *Consider the abstract elementary class $(\mathbf{K}, \prec)$ where $\mathbf{K}$ is the class of well-order types of length $\leq \omega_1$ and $\prec$ is the initial segment relation. Then $(\mathbf{K}, \prec)$ satisfies amalgamation and joint embedding in $\aleph_0$, and is almost Galois ω-stable but not Galois ω-stable, despite being $\aleph_1$-categorical.*

In view of Example 5.2, there is no hope of a direct generalization of Theorem 0.2 to arbitrary Abstract Elementary Classes. The existence of almost Galois ω-stable but not Galois ω-stable classes is one obstruction. This example seems extreme as there are no models beyond $\aleph_1$ and no nice syntactic description of the class. In particular it is not analytically presented. But, we can find apparently more tractable examples of almost ω-Galois stability (without ω-Galois stability).

A linear order L is 1-transitive (equivalently, groupable, i.e admits a compatible group structure) if for any a, b in L, there is an automorphism of L taking a to b. The class of groupable linear orders has exactly $\aleph_1$ countable models. (See Corollary 8.6 of [32].) The following example is a variant by Jarden of a somewhat less natural version in Chapter 1 of [35].

Example 5.3. *Let $(\mathbf{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended. Since there are only $\aleph_1$ isomorphism types of components this class is almost Galois ω-stable. This AEC is analytically presented and definable as a reduct of a class in $L(Q)$. But it has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.*

We sketch an argument (told to us by Kesälä) that implies that every almost ω-Galois stable sentence of $L_{\omega_1,\omega}$ with the amalgamation property and JEP is ω-Galois stable. Hyttinen and Kesälä [15] introduced the important notions: finite character and weak Galois type. An AEC $\mathbf{K}$ has *finite character* if for $M \subseteq N$ with $M, N \in \mathbf{K}$: if for every finite $\boldsymbol{a} \in M$ there is a $\mathbf{K}$-embedding of M

[12]We make the definition this way to avoid the awkwardness that if almost Galois ω-stable is defined as having only $\aleph_1$ classes, then under CH every AEC is almost Galois ω-stable. It is not clear to us which notion is more natural for larger κ.

into N fixing $\boldsymbol{a}$, then $M \prec_{\mathbf{K}} N$. The key point is that any sentence of $L_{\omega_1,\omega}$ has finite character and any such AEC is very close to $L_{\omega_1,\omega}$. Generally speaking, sentences of $L_{\omega_1,\omega}(Q)$ do not have finite character. Two points have the same *weak Galois type* over a model M if they have the same Galois type over every finite subset of M.

It follows easily from work of Kueker [25] and Hyttinen-Kesälä [15] that *for countable models* of an AEC with finite character satisfying the amalgamation and joint embedding properties, almost Galois ω-stability implies Galois ω-stability. Here is the argument. Hyttinen and Kesala call an AEC satisfying these conditions *weakly Galois ω-stable* if there are only countably many weak types over each countable model. For such classes, Hyttinen and Kesala show, if two elements have the same weak Galois type over a *countable* model M they have the same Galois type over M. Kueker proves (Corollary 4.9 of [25]) that for finitary AEC (with ap) points a and b have the same weak-Galois type over a countable model M if and only if $\mathrm{tp}_{\infty,\omega}(a/M) = \mathrm{tp}_{\infty,\omega}(a/M)$[13]. Thus for countable models of such sentences, syntactic ω-stability implies Galois ω-stability. Since we noted above that almost Galois ω-stability implies syntactic ω-stability (if there were a model M with uncountably many syntactic types, it would have a perfect set of syntactic types and thus there would be a perfect set of Galois types over M), we get the following.

5.4 Fact. If an $L_{\omega_1,\omega}$-sentence, satisfying amalgamation and joint embedding, is almost Galois ω-stable then it is Galois ω-stable.

Baldwin, Larson, and Shelah [5] have shown a related fact, whose proof is used in the proof of Theorem 0.6.

Theorem 5.5. *Suppose that* $\mathbf{K}$ *is an analytically-presented AEC which satisfies amalgamation, and that* $\mathbf{K}$ *is almost Galois ω-stable. If* $\mathbf{K}$ *has only countably many models in* $\aleph_1$*, then* $\mathbf{K}$ *is Galois ω-stable.*

We deal here with the case that there is a perfect set of E_M-inequivalent reals, for some M (i.e., the case where almost Galois ω-stability fails). This perfect set plays roughly the role that the uncountably many syntactic types played in Theorem 2.3.

Since a Galois type is not represented by a unique real but by uncountably many reals (so in general no individual representing real can not be recovered from a triple in given Galois type), we cannot adapt the argument from the proof of Theorem 0.3 to produce many models from an uncountable set of Galois types. Instead, we use a perfect set as a sufficiently absolute representation of a set of Galois types by reals.

The following generalization of Keisler's Theorem 0.3 gives a uniform proof of the results for various logics. We do not assume that $\mathbf{K}$ satisfies amalgamation or the joint embedding property. However, one would typically use amalgamation to obtain hypothesis (5) of the theorem. The result is more general than Keisler's in that it deals with Galois types. It is somewhat weaker as the conclusion is

[13] Note this type is evaluated in a fixed Galois-saturated monster model.

not $2^{\aleph_1}$ models in $\aleph_1$ but rather a criteria which gives many models under the assumption $2^{\aleph_0} < 2^{\aleph_1}$. This assumption on cardinal arithmetic is already needed in the $L_{\omega_1,\omega}$-case to apply Keisler's result to get the weaker syntactic ω-stability.

The gist of the following theorem is: If $\mathbf{K}$ is not almost Galois-ω-stable, but satisfies amalgamation in $\aleph_0$, then for some countable N_0 there are $2^{\aleph_1}$ models in $\mathbf{K}$ of cardinality of $\aleph_1$ that are pairwise non-isomorphic over N_0. The theorem is stated more technically to permit applications where amalgamation is not assumed.

Theorem 5.6. *Suppose that*

1. $\mathbf{K}$ *is an analytically presented abstract elementary class;*
2. N_0 *is a countable structure in* $\mathbf{K}$*;*
3. P *is a perfect set of* E_{N_0}*-inequivalent members of* B_{N_0}*;*
4. N *is a* $\mathbf{K}$*-structure of cardinality* $\aleph_1$*, with* $N_0 \prec_{\mathbf{K}} N$*;*
5. N *realizes uncountably many Galois types over* N_0 *represented by members of* P*.*

Then there exists a family $\{N^\alpha : \alpha \in 2^{\aleph_1}\}$ *of* $\mathbf{K}$*-structures of cardinality* $\aleph_1$ *such that*

- *for each* $\alpha \in 2^{\aleph_1}$, $N_0 \prec_{\mathbf{K}} N^\alpha$*;*
- *for each* $\alpha \in 2^{\aleph_1}$, N^α *realizes uncountably many Galois types over* N_0 *represented by members of* P*.*
- *for each distinct pair* α, α' *from* $2^{\aleph_1}$*, the set of* $x \in P$ *for which both* N^α *and* $N^{\alpha'}$ *realize the Galois type over* N_0 *represented by* x *is countable.*

If, in addition, $2^{\aleph_0} < 2^{\aleph_1}$*, then* $\mathbf{K}$ *has* $2^{\aleph_1}$ *many nonisomorphic models of cardinality* $\aleph_1$*. If* $\mathbf{K}$ *satisfies amalgamation in* $\aleph_0$ *then the existence of an* N *as in hypotheses 4 and 5 follows from the other assumptions.*

Proof. Fix a regular $\kappa > 2^{2^{\aleph_1}}$, let f' be a Borel partial function on ω^ω extending $f_{E_{N_0},P}$ (as in Theorem 4.1), and let Y be a countable elementary submodel of $H(\kappa)$ with $\mathbf{K} \cap H(\aleph_1)$, N_0, N, P and f' in Y. Let M^* be the transitive collapse of Y, and let N^* be the image of N under this collapse. Observe that $P \cap Y$ and $f' \cap Y$ are the images of P and Y under the collapse, and that Y (and thus M^*) contains hereditarily countable codes for P and f'. Let M_0 be a forcing extension of M^* (by a c.c.c. partial order of cardinality at most $2^{\aleph_1}$ in M^*) satisfying $\mathrm{MA}_{\aleph_1}$. By the elementarity of the collapsing map on Y, there exists in M^* a continuous increasing chain $\langle N^*_\alpha : \alpha < \omega_1^{M^*} \rangle$ such that, for each $\alpha \in \omega_1^{M^*}$, N^*_α is countable in M^* and in $\mathbf{K}$, and $N^*_\alpha \prec_{\mathbf{K}} N^*$. For each $\alpha \in \omega_1^{M^*}$, let X_α be the set of reals of $M_0 \cap P$ coding triples which are $\sim$-equivalent to triples (N_0, a, N') with $N' \prec_{\mathbf{K}} N^*_\alpha$. Let $X = \bigcup_{\alpha \in \omega_1^{M^*}} X_\alpha$. Then $X \in M_0$, since for each α, X_α is Σ^1_1 in any real coding N^*_α (such reals exist in M^* since N^*_α is countable there), and M_0, being well-founded, computes Σ^1_1-truth correctly. The set X is uncountable in M^* by the elementarity of the collapsing map, and therefore also uncountable in M_0, as $\omega_1^{M_0} = \omega_1^{M^*}$. By Theorem 1.13, there are $2^{\aleph_1}$ many iterates $\{M^\alpha : \alpha \in 2^{\aleph_1}\}$ of M_0 pairwise having just countably many reals in common.

Let M^α be such an iterate via an iteration j^α, and let N^α be the corresponding image of N^*. Then in M^α, N^α realizes the Galois types of uncountably many members of $j(P \cap Y)$ over N_0. Since M^* contains a code for P, $j(P \cap Y) \subseteq P$. Furthermore, M^α is correct about uncountability, so N^α realizes (in V) the Galois types of uncountably many members of P over N_0. For each countable $N' \prec_{\mathbf{K}} N^\alpha$, there is a countable $N'' \prec_{\mathbf{K}} N^\alpha$ in M^α with $N' \prec_{\mathbf{K}} N''$. In this case, if $N_0 \prec_{\mathbf{K}} N'$ and $a \in N' \setminus N_0$, then $(N_0, a, N') \sim_0 (N_0, a, N'')$ via the identity map on N''. Since M^* contains a code for f', Lemma 4.2 implies that for each $y \in P$ representing a Galois type realized by N^α over N_0, $y \in M^\alpha$. Since M^α and $M^{\alpha'}$ have just countably many reals in common for any distinct pair α, α' in $2^{\aleph_1}$,the set of $x \in P$ for which both N^α and $N^{\alpha'}$ realize the Galois type of x over N_0 is countable.

The last two claims in the theorem follow easily. □

5.7 Remark. As with Theorem 2.3 (see the remarks immediately before it), one could use the version of Keisler's omitting types construction given in Theorem 1.12 in place of Theorem 1.13 to get a weaker theorem with an easier proof.

5.8 Remark. The proof of Theorem 5.6 gives a slightly stronger conclusion. One can get, for instance, that the set of $x \in P$ for which there exist $N_1 \in M^\alpha$ and $N_2 \in M^{\alpha'}$ such that N^α realizes the Galois type represented by x over N_1 and $N^{\alpha'}$ realizes the Galois type represented by x over N_2 is countable.

5.9 Remark. The assumption in Theorem 5.6 that the set of reals coding countable structures in $\mathbf{K}$ be analytic can be relaxed to the requirement this set of codes be universally Baire (see [12]), if one is willing to assume the existence of a Woodin cardinal with a measurable cardinal above it (see [9, 10]). However, the corresponding versions of Burgess's Theorem are weaker (see [14]), which means that the range of applications should be narrower.

6 Absoluteness of $\aleph_1$-categoricity

In first order logic, the Baldwin-Lachlan equivalence between '$\aleph_1$-categorical' and 'ω-stable with no two-cardinal models' makes the notion of $\aleph_1$-categoricity Π^1_1 and hence absolute. Shelah has provided an example of an AEC, definable in $L(Q)$, which is $\aleph_1$-categorical under MA and has $2^{\aleph_1}$ models in $\aleph_1$ under $2^{\aleph_0} < 2^{\aleph_1}$. It is an open question whether there is such a non-absolute example in $L_{\omega_1,\omega}$. Theorem 6.3.2 of [1] shows that if ϕ is an $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ (with an uncountable model) then there is a complete $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ (with an uncountable model) which implies ϕ. The remainder of the analysis in [1] and in Shelah's work on which it is based restricts to complete sentences. There is a simple argument (25.19 of [1]) that the categoricity characterization extends to incomplete sentences; but the characterization is ostensibly very dependent on assuming $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$. The importance of the completeness hypothesis manifests itself in considering amalgamation. Theorem 0.1 implies in particular the consistency of: for a complete $L_{\omega_1,\omega}$-sentence, $\aleph_1$-categoricity implies amalgamation in $\aleph_0$. But an easy example in [4] shows (in ZFC) that there is an $\aleph_1$-categorical (but not complete) $L_{\omega_1,\omega}$-sentence which fails amalgamation in $\aleph_0$.

It is shown in [2] that $\aleph_1$-categoricity is absolute for any complete sentence of $L_{\omega_1,\omega}$ which satisfies amalgamation and JEP in $\aleph_0$ and is ω-stable.[14] In this statement JEP is redundant since such a sentence is $\aleph_0$-categorical. Moreover, for such complete sentences, since ω-stability implies amalgamation in $\aleph_0$ (Corollary 19.14.3 of [1]), the result yields absoluteness of $\aleph_1$-categoricity for ω-stable sentences of $L_{\omega_1,\omega}$. The notion of ω-stability in that analysis of complete sentences (atomic classes) is a syntactic one. Here we generalize this analysis to analytically presented AEC and (almost) Galois ω-stability. Thus we avoid the completeness hypothesis. However, the study of Galois ω-stability in AEC is not sufficiently advanced as to deduce $\aleph_0$-amalgamation from Galois ω-stability.

We show here that amalgamation plus almost Galois ω-stability is enough to make $\aleph_1$-categoricity absolute for analytically presented AEC. The argument for Theorem 2.1 shows that the existence of an uncountable model for such an AEC is Σ^1_1 in a hereditarily countable parameter for the AEC in question, and therefore absolute. Amalgamation for countable models in an analytically presented AEC is similarly Π^1_2 and therefore also absolute.[15]

6.1 Remark. If $\mathbf{K}$ is an almost Galois ω-stable AEC satisfying amalgamation for countable structures, then there is a model in $\mathbf{K}$ of size $\aleph_1$ which realizes every Galois type over every one of its countable substructures (i.e., it is $(\aleph_1, \mathbf{K})$-*Galois saturated*). If an AEC $\mathbf{K}$ satisfies the joint embedding property for countable structures, then all $(\aleph_1, \mathbf{K})$-Galois saturated structures of cardinality $\aleph_1$ are isomorphic.

6.2 Remark. For cardinals $\kappa < \lambda$, we say that an AEC $\mathbf{K}$ satisfies (κ, λ)-joint embedding if for any two M_0, M_1 in $\mathbf{K}$ of cardinality κ, if there exist N_0, N_1 of cardinality λ such that $M_0 \prec_{\mathbf{K}} N_0$ and $M_1 \prec_{\mathbf{K}} N_1$, then there exist a $M_2 \in \mathbf{K}$ of cardinality κ and strong embeddings of M_0 and M_1 into M_2. The assertion that an analytically presented AEC satisfies $(\aleph_0, \aleph_1)$-joint embedding is easily

[14] Shelah's $L(Q)$-example fails amalgamation in $\aleph_0$ and is not ω-stable. The variant of the example detailed in Chapter 17 of [1] can be seen to be analytically presented using the discussion on page 55 and the interpretation of $\neg Q$ in Theorem 5.1.8 of that book.

[15] This claim does not extend to AEC which are not analytically presented: if membership in $\mathbf{K}$ were Π^1_1 or more complicated, amalgamation would not automatically satisfy Shoenfield absoluteness. As a (presumably, non-optimal) example, consider the class of linear orders of subsets $\mathcal{P}(\omega)$ which are either initial segments of the constructibility order on $\mathcal{P}(\omega)$ in L or contain a nonconstructible subset of ω, ordered by end-extension. Membership in this AEC is defined by the disjunction of a Σ^1_2 sentence and a Π^1_2 sentence, and therefore absolute. The AEC satisfies amalgamation if and only if $\mathcal{P}(\omega)$ is contained in L.

seen to be Π^1_2 in a code for $\mathbf{K}$. One can show this, for instance, by using the fact that for a given countable $M \in \mathbf{K}$, the existence of an uncountable $N \in \mathbf{K}$ with $M \prec_{\mathbf{K}} N$ is Π^1_2 in codes for $\mathbf{K}$ and M, as it is equivalent to the existence of an ω-model of ZFC$^\circ$ in which such an N exists (by the iterated ultrapower construction).

The question of $\aleph_1$-categoricity for an almost Galois ω-stable AEC $\mathbf{K}$ satisfying amalgamation then just depends on whether $\mathbf{K}$ satisfies $(\aleph_0, \aleph_1)$-joint embedding (if this fails this then $\mathbf{K}$ is clearly not $\aleph_1$-categorical) and whether it has a model of size $\aleph_1$ omitting some Galois type over some countable substructure. Theorem 6.3 below shows that a variant of this last property is equivalent to a $\undertilde{\Sigma}^1_2$ sentence, and thus absolute. In Theorem 6.10 we show that this variant, along $(\aleph_0, \aleph_1)$-JEP, is enough to characterize $\aleph_1$-categoricity.

Recall that each analytic set is Σ^1_1 definable from a hereditarily countable code (see the discussion before Lemma 4.2 and after Theorem 4.3). We use the following notion to formulate Theorems 6.3 and 6.6.

Notation. We say that model P of ZFC$^\circ$ *contains hereditarily countable codes for* $\mathbf{K}$ if P contains hereditarily countable codes for

- the set of elements of 2^ω coding countable structures in $\mathbf{K}$;
- the relation on 2^ω corresponding to $\prec_{\mathbf{K}}$.

Theorem 6.3. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC. Then the following statements are equivalent.*

1. *There exist a countable* $M \in \mathbf{K}$ *and an* $N \in \mathbf{K}$ *of cardinality* $\aleph_1$ *such that*
   (a) $M \prec_{\mathbf{K}} N$*;*
   (b) *the set of Galois types over* M *realized in* N *is countable;*
   (c) *some Galois type over* M *is not realized in* N*.*
2. *There is a countable model* P *of* ZFC$^\circ$ *, with* ω_1^P *wellfounded, which satisfies statement (1) and contains hereditarily countable codes for* $\mathbf{K}$*.*

Proof. The implication from (1) to (2) just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix M and N witnessing (1) in P. In P, there exists a countable set S containing a member of each Galois type over M realized in N, and a member t of a Galois type over M not realized in N. Fixing elements of $\omega^\omega \cap P$ coding t and the elements of S, the statement that t is not Galois-equivalent to any member of S is Π^1_1 in these codes. Since P believes this statement, and since ω_1^P is well-founded, it is true in V also that t is not Galois-equivalent to any member of S.

Then, if P' is an iterate of P by an iteration of length ω_1, and N' is the corresponding image of N, P' thinks that every element of $N' \setminus M$ satisfies a Galois type corresponding to a member of S, which, being countable in P, was fixed by this iteration. It follows that no member of N' satisfies the Galois type corresponding to t. $\square$

6.4 Remark. A simpler version of the proof of Theorem 6.3 gives that statement (1) remains absolute if clause (1c) is omitted. For each version the corresponding statement for a fixed countable $M \in \mathbf{K}$ is similarly absolute. Theorem 6.7 below shows that for a fixed countable M in an analytically presented AEC $\mathbf{K}$, the existence of an uncountable $N \in \mathbf{K}$ such that $M \prec_{\mathbf{K}} N$ and N realizes just countably many Galois types over M is equivalent to the existence of an uncountable $N \in \mathbf{K}$ such that $M \prec_{\mathbf{K}} N$ (which, as remarked above, is absolute).

6.5 Remark. By Theorem 6.3 and the paragraph before it, if $\mathbf{K}$ is an analytically presented AEC satisfying amalgamation, then $\aleph_1$-categority of $\mathbf{K}$ is absolute between models (containing ω_1^V) of a sufficient fragment of set theory in which K is Galois ω-stable. In conjunction with Fact 5.4 and Theorem 6.6 below, this gives another proof of the $\aleph_1$-categoricity fact for $L_{\omega_1,\omega}$ mentioned at the beginning of the second paragraph of this section.

In general, Galois ω-stability is *not* absolute. In more detail, arguments in [28] show that Galois ω-stability is upwards absolute for analytically presented AEC satisfying amalgamation and joint embedding, so it suffices to consider whether $\mathbf{K}$ is Galois ω-stable in $L[r]$, where r is a real parameter code for the analytic definition of $\mathbf{K}$. However, the same paper shows that Galois ω-stability is not in general downwards absolute for analytically presented AEC satisfying amalgamation and joint embedding.

However, *almost* Galois ω-stability is absolute.

Theorem 6.6. *If $\mathbf{K}$ is an analytically presented AEC, then the assertion that $\mathbf{K}$ is almost Galois ω-stable is equivalent to a Π^1_2 statement in a code for $\mathbf{K}$, and thus absolute.*

Proof. The theorem follows from the equivalence of the following two statements, which in turn is a special case of Theorem 4.6.

1. $\mathbf{K}$ is not almost Galois ω-stable.
2. There is a countable model M of ZFC$^\circ$ such that
    (a) ω_1^M is well-founded;
    (b) M contains hereditarily countable codes for $\mathbf{K}$;
    (c) in M, $\mathbf{K}$ is not almost Galois ω-stable.

□

Theorem 6.6 has an easier proof in the case where the set of codes for countable models in $\mathbf{K}$ is Borel (the set of countable models of a sentence of $L_{\omega_1,\omega}$ being one such example). In this case, almost Galois ω-stability of $\mathbf{K}$ is equivalent to the assertion that for each countable $M \in \mathbf{K}$ and each subtree of $2^{<\omega}$ without terminal nodes, either there is a path through the tree not representing a Galois type over M, or there exist distinct E_M-equivalent paths through the tree.

We return to the issue of $\aleph_1$-categoricity. The proof of Theorem 6.7 draws on both the notation and details of the proof of Burgess's Theorem 4.3.

Theorem 6.7. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC which is almost Galois* ω*-stable, and that there exist structures* M *and* N *in* $\mathbf{K}$ *such that* M *is countable,* $M \prec_{\mathbf{K}} N$ *and* N *realizes uncountably many Galois types over* M*. Then there is an uncountable* $N' \in \mathbf{K}$ *such that* $M \prec_{\mathbf{K}} N'$ *and* N' *realizes only countably many Galois types over* M*.*

Theorem 6.7 follows from the division into cases in the following remark. Case (1) contradicts almost Galois ω-stability, while Case (2) finds an uncountable model realizing only countably many Galois types over M.

6.8 Remark. Suppose that $\mathbf{K}$ is an analytically presented AEC, that $M \prec_{\mathbf{K}} N$ are structures in K with M countable and N uncountable, and N realizes uncountably many Galois types over M. Let $f\colon (\omega^\omega \times \omega^\omega) \to \mathrm{LO}$ be a Borel function such that $E_M = (\omega^\omega \times \omega^\omega) \setminus f^{-1}[WO]$. Following Burgess's proof of Theorem 4.3, we may fix a club $C \subseteq \omega_1$ such that for each $\alpha \in C$, $E_\alpha = (\omega^\omega \times \omega^\omega) \setminus f^{-1}[\mathrm{WO}_\alpha]$ is an equivalence relation. One of two possibilities holds

1. There exists an $\alpha \in C$ and an uncountable set of Galois types over M realized in N whose corresponding representing reals are E_α-inequivalent.
2. There is no such α.

In the first case, by Silver's theorem on coanalytic equivalence relations (see page 126 of [13]), there exists a perfect set of elements of 2^ω representing distinct Galois types over M; in particular, $\mathbf{K}$ is not almost Galois ω-stable. Conversely, if $\mathbf{K}$ is not almost Galois stable, and satisfies amalgamation, then there exist M and N making the first case hold. To see this, note first of all that the failure of almost Galois ω-stability gives a perfect set of (representatives for) inequivalent Galois type over some fixed countable M. By Remark 4.5 (using f and C as in the beginning of this remark, for this M) there is a perfect set of these representatives which are E_α-inequivalent for some fixed $\alpha \in C$. Amalgamation then implies that an N as in the first case exists.

In the second case, we can find a countable elementary submodel Y of $H((2^{2^{\aleph_1}})^+)$ with M, N, C and codes for f, $\mathbf{K}$ and $\prec_{\mathbf{K}}$ as elements. Let P be transitive collapse of Y, and let N_0 be the image of N under this collapse. Then P is a model of ZFC°. Let $j\colon P \to P'$ be an iteration of P of length ω_1, using the ideal `Ctble`. Then P' is correct about uncountability, and the longest well-founded initial segment of $\omega_1^{P'}$ is isomorphic to ω_1^P. Let t be an ill-founded element of $j(C \cap \omega_1^P)$. Then, in P' there are just countably many E_t-classes represented by Galois types over M realized in $j(N)$ (where E_t is, in P', the set of pairs from ω^ω mapped by f into WO_t). For any E_t-equivalent pair x, y, $f(x, y)$ is a linear order in P' not isomorphic to any initial segment of t, and therefore not to any ordinal in the wellfounded part of $\omega_1^{P'}$. It follows that $f(x, y)$ is (in V) ill-founded, and that xE_My (as in Remark 4.4). This shows that (in V) $j(N)$ realizes just countably many Galois types over M.[16]

[16] This argument is closely related to the proof of a theorem of Shelah [34] which appears as Theorem 6.3.1 of [1] and is used heavily in [5] (and also to the argument in Remark 4.5).

Theorem 6.7, along with the remarks before Theorem 6.3 gives the following (weak) variation of Theorem 5.5 (which appears in [5]). Again, instead of assuming amalgamation, we assume its relevant consequence.

Theorem 6.9. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC which is almost Galois* ω*-stable, and that there exist structures* M *and* N *in* $\mathbf{K}$ *such that* M *is countable,* $M \prec_{\mathbf{K}} N$ *and* N *realizes uncountably many Galois types over* M*. Then* $\mathbf{K}$ *is not* $\aleph_1$*-categorical.*

We can also prove the following improvement of Theorem 4.2 of [5]. We now show for an analytically presented almost Galois ω-stable AEC satisfying amalgamation in $\aleph_0$, $\aleph_1$-categoricity is equivalent to a Π^1_2 statement in a hereditarily countable code for $\mathbf{K}$; in the earlier paper the absolute statement was a conjunction of a Π^1_2 statement and a Σ^1_2 statement.

Theorem 6.10. *If* $\mathbf{K}$ *is an analytically presented almost Galois* ω*-stable AEC satisfying amalgamation in* $\aleph_0$*, then the* $\aleph_1$*-categoricity of* $\mathbf{K}$ *is equivalent to a* Π^1_2 *statement in a code for* $\mathbf{K}$*, and thus absolute.*

Proof. The Π^1_2 statement is the conjunction of the $(\aleph_0, \aleph_1)$-joint embedding property (see Remark 6.2) with the nonexistence of a P as in part (2) of Theorem 6.3. As noted after Remark 6.2, the failure of $(\aleph_0, \aleph_1)$-joint embedding gives the failure of $\aleph_1$-categoricity. Since an almost Galois ω-stable AEC satisfying amalgamation has an $(\aleph_1, \mathbf{K})$-Galois saturated model of cardinality $\aleph_1$, the existence of a P as in part (2) of Theorem 6.3 (or, more to the point, part (1) of Theorem 6.3 to which it is equivalent) also implies that $\mathbf{K}$ is not $\aleph_1$-categorical.

For the other direction, suppose that $\mathbf{K}$ is not $\aleph_1$-categorical. If $\mathbf{K}$ is Galois ω-stable and satisfies $(\aleph_0, \aleph_1)$-joint embedding, then $\mathbf{K}$ has a non-$(\aleph_1, \mathbf{K})$-saturated model of cardinality $\aleph_1$, so there exist M and N as in part (1) of Theorem 6.3. If $\mathbf{K}$ is not Galois ω-stable, then there exist $\mathbf{K}$-structures $M \prec_{\mathbf{K}} N$ with M countable and N uncountable, and N realizing uncountably many Galois types over N. Then Theorem 6.7 gives a pair of models as in in part (1) of Theorem 6.3. ☐

6.11 Remark. Suppose that $\mathbf{K}$ is an AEC, and let $\mathbf{K}'$ be the class of structures from $\mathbf{K}$ which are either uncountable or have uncountable $\prec_{\mathbf{K}}$-extensions. It can happen (for instance, if $\mathbf{K}$ satisfies amalgamation for countable structures, although this is not necessary) that $\mathbf{K}'$ is an AEC, with $\prec_{\mathbf{K}'}$ as the restriction of $\prec_{\mathbf{K}}$ to $\mathbf{K}'$. If $\mathbf{K}$ is analytically presented, then $\mathbf{K}'$ is also. If $\mathbf{K}$ satisfies amalgamation in $\aleph_0$ then so does $\mathbf{K}'$ (the converse need not hold). Similarly, JEP in $\aleph_0$ for $\mathbf{K}'$ is equivalent to $(\aleph_0, \aleph_1)$-JEP for $\mathbf{K}$ (if $\mathbf{K}$ satisfies both JEP and amalgamation in $\aleph_0$ then $\mathbf{K}$ and $\mathbf{K}'$ are the same). Clearly, $\mathbf{K}'$ is $\aleph_1$-categorical if and only if $\mathbf{K}$ is. There are cases in which Theorem 6.10 applies to $\mathbf{K}'$ but not to $\mathbf{K}$; for instance, if $\mathbf{K}'$ is almost Galois ω-stable, while $\mathbf{K}$ is not (which implies that $\mathbf{K}$ fails amalgamation in $\aleph_0$), and $\mathbf{K}'$ satisfies amalgamation in $\aleph_0$. The proof of Theorem 4.2 of [5] works with $\mathbf{K}'$ in place of $\mathbf{K}$.

The following is an example of an AEC where Theorem 6.10 applies to $\mathbf{K}'$ (as above) but not to $\mathbf{K}$. It shows that some additional conditions must be placed on an AEC for $\aleph_1$-categoricity to be well behaved.

Example 6.12. Let τ contain unary functions p and s, a unary predicate P and a constant 0. Suppose the axioms of an $L_{\omega_1,\omega}$ defined class $\mathbf{K}$ say that that p and s are inverses and there are no p-cycles. Suppose further that no element that is a finite distance from 0 is in P and that any two elements satisfying P are finitely far apart. And finally that if P is not empty then all elements are finitely distant from either that element or from 0.

The countable members of $\mathbf{K}$ are a 'prime' model M consisting of a copy of (Z, p, s) with no element satisfying P, $2^{\aleph_0}$ models consisting of a copy of M and a Z-chain with some elements realizing P and any countable number of Z-chains with P empty.

This class is not almost Galois ω-stable but is $\aleph_1$ categorical. It fails amalgamation and jep in $\aleph_0$ but they hold in all larger cardinals.

This paper was motivated by the question: 'Is $\aleph_1$ categoricity absolute for sentences of $L_{\omega_1,\omega}$?' The following questions explore related issues and extend them to analytically presented AEC when appropriate.

6.13 Question. It is shown in [5] (Theorem 3.18) that if an almost Galois ω-stable analytically presented class satisfying amalgamation and joint embedding *has only countably many models in* $\aleph_1$, then it is Galois ω-stable. Can countably many be relaxed to $< 2^{\aleph_1}$? Alternately, is it consistent with ZFC that there is an analytically presented AEC with amalgamation and joint embedding in $\aleph_0$, that is almost Galois ω-stable but not Galois ω-stable, despite having less than $2^{\aleph_1}$ models in $\aleph_1$?

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